LIPSCHITZ TYPE p-SCHATTEN NORM INEQUALITIES FOR ANALYTIC FUNCTIONS OF OPERATORS IN HILBERT SPACES

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Abstract. Let H be a complex Hilbert space. A bounded linear operator $A \in \mathcal{B}(H)$ is said to belong to the *von Neumann-Schatten class* $\mathcal{B}_p(H)$, $1 \leq p < \infty$ if the *p-Schatten norm* is finite

$$\|A\|_p := [\operatorname{tr}(|A|^p)]^{1/p} < \infty,$$

where $\operatorname{tr}(\cdot)$ is the operator trace functional.

Let $f : D \subset \mathbb{C} \to \mathbb{C}$ be an analytic function on the convex domain D and $X, Y \in \mathcal{B}_p(H)$, $p \geq 1$ with $\sigma(X), \sigma(Y) \subset D$ and γ a closed rectifiable path in D and such that $\sigma(X), \sigma(Y) \subset \operatorname{ins}(\gamma)$, then we have

$$\|f(Y) - f(X)\|_p \leq \frac{1}{2\pi} \|Y - X\|_p \int_\gamma \frac{|f(\xi)|}{(|\xi| - \|Y\|)(|\xi| - \|X\|)} |d\xi|,$$

where $\|\cdot\|$ is the usual operator norm. Some examples for power series of operators are also provided.

1. Introduction

Let $(H; \langle ., . \rangle)$ be a complex Hilbert space and $\mathcal{B}(H)$ the Banach algebra of all bounded linear operators on H. If $\{e_i\}_{i \in I}$ an orthonormal basis of H, we say that $A \in \mathcal{B}(H)$ is of *trace class* if

$$\|A\|_1 := \sum_{i \in I} \langle |A| e_i, e_i \rangle < \infty. \tag{1.1}$$

The definition of $\|A\|_1$ does not depend on the choice of the orthonormal basis $\{e_i\}_{i \in I}$. We denote by $\mathcal{B}_1(H)$ the set of trace class operators in $\mathcal{B}(H)$.

We define the *trace* of a trace class operator $A \in \mathcal{B}_1(H)$ to be

$$\operatorname{tr}(A) := \sum_{i \in I} \langle A e_i, e_i \rangle, \tag{1.2}$$

where $\{e_i\}_{i \in I}$ an orthonormal basis of H. Note that this coincides with the usual definition of the trace if H is finite-dimensional. We observe that the series (1.2) converges absolutely and it is independent from the choice of basis.

The following result collects some properties of the trace:

Theorem 1. *We have:*

(i) If $A \in \mathcal{B}_1(H)$ then $A^* \in \mathcal{B}_1(H)$ and

$$\operatorname{tr}(A^*) = \overline{\operatorname{tr}(A)}; \tag{1.3}$$

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(ii) If $A \in \mathcal{B}_1(H)$ *and* $T \in \mathcal{B}(H)$, *then* $AT, TA \in \mathcal{B}_1(H)$ *and*

$$\operatorname{tr}(AT) = \operatorname{tr}(TA) \text{ and } |\operatorname{tr}(AT)| \le \|A\|_1 \|T\|; \tag{1.4}$$

(iii) $\operatorname{tr}(\cdot)$ *is a bounded linear functional on* $\mathcal{B}_1(H)$ *with* $\|\operatorname{tr}\| = 1$;
(iv) If $A, B \in \mathcal{B}_2(H)$ *then* $AB, BA \in \mathcal{B}_1(H)$ *and* $\operatorname{tr}(AB) = \operatorname{tr}(BA)$;
(v) $\mathcal{B}_{fin}(H)$ *is a dense subspace of* $\mathcal{B}_1(H)$.

An operator $A \in \mathcal{B}(H)$ is said to belong to the *von Neumann-Schatten class* $\mathcal{B}_p(H)$, $1 \le p < \infty$ if the *p-Schatten norm* is finite [16, p. 60-64]

$$\|A\|_p := [\operatorname{tr}(|A|^p)]^{1/p} < \infty.$$

For $1 < p < q < \infty$ we have that

$$\mathcal{B}_1(H) \subset \mathcal{B}_p(H) \subset \mathcal{B}_q(H) \subset \mathcal{B}(H) \tag{1.5}$$

and

$$\|A\|_1 \ge \|A\|_p \ge \|A\|_q \ge \|A\|. \tag{1.6}$$

For $p \ge 1$ the functional $\|\cdot\|_p$ is a *norm* on the $*$-ideal $\mathcal{B}_p(H)$ and $\left(\mathcal{B}_p(H), \|\cdot\|_p\right)$ is a Banach space.

Also, see for instance [16, p. 60-64],

$$\|A\|_p = \|A^*\|_p, \ A \in \mathcal{B}_p(H) \tag{1.7}$$

$$\|AB\|_p \le \|A\|_p \|B\|_p, \ A, B \in \mathcal{B}_p(H) \tag{1.8}$$

and

$$\|AB\|_p \le \|A\|_p \|B\|, \ \|BA\|_p \le \|B\| \|A\|_p, \ A \in \mathcal{B}_p(H), \ B \in \mathcal{B}(H). \tag{1.9}$$

This implies that

$$\|CAB\|_p \le \|C\| \|A\|_p \|B\|, \ A \in \mathcal{B}_p(H), \ B, \ C \in \mathcal{B}(H). \tag{1.10}$$

In terms of *p-Schatten norm* we have the *Hölder inequality* for $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$

$$(|\operatorname{tr}(AB)| \le) \|AB\|_1 \le \|A\|_p \|B\|_q, \ A \in \mathcal{B}_p(H), \ B \in \mathcal{B}_q(H). \tag{1.11}$$

For the theory of trace functionals and their applications the reader is referred to [15] and [16].

For some classical trace inequalities see [4], [6], and [14], which are continuations of the work of Bellman [2].

Consider a complex Hilbert space $(H, \langle \cdot, \cdot \rangle)$. An operator T is said to be positive (denoted by $T \ge 0$) if $\langle Tx, x \rangle \ge 0$ for all $x \in H$ and also an operator T is said to be *strictly positive* (denoted by $T > 0$) if T is positive and invertible. A real valued continuous function f on $(0, \infty)$ is said to be operator monotone if $f(A) \ge f(B)$ holds for any $A \ge B > 0$. By $A \ge B$ we understand that $A - B \ge 0$

In 1934, K. Löwner [12] had given a definitive characterization of operator monotone functions as follows, see for instance [3, p. 144-145]:

Theorem 2. *A function* $f : [0, \infty) \to \mathbb{R}$ *is operator monotone in* $[0, \infty)$ *if and only if it has the representation*

$$f(t) = f(0) + bt + \int_0^\infty \frac{t\lambda}{t + \lambda} d\mu(\lambda) \tag{1.12}$$

where μ is a positive measure on $(0,\infty)$ such that

$$\int_0^\infty \frac{\lambda}{1+\lambda} d\mu(\lambda) < \infty.$$

We recall the important fact proved by Löwner and Heinz that states that the power function $f:(0,\infty)\to\mathbb{R}$, $f(t)=t^\alpha$ is an operator monotone function for any $\alpha\in[0,1]$, [11]. The function ln is also operator monotone on $(0,\infty)$. For other examples of operator monotone functions, see [9] and [10].

A real valued continuous function f on an interval I is said to be *operator convex (operator concave)* on I if

$$f((1-\lambda)A+\lambda B) \le (\ge)\,(1-\lambda) f(A) + \lambda f(B) \tag{OC}$$

in the operator order, for all $\lambda\in[0,1]$ and for every selfadjoint operator A and B on a Hilbert space H whose spectra are contained in I. Notice that a function f is operator concave if $-f$ is operator convex.

We have the following representation of operator convex functions [3, p. 147]:

Theorem 3. *A function $f:[0,\infty)\to\mathbb{R}$ is operator convex in $[0,\infty)$ with $f'_+(0)\in\mathbb{R}$ if and only if it has the representation*

$$f(t) = f(0) + f'_+(0)\,t + ct^2 + \int_0^\infty \frac{t^2\lambda}{t+\lambda} d\mu(\lambda), \tag{1.13}$$

where $c\ge 0$ and a positive measure μ on $(0,\infty)$ such that (1.13) holds.

In the recent paper [8] we obtained the following Lipschitz type inequalities for p-Schatten norm.

Assume that $f:[0,\infty)\to\mathbb{R}$ is an operator monotone function. If $A, B\in\mathcal{B}_p(H)$, $p\ge 1$ and $A\ge m_1>0$, $B\ge m_2>0$, then

$$\|f(B)-f(A)\|_p \le \|B-A\|_p \times \begin{cases} \frac{f(m_2)-f(m_1)}{m_2-m_1} & \text{if } m_1\ne m_2, \\ \\ f'(m) & \text{if } m_1=m_2=m. \end{cases}$$

We also have the trace inequality

$$\begin{aligned} &|\operatorname{tr}[f(B)] - \operatorname{tr}[f(A)]| \\ &\le \|B-A\|_1 \times \begin{cases} \frac{f(m_2)-f(m_1)}{m_2-m_1} & \text{if } m_1\ne m_2, \\ \\ f'(m) & \text{if } m_1=m_2=m, \end{cases} \end{aligned} \tag{1.14}$$

provided that $A, B\in\mathcal{B}_1(H)$ with $A\ge m_1>0$, $B\ge m_2>0$.

Assume that $f:[0,\infty)\to\mathbb{R}$ is operator convex in $[0,\infty)$. If $A, B\in\mathcal{B}_p(H)$, $p\ge 1$ and $A\ge m_1>0$, $B\ge m_2>0$, then

$$\begin{aligned} &\left\|f(B)B^{-1} - f(A)A^{-1} - f(0)\left(B^{-1}-A^{-1}\right)\right\|_p \\ &\le \|B-A\|_p \times \begin{cases} \frac{f(m_2)m_2^{-1} - f(m_1)m_1^{-1} - f(0)\left(m_2^{-1}-m_1^{-1}\right)}{m_2-m_1} & \text{if } m_1\ne m_2, \\ \\ \frac{f'(m)m - f(m) + f(0)}{m^2} & \text{if } m_1=m_2=m. \end{cases} \end{aligned} \tag{1.15}$$

In particular, if $f(0)=0$, then we get the simpler inequality

$$\left\|f(B)B^{-1}-f(A)A^{-1}\right\|_p \leq \|B-A\|_p \times \begin{cases} \frac{f(m_2)m_2^{-1}-f(m_1)m_1^{-1}}{m_2-m_1} \text{ if } m_1 \neq m_2, \\ \\ \frac{f'(m)m-f(m)}{m^2} \text{ if } m_1 = m_2 = m. \end{cases} \tag{1.16}$$

We also have the trace inequality

$$\left|\operatorname{tr}\left[f(B)B^{-1}\right]-\operatorname{tr}\left[f(A)A^{-1}\right]\right| \leq \|B-A\|_1 \times \begin{cases} \frac{f(m_2)m_2^{-1}-f(m_1)m_1^{-1}}{m_2-m_1} \text{ if } m_1 \neq m_2, \\ \\ \frac{f'(m)m-f(m)}{m^2} \text{ if } m_1 = m_2 = m, \end{cases} \tag{1.17}$$

provided that $A,\ B\in\mathcal{B}_1(H)$ with $A\geq m_1>0$, $B\geq m_2>0$.

For similar results concerning the operator norm, see [7].

In this paper we show among others that, if $f: D\subset\mathbb{C}\to\mathbb{C}$ is an analytic function on the convex domain D and $X,Y\in\mathcal{B}_p(H)$, $p\geq 1$ with $\sigma(X),\sigma(Y)\subset D$ and γ is a closed rectifiable path in D and such that $\sigma(X),\ \sigma(Y)\subset \operatorname{ins}(\gamma)$, then we have

$$\|f(Y)-f(X)\|_p \leq \frac{1}{2\pi}\|Y-X\|_p \int_\gamma \frac{|f(\xi)|}{(|\xi|-\|Y\|)(|\xi|-\|X\|)}|d\xi|,$$

where $\|\cdot\|$ is the usual operator norm. Some examples for power series of operators are also provided.

2. Main Results

We have the following simple identity:

Lemma 1. *Let $A,\ B\in B(H)$ with $A\neq B$ and assume that the closed segment $[A,B]:=\{(1-t)A+tB,\ t\in[0,1]\}\subset\operatorname{Inv}(\mathcal{B}(H))$. Then we have the representation*

$$B^{-1}-A^{-1}=\int_0^1((1-t)A+tB)^{-1}(A-B)((1-t)A+tB)^{-1}dt. \tag{2.1}$$

Proof. Consider the function with values in Banach algebra $\mathcal{B}(H)$, $\varphi_{A,B}:[0,1]\to\mathcal{B}(H)$,

$$\varphi_{A,B}(t)=((1-t)A+tB)^{-1}.$$

This is well defined for $t\in[0,1]$.

We recall the following simple identity for the two invertible elements A, B in $\mathcal{B}(H)$

$$A^{-1}-B^{-1}=A^{-1}(B-A)B^{-1}.$$

For $t \in (0,1)$ and h in a neighborhood of 0 such that $t+h \in (0,1)$, we have

$$\begin{aligned}
&\varphi_{A,B}(t+h) - \varphi_{A,B}(t) \\
&= ((1-(t+h))A + (t+h)B)^{-1} - ((1-t)A + tB)^{-1} \\
&= ((1-(t+h))A + (t+h)B)^{-1}((1-t)A + tB - (1-(t+h))A - (t+h)B) \\
&\times ((1-t)A + tB)^{-1} \\
&= h((1-(t+h))A + (t+h)B)^{-1}(A-B)((1-t)A + tB)^{-1}.
\end{aligned}$$

This implies, for $h \neq 0$, that

$$\begin{aligned}
&\frac{\varphi_{A,B}(t+h) - \varphi_{A,B}(t)}{h} \\
&= ((1-(t+h))A + (t+h)B)^{-1}(A-B)((1-t)A + tB)^{-1}
\end{aligned}$$

and by taking the limit over $h \to 0$, we get

$$\varphi'_{A,B}(t) = ((1-t)A + tB)^{-1}(A-B)((1-t)A + tB)^{-1}$$

that holds for $t \in (0,1)$.

By the use of Bochner's integral, [13], for functions with values in Banach algebras, we have

$$\varphi_{A,B}(1) - \varphi_{A,B}(0) = \int_0^1 \varphi'_{A,B}(t)\,dt,$$

which is the desired equality (2.1). □

Corollary 1. *Let $X, Y \in \mathcal{B}(H)$ with $X \neq Y$ and assume that $\|X\|, \|Y\| < 1$, then*

$$\begin{aligned}
&(1-Y)^{-1} - (1-X)^{-1} \\
&= \int_0^1 (1-(1-t)X - tY)^{-1}(Y-X)(1-(1-t)X - tY)^{-1}\,dt.
\end{aligned} \tag{2.2}$$

Follows by Lemma 1 by taking $B = 1-Y$ and $A = 1-X$.

Let $\mathcal{B}$ be a unital Banach algebra, $a \in \mathcal{B}$ and G be a domain of $\mathbb{C}$ with $\sigma(a) \subset G$. If $f : G \to \mathbb{C}$ is analytic on G, we define an element $f(a)$ in $\mathcal{B}$ by

$$f(a) := \frac{1}{2\pi i}\int_\delta f(\xi)(\xi - a)^{-1}\,d\xi, \tag{2.3}$$

where $\delta \subset G$ is taken to be close rectifiable curve in G and such that $\sigma(a) \subset \operatorname{ins}(\delta)$, the inside of δ.

It is well known (see for instance [6, pp. 201-204]) that $f(a)$ does not depend on the choice of δ and the *Spectral Mapping Theorem* (SMT)

$$\sigma(f(a)) = f(\sigma(a)) \tag{2.4}$$

holds.

We have the following representation result for the difference $f(Y) - f(X)$ for X, $Y \in$B(H).

Theorem 4. *Let $f : D \subset \mathbb{C} \to \mathbb{C}$ be an analytic function on the convex domain D and X, $Y \in B(H)$ with $\sigma(X), \sigma(Y) \subset D$ and γ a closed rectifiable path in D and* *such that* $\sigma(X), \sigma(Y) \subset \operatorname{ins}(\gamma)$. *Then we have*

$$f(Y) - f(X) = \frac{1}{2\pi i}\int_0^1 \left(\int_\gamma f(\xi)(\xi - (1-t)X - tY)^{-1} \times (Y - X)(\xi - (1-t)X - tY)^{-1} d\xi\right) dt = \frac{1}{2\pi i}\int_\gamma f(\xi)\left(\int_0^1 (\xi - (1-t)X - tY)^{-1} \times (Y - X)(\xi - (1-t)X - tY)^{-1} dt\right) d\xi. \tag{2.5}$$

Proof. Using the Riesz functional calculus (2.3) for f and $B(H)$, we have

$$f(Y) - f(X) = \frac{1}{2\pi i}\left(\int_\gamma f(\xi)(\xi - Y)^{-1} d\xi - \int_\gamma f(\xi)(\xi - X)^{-1} d\xi\right) = \frac{1}{2\pi i}\int_\gamma f(\xi)\left[(\xi - Y)^{-1} - (\xi - X)^{-1}\right] d\xi = \frac{1}{2\pi i}\int_\gamma f(\xi)\xi^{-1}\left[\left(1 - \frac{Y}{\xi}\right)^{-1} - \left(1 - \frac{X}{\xi}\right)^{-1}\right] d\xi. \tag{2.6}$$

Since $\left\|\frac{Y}{\xi}\right\|, \left\|\frac{X}{\xi}\right\| < 1$ for $\xi \in \gamma$, then we can apply Corollary 1 to get

$$\left(1 - \frac{Y}{\xi}\right)^{-1} - \left(1 - \frac{X}{\xi}\right)^{-1} = \int_0^1 \left(1 - (1-t)\frac{X}{\xi} - t\frac{Y}{\xi}\right)^{-1}\left(\frac{Y}{\xi} - \frac{X}{\xi}\right)\left(1 - (1-t)\frac{X}{\xi} - t\frac{Y}{\xi}\right)^{-1} dt = \int_0^1 \xi(\xi - (1-t)X - tY)^{-1}(Y - X)(\xi - (1-t)X - tY)^{-1} dt.$$

Therefore

$$\int_\gamma f(\xi)\xi^{-1}\left[\left(1 - \frac{Y}{\xi}\right)^{-1} - \left(1 - \frac{X}{\xi}\right)^{-1}\right] d\xi = \int_\gamma f(\xi)\xi^{-1}\left(\int_0^1 \xi(\xi - (1-t)X - tY)^{-1}(Y - X)(\xi - (1-t)X - tY)^{-1} dt\right) d\xi = \int_\gamma f(\xi)\left(\int_0^1 (\xi - (1-t)X - tY)^{-1}(Y - X)(\xi - (1-t)X - tY)^{-1} dt\right) d\xi = \int_0^1 \left(\int_\gamma f(\xi)(\xi - (1-t)X - tY)^{-1}(Y - X)(\xi - (1-t)X - tY)^{-1} d\xi\right) dt,$$

where for the last equality we used Fubini's theorem.

By utilising (2.6), we get (2.5). $\square$

Theorem 5. *With the assumptions of Theorem 4 and if* $X,\ Y \in \mathcal{B}_p(H)$, $p \geq 1$, *then we have*

$$\begin{aligned} (2.7)\qquad & \|f(Y)-f(X)\|_p \\ & \leq \frac{1}{2\pi}\|Y-X\|_p \int_\gamma |f(\xi)| \left(\int_0^1 \left\|(\xi-(1-t)X-tY)^{-1}\right\|^2 dt\right)|d\xi| \\ & \leq \frac{1}{2\pi}\|Y-X\|_p \int_\gamma |f(\xi)| \left(\int_0^1 (|\xi|-\|(1-t)X+tY\|)^{-2} dt\right)|d\xi| \\ & \leq \frac{1}{2\pi}\|Y-X\|_p \int_\gamma \frac{|f(\xi)|}{(|\xi|-\|Y\|)(|\xi|-\|X\|)}|d\xi|, \end{aligned}$$

where $\|\cdot\|$ *is the usual operator norm.*

Proof. We have, by taking the p-Schatten norm in (2.5) and using the property (1.8), that

$$\begin{aligned} (2.8)\qquad & \|f(Y)-f(X)\|_p \\ & \leq \frac{1}{2\pi}\int_0^1 \left(\int_\gamma |f(\xi)| \right. \\ & \left. \times \left\|(\xi-(1-t)X-tY)^{-1}(Y-X)(\xi-(1-t)X-tY)^{-1}\right\|_p |d\xi|\right) dt \\ & \leq \frac{1}{2\pi}\int_0^1 \left(\int_\gamma |f(\xi)| \right. \\ & \left. \times \left\|(\xi-(1-t)X-tY)^{-1}\right\| \|Y-X\|_p \left\|(\xi-(1-t)X-tY)^{-1}\right\| |d\xi|\right) dt \\ & = \frac{1}{2\pi}\|Y-X\|_p \int_\gamma |f(\xi)| \int_0^1 \left(\left\|(\xi-(1-t)X-tY)^{-1}\right\|^2 dt\right)|d\xi| \\ & =: \|Y-X\|_p \mathcal{K}(f,X,Y), \end{aligned}$$

which proves the first inequality in (2.7).

We have

$$\begin{aligned} \mathcal{K}(f,X,Y) &:= \frac{1}{2\pi}\int_\gamma |f(\xi)| \int_0^1 \left(\left\|(\xi-(1-t)X-tY)^{-1}\right\|^2 dt\right)|d\xi| \\ &= \frac{1}{2\pi}\int_\gamma |f(\xi)|\,|\xi|^{-2} \left(\int_0^1 \left\|\left(1-(1-t)\frac{X}{\xi}-t\frac{Y}{\xi}\right)^{-1}\right\|^2 dt\right)|d\xi|. \end{aligned}$$

Since

$$\left\|(1-t)\frac{X}{\xi}+t\frac{Y}{\xi}\right\| \leq (1-t)\left\|\frac{X}{\xi}\right\| + t\left\|\frac{Y}{\xi}\right\| < 1-t+t = 1$$

for $\xi \in \gamma$, hence

$$\left(1-(1-t)\frac{X}{\xi}-t\frac{Y}{\xi}\right)^{-1} = \sum_{k=0}^{\infty}\left[(1-t)\frac{X}{\xi}+t\frac{Y}{\xi}\right]^k.$$

Therefore

$$\begin{aligned}\left\|\left(1-(1-t)\frac{X}{\xi}-t\frac{Y}{\xi}\right)^{-1}\right\| &\leq \sum_{k=0}^{\infty}\left\|(1-t)\frac{X}{\xi}+t\frac{Y}{\xi}\right\|^{k} \\ &= \left(1-\left\|(1-t)\frac{X}{\xi}+t\frac{Y}{\xi}\right\|\right)^{-1} \\ &= \left(\frac{|\xi|}{|\xi|}-\left\|(1-t)\frac{X}{\xi}+t\frac{Y}{\xi}\right\|\right)^{-1} \\ &= |\xi|\left(|\xi|-\|(1-t)X+tY\|\right)^{-1}\end{aligned}$$

for $\xi \in \gamma$, which implies that

$$\left\|\left(1-(1-t)\frac{X}{\xi}-t\frac{Y}{\xi}\right)^{-1}\right\|^{2} \leq |\xi|^{2}\left(|\xi|-\|(1-t)X+tY\|\right)^{-2}$$

for $\xi \in \gamma$.

Therefore

$$\begin{aligned}&\int_{\gamma}|f(\xi)||\xi|^{-2}\left(\int_{0}^{1}\left\|\left(1-(1-t)\frac{X}{\xi}-t\frac{Y}{\xi}\right)^{-1}\right\|^{2}dt\right)|d\xi| \\ &\leq \int_{\gamma}|f(\xi)|\left(\int_{0}^{1}\left(|\xi|-\|(1-t)X+tY\|\right)^{-2}dt\right)|d\xi|\end{aligned}$$

and by (2.8) we derive the second inequality in (2.7).

By the triangle inequality we have

$$\begin{aligned}|\xi|-\|(1-t)X+tY\| &\geq |\xi|-(1-t)\|X\|-t\|Y\| \\ &= (1-t)\left(|\xi|-\|X\|\right)+t\left(|\xi|-\|Y\|\right)>0\end{aligned}$$

for $\xi \in \gamma$.

This implies that

$$\left(|\xi|-\|(1-t)X+tY\|\right)^{-1} \leq \left[(1-t)\left(|\xi|-\|X\|\right)+t\left(|\xi|-\|Y\|\right)\right]^{-1}$$

and thus

$$\left(|\xi|-\|(1-t)X+tY\|\right)^{-2} \leq \left[(1-t)\left(|\xi|-\|X\|\right)+t\left(|\xi|-\|Y\|\right)\right]^{-2}$$

for $\xi \in \gamma$ and $t \in [0,1]$.

Taking the integral over $t \in [0,1]$, we get

$$\begin{aligned}
&\int_0^1 \left(|\xi| - \|(1-t)X + tY\|\right)^{-2} dt \\
&\leq \int_0^1 \left[(1-t)\left(|\xi| - \|X\|\right) + t\left(|\xi| - \|Y\|\right)\right]^{-2} dt \\
&= -\frac{1}{\|X\| - \|Y\|} \int_0^1 \frac{d}{dt}\left[(1-t)\left(|\xi| - \|X\|\right) + t\left(|\xi| - \|Y\|\right)\right]^{-1} dt \\
&= -\frac{1}{\|X\| - \|Y\|} \left[(1-t)\left(|\xi| - \|X\|\right) + t\left(|\xi| - \|Y\|\right)\right]^{-1}\Big|_0^1 \\
&= \frac{1}{\|Y\| - \|X\|}\left[\left(|\xi| - \|Y\|\right)^{-1} - \left(|\xi| - \|X\|\right)^{-1}\right] \\
&= \frac{1}{\|Y\| - \|X\|} \frac{\|Y\| - \|X\|}{\left(|\xi| - \|Y\|\right)\left(|\xi| - \|X\|\right)} = \frac{1}{\left(|\xi| - \|Y\|\right)\left(|\xi| - \|X\|\right)},
\end{aligned}$$

for $\|Y\| \neq \|X\|$, which proves the last part of (2.7).

If $\|Y\| = \|X\|$, then we have

$$\begin{aligned}
&\int_0^1 \left(|\xi| - \|(1-t)X + tY\|\right)^{-2} dt \\
&\leq \int_0^1 \left[(1-t)\left(|\xi| - \|X\|\right) + t\left(|\xi| - \|X\|\right)\right]^{-2} dt = \left(|\xi| - \|X\|\right)^{-2},
\end{aligned}$$

which also gives the last bound for $\|Y\| = \|X\|$. □

Remark 1. *If $X, Y \in \mathcal{B}_1(H)$, then we have the trace inequality*

$$|\operatorname{tr}[f(Y)] - \operatorname{tr}[f(X)]| \leq \frac{1}{2\pi}\|Y - X\|_1 \int_\gamma \frac{|f(\xi)|}{\left(|\xi| - \|Y\|\right)\left(|\xi| - \|X\|\right)} |d\xi|. \tag{2.9}$$

Corollary 2. *With the assumptions of Theorem 4 and if*

$$\|f\|_{\gamma,\infty} := \sup_{\xi \in \gamma} |f(\xi)| < \infty,$$

then

$$\begin{aligned}
&\|f(Y) - f(X)\|_p \\
&\leq \frac{1}{2\pi}\|Y - X\|_p \|f\|_{\gamma,\infty} \int_\gamma \left(\int_0^1 \left\|\left(\xi - (1-t)X - tY\right)^{-1}\right\|^2 dt\right)|d\xi| \\
&\leq \frac{1}{2\pi}\|Y - X\|_p \|f\|_{\gamma,\infty} \int_\gamma \left(\int_0^1 \left(|\xi| - \|(1-t)X + tY\|\right)^{-2} dt\right)|d\xi| \\
&\leq \frac{1}{2\pi}\|Y - X\|_p \|f\|_{\gamma,\infty} \int_\gamma \frac{|d\xi|}{\left(|\xi| - \|Y\|\right)\left(|\xi| - \|X\|\right)}.
\end{aligned} \tag{2.10}$$

Remark 2. *Assume that $f : D \subset \mathbb{C} \to \mathbb{C}$ is an analytic function on the convex domain D and $X, Y \in \mathcal{B}_p(H)$, $p \geq 1$ with $\sigma(X)$, $\sigma(Y) \subset D(0,R) \subset D$ where $D(0,R)$ is an open disk centered in 0 and of radius R. By taking γ parametrized by $\xi(t) = Re^{2\pi i t}$ where $t \in [0,1]$, then $d\xi(t) = 2\pi i R e^{2\pi i t} dt$, $|d\xi(t)| = 2\pi R dt$, $|\xi| = R$*

and by (2.7) we get

$$\begin{aligned}
&\|f(Y)-f(X)\|_p \\
&\leq R\|Y-X\|_p \\
&\times \int_0^1 \left|f\left(Re^{2\pi it}\right)\right| \left(\int_0^1 \left\|\left(Re^{2\pi it}-(1-s)X-sY\right)^{-1}\right\|^2 ds\right) dt \\
&\leq R\|Y-X\|_p \int_0^1 (R-\|(1-s)X+sY\|)^{-2} ds \int_0^1 \left|f\left(Re^{2\pi it}\right)\right| dt \\
&\leq \frac{R\|Y-X\|_p}{(R-\|Y\|)(R-\|X\|)} \int_0^1 \left|f\left(Re^{2\pi it}\right)\right| dt.
\end{aligned} \tag{2.11}$$

Moreover, if $\|f\|_{R,\infty} := \sup_{t\in[0,1]} \left|f\left(Re^{2\pi it}\right)\right| < \infty$, *then we have the simpler inequalities*

$$\begin{aligned}
&\|f(Y)-f(X)\|_p \\
&\leq R\|Y-X\|_p \|f\|_{R,\infty} \int_0^1 \left(\int_0^1 \left\|\left(Re^{2\pi it}-(1-s)X-sY\right)^{-1}\right\|^2 ds\right) dt \\
&\leq R\|Y-X\|_p \|f\|_{R,\infty} \int_0^1 (R-\|(1-s)X+sY\|)^{-2} ds \\
&\leq \frac{R\|Y-X\|_p \|f\|_{R,\infty}}{(R-\|Y\|)(R-\|X\|)}.
\end{aligned} \tag{2.12}$$

We also have the following upper bounds:

Theorem 6. *With the assumptions of Theorem 5 we have*

$$\begin{aligned}
&\|f(Y)-f(X)\|_p \\
&\leq \frac{1}{2\pi}\|Y-X\|_p \int_\gamma |f(\xi)| \left(\int_0^1 \left\|(\xi-(1-t)X-tY)^{-1}\right\|^2 dt\right) |d\xi| \\
&\leq \frac{1}{2\pi}\|Y-X\|_p \int_\gamma |f(\xi)| \left(\int_0^1 (|\xi|-\|(1-t)X+tY\|)^{-2} dt\right) |d\xi| \\
&\leq \frac{1}{4\pi}\|Y-X\|_p \left\{\frac{1}{2}\int_\gamma |f(\xi)| \left[(|\xi|-\|Y\|)^{-2}+(|\xi|-\|X\|)^{-2}\right] |d\xi|\right. \\
&\left. + \int_\gamma |f(\xi)| \left(|\xi|-\left\|\frac{X+Y}{2}\right\|\right)^{-2} |d\xi|\right\} \\
&\leq \frac{1}{4\pi}\|Y-X\|_p \int_\gamma |f(\xi)| \left[(|\xi|-\|Y\|)^{-2}+(|\xi|-\|X\|)^{-2}\right] |d\xi|.
\end{aligned} \tag{2.13}$$

Proof. Let $\xi \in \gamma$. For $\alpha,\ \beta \geq 0$ with $\alpha+\beta=1$ and $t_1,\ t_2 \in [0,1]$, then for $g_\xi(t) := (|\xi|-\|(1-t)X+tY\|)^{-2}$, $t\in[0,1]$ we get

$$\begin{aligned}
&g_\xi(\alpha t_1+\beta t_2) \\
&= (|\xi|-\|(1-(\alpha t_1+\beta t_2))X+(\alpha t_1+\beta t_2)Y\|)^{-2} \\
&= (|\xi|-\|\alpha[(1-t_1)X+t_1Y]+\beta[(1-t_2)X+t_2Y]\|)^{-2}
\end{aligned}$$

By the properties of the norm, we have

$$\begin{aligned}&\left\|\alpha\left[(1-t_1)X+t_1Y\right]+\beta\left[(1-t_2)X+t_2Y\right]\right\|\\&\leq\alpha\left\|(1-t_1)X+t_1Y\right\|+\beta\left\|(1-t_2)X+t_2Y\right\|\end{aligned}$$

which gives that

$$\begin{aligned}&|\xi|-\left\|\alpha\left[(1-t_1)X+t_1Y\right]+\beta\left[(1-t_2)X+t_2Y\right]\right\|\\&\geq|\xi|-\alpha\left\|(1-t_1)X+t_1Y\right\|+\beta\left\|(1-t_2)X+t_2Y\right\|>0\end{aligned}$$

for $\xi\in\gamma$.

This implies that

$$\begin{aligned}&\left(|\xi|-\left\|\alpha\left[(1-t_1)X+t_1Y\right]+\beta\left[(1-t_2)X+t_2Y\right]\right\|\right)^{-1}\\&\leq\left(|\xi|-\alpha\left\|(1-t_1)X+t_1Y\right\|+\beta\left\|(1-t_2)X+t_2Y\right\|\right)^{-1}\end{aligned}$$

giving that

$$\begin{aligned}&g_\xi\left(\alpha t_1+\beta t_2\right)\\&\leq\left(|\xi|-\alpha\left\|(1-t_1)X+t_1Y\right\|+\beta\left\|(1-t_2)X+t_2Y\right\|\right)^{-2}\\&=\left(\alpha\left[|\xi|-\left\|(1-t_1)X+t_1Y\right\|\right]+\beta\left[|\xi|-\left\|(1-t_2)X+t_2Y\right\|\right]\right)^{-2}.\end{aligned}$$

By using the convexity of the function $(\cdot)^{-2}$ we have

$$\begin{aligned}&\left(\alpha\left[|\xi|-\left\|(1-t_1)X+t_1Y\right\|\right]+\beta\left[|\xi|-\left\|(1-t_2)X+t_2Y\right\|\right]\right)^{-2}\\&\leq\alpha\left[|\xi|-\left\|(1-t_1)X+t_1Y\right\|\right]^{-2}+\beta\left[|\xi|-\left\|(1-t_2)X+t_2Y\right\|\right]^{-2}\\&=\alpha g_\xi\left(t_1\right)+\beta g_\xi\left(t_2\right).\end{aligned}$$

Therefore

$$g_\xi\left(\alpha t_1+\beta t_2\right)\leq\alpha g_\xi\left(t_1\right)+\beta g_\xi\left(t_2\right),$$

which proves the convexity of g_ξ on $[0,1]$.

By using the Hermite-Hadamard type inequality, for g_ξ on $[0,1]$ we get

$$\int_0^1 g_\xi(t)\,dt\leq\frac{1}{2}\left\{\frac{1}{2}\left[g_\xi(1)+g_\xi(0)\right]+g_\xi\left(\frac{1}{2}\right)\right\}\leq\frac{1}{2}\left[g_\xi(1)+g_\xi(0)\right]$$

namely

$$\begin{aligned}&\int_0^1\left(|\xi|-\left\|(1-t)X+tY\right\|\right)^{-2}dt\\&\leq\frac{1}{2}\left\{\frac{1}{2}\left[\left(|\xi|-\|Y\|\right)^{-2}+\left(|\xi|-\|X\|\right)^{-2}\right]+\left(|\xi|-\left\|\frac{X+Y}{2}\right\|\right)^{-2}\right\}\\&\leq\frac{1}{2}\left[\left(|\xi|-\|Y\|\right)^{-2}+\left(|\xi|-\|X\|\right)^{-2}\right]\end{aligned}$$

for $\xi\in\gamma$.

By making use of this inequality, we have

$$\begin{aligned}
&\int_\gamma |f(\xi)| \left(\int_0^1 (|\xi| - \|(1-t)X + tY\|)^{-2} dt\right) |d\xi| \\
&\le \frac{1}{4}\int_\gamma |f(\xi)| \left[(|\xi| - \|Y\|)^{-2} + (|\xi| - \|X\|)^{-2}\right] |d\xi| \\
&+ \frac{1}{2}\int_\gamma |f(\xi)| \left(|\xi| - \left\|\frac{X+Y}{2}\right\|\right)^{-2} |d\xi| \\
&\le \frac{1}{2}\int_\gamma |f(\xi)| \left[(|\xi| - \|Y\|)^{-2} + (|\xi| - \|X\|)^{-2}\right] |d\xi| .
\end{aligned}$$

By making use of Theorem 4 we then derive the desired result (2.13). □

Remark 3. *If $X, Y \in \mathcal{B}_1(H)$, then we have the trace inequalities:*

$$\begin{aligned}
&|\operatorname{tr}[f(Y)] - \operatorname{tr}[f(X)]| \\
&\le \frac{1}{4\pi}\|Y - X\|_1 \int_\gamma |f(\xi)| \left[(|\xi| - \|Y\|)^{-2} + (|\xi| - \|X\|)^{-2}\right] |d\xi| .
\end{aligned} \tag{2.14}$$

Corollary 3. *With the assumptions of Theorem 5 and if*

$$\|f\|_{\gamma,\infty} := \sup_{\xi\in\gamma} |f(\xi)| < \infty,$$

then

$$\begin{aligned}
&\|f(Y) - f(X)\|_p \\
&\le \frac{1}{2\pi}\|Y - X\|_p \|f\|_{\gamma,\infty} \int_\gamma \left(\int_0^1 \left\|(\xi - (1-t)X - tY)^{-1}\right\|^2 dt\right) |d\xi| \\
&\le \frac{1}{2\pi}\|Y - X\|_p \|f\|_{\gamma,\infty} \int_\gamma \left(\int_0^1 (|\xi| - \|(1-t)X + tY\|)^{-2} dt\right) |d\xi| \\
&\le \frac{1}{4\pi}\|Y - X\|_p \|f\|_{\gamma,\infty} \\
&\times \int_\gamma \left[\frac{(|\xi| - \|Y\|)^{-2} + (|\xi| - \|X\|)^{-2}}{2} + \left(|\xi| - \left\|\frac{X+Y}{2}\right\|\right)^{-2}\right] |d\xi| \\
&\le \frac{1}{4\pi}\|Y - X\|_p \|f\|_{\gamma,\infty} \int_\gamma \left[(|\xi| - \|Y\|)^{-2} + (|\xi| - \|X\|)^{-2}\right] |d\xi| .
\end{aligned} \tag{2.15}$$

Remark 4. *Assume that $f : D \subset \mathbb{C} \to \mathbb{C}$ is an analytic function on the convex domain D and $X, Y \in \mathcal{B}_p(H)$, $p \ge 1$ with $\sigma(X), \sigma(Y) \subset D(0, R) \subset D$ where*

$D(0,R)$ is an open disk centered in 0 *and of radius* R*. Then by (2.13),*

$$\begin{aligned}(2.16)\qquad & \|f(Y)-f(X)\|_p \\ & \le R\|Y-X\|_p \\ & \times \int_0^1 \left|f\left(Re^{2\pi it}\right)\right| \left(\int_0^1 \left\|\left(Re^{2\pi it}-(1-s)X-sY\right)^{-1}\right\|^2 ds\right) dt \\ & \le R\|Y-X\|_p \int_0^1 (R-\|(1-s)X+sY\|)^{-2} ds \int_\gamma \left|f\left(Re^{2\pi it}\right)\right| dt \\ & \le \frac{1}{2}R\|Y-X\|_p \\ & \times \left\{\frac{1}{2}\left[(R-\|Y\|)^{-2}+(R-\|X\|)^{-2}\right]+\left(R-\left\|\frac{X+Y}{2}\right\|\right)^{-2}\right\} \\ & \times \int_0^1 \left|f\left(Re^{2\pi it}\right)\right| dt \\ & \le \frac{1}{2}R\|Y-X\|_p \left[(R-\|Y\|)^{-2}+(R-\|X\|)^{-2}\right] \int_0^1 \left|f\left(Re^{2\pi it}\right)\right| dt\end{aligned}$$

and, by (2.15),

$$\begin{aligned}(2.17)\qquad & \|f(Y)-f(X)\|_p \\ & \le R\|Y-X\|_p \|f\|_{R,\infty} \int_0^1 \left(\int_0^1 \left\|\left(Re^{2\pi it}-(1-s)X-sY\right)^{-1}\right\|^2 ds\right) dt \\ & \le R\|Y-X\|_p \|f\|_{R,\infty} \int_0^1 (R-\|(1-s)X+sY\|)^{-2} ds \\ & \le \frac{1}{2}R\|Y-X\|_p \|f\|_{R,\infty} \\ & \times \left\{\frac{1}{2}\left[(R-\|Y\|)^{-2}+(R-\|X\|)^{-2}\right]+\left(R-\left\|\frac{X+Y}{2}\right\|\right)^{-2}\right\} \\ & \le \frac{1}{2}R\|Y-X\|_p \|f\|_{R,\infty} \left[(R-\|Y\|)^{-2}+(R-\|X\|)^{-2}\right].\end{aligned}$$

Remark 5. *From inequality (2.12) we have the following bound for the quantity* $\frac{1}{\|f\|_{R,\infty}}\|f(Y)-f(X)\|$

$$\mathcal{L}_{1,R}(X,Y) := \frac{R\|Y-X\|_p}{(R-\|Y\|)(R-\|X\|)}$$

while from (2.17),

$$\begin{aligned}\mathcal{L}_{2,R}(X,Y) &:= \frac{1}{2}R\|Y-X\|_p \\ & \times \left\{\frac{1}{2}\left[(R-\|Y\|)^{-2}+(R-\|X\|)^{-2}\right]+\left(R-\left\|\frac{X+Y}{2}\right\|\right)^{-2}\right\}\end{aligned}$$

where $\|X\|, \|Y\| < R$.

By taking $R=1$, $X, Y \in (-1,1)$, $\|\cdot\| = |\cdot|$ *and doing a 3-dimensional plot for the difference* $\mathcal{L}_{1,R}(X,Y)-\mathcal{L}_{2,R}(X,Y)$ *on the box* $(X,Y)\in(-1,1)\times(-1,1)$, *we observe that some time one bound is better than the other.*

The same conclusion then applies to the inequalities

$$\|f(Y)-f(X)\|_p \le \frac{R\|Y-X\|_p \|f\|_{R,\infty}}{(R-\|Y\|)(R-\|X\|)} \tag{2.18}$$

and

$$\begin{aligned}\|f(Y)-f(X)\|_p &\\ \le \frac{1}{2}R\|Y-X\|_p \|f\|_{R,\infty} &\\ \times \left\{\frac{1}{2}\left[(R-\|Y\|)^{-2}+(R-\|X\|)^{-2}\right]+\left(R-\left\|\frac{X+Y}{2}\right\|\right)^{-2}\right\}, &\end{aligned} \tag{2.19}$$

provided that $f: D\subset\mathbb{C}\to\mathbb{C}$ *is an analytic function on the domain* D *and* X, $Y\in\mathcal{B}_p(H)$, $p\ge 1$ *with* $\sigma(X)$, $\sigma(Y)\subset D(0,R)\subset D$ *where* $D(0,R)$ *is an open disk centered in* 0 *and of radius* R.

3. Some Examples

The *modified Bessel function of the first kind* $I_\nu(z)$ for real number ν can be defined by the power series as [1, p. 376]

$$I_\nu(z)=\left(\frac{1}{2}z\right)^\nu\sum_{k=0}^{\infty}\frac{\left(\frac{1}{4}z^2\right)^k}{k!\Gamma(\nu+k+1)},$$

where Γ is the *gamma function*. For $n=0$ we have $I_0(z)$ given by

$$I_0(z)=\sum_{k=0}^{\infty}\frac{\left(\frac{1}{4}z^2\right)^k}{(k!)^2}.$$

An integral formula for real number ν is

$$I_\nu(z)=\frac{1}{\pi}\int_0^\pi e^{z\cos\theta}\cos(\nu\theta)\,d\theta-\frac{\sin(\nu\pi)}{\pi}\int_0^\infty e^{-z\cosh t-\nu t}dt,$$

which simplifies for ν an integer n to

$$I_n(z)=\frac{1}{\pi}\int_0^\pi e^{z\cos\theta}\cos(n\theta)\,d\theta.$$

For $n=0$ we have

$$I_0(z)=\frac{1}{\pi}\int_0^\pi e^{z\cos\theta}d\theta.$$

Consider the exponential function $f(A)=\exp A$, $A\in\mathrm{B}(H)$. Assume that X, $Y\in\mathrm{B}(H)$ and $\|X\|$, $\|Y\|<R$ for some $R>0$. Observe that

$$\left|\exp\left(Re^{2\pi it}\right)\right|=\left|\exp\left[R\left(\cos(2\pi t)+i\sin(2\pi t)\right)\right]\right|=\exp\left[R\cos(2\pi t)\right]$$

and, by changing the variable $\theta = 2\pi t$, we get $dt = \frac{1}{2\pi} d\theta$ and

$$\begin{aligned}
(3.1)\qquad & \int_0^1 \exp\left[R\cos(2\pi t)\right] dt \\
&= \frac{1}{2\pi}\int_0^{2\pi} \exp\left[R\cos\theta\right] d\theta \\
&= \frac{1}{2}\left(\frac{1}{\pi}\int_0^{\pi} \exp\left[R\cos\theta\right] d\theta + \frac{1}{\pi}\int_{\pi}^{2\pi} \exp\left[R\cos\theta\right] d\theta\right) \\
&= \frac{1}{2}\left(I_0(R) + I_0(-R)\right) = I_0(R).
\end{aligned}$$

Assume that $X,\, Y \in \mathcal{B}_p(H)$, $p \geq 1$ with $\sigma(X),\, \sigma(Y) \subset D(0,R) \subset D$ where $D(0,R)$ is an open disk centered in 0 and of radius R. By (2.11) for the exponential function we get

$$\begin{aligned}
(3.2)\qquad & \left\|\exp Y - \exp Y\right\|_p \\
&\leq R\left\|Y - X\right\|_p \\
&\times \int_0^1 \exp\left[R\cos(2\pi t)\right]\left(\int_0^1 \left\|\left(Re^{2\pi i t} - (1-s)X - sY\right)^{-1}\right\|^2 ds\right) dt \\
&\leq R I_0(R)\left\|Y - X\right\|_p \int_0^1 \left(R - \left\|(1-s)X + sY\right\|\right)^{-2} ds \\
&\leq \frac{R I_0(R)\left\|Y - X\right\|_p}{\left(R - \|Y\|\right)\left(R - \|X\|\right)}
\end{aligned}$$

and from (2.16),

$$\begin{aligned}
(3.3)\qquad & \left\|\exp Y - \exp Y\right\|_p \\
&\leq R\left\|Y - X\right\|_p \\
&\times \int_0^1 \exp\left[R\cos(2\pi t)\right]\left(\int_0^1 \left\|\left(Re^{2\pi i t} - (1-s)X - sY\right)^{-1}\right\|^2 ds\right) dt \\
&\leq R I_0(R)\left\|Y - X\right\|_p \int_0^1 \left(R - \left\|(1-s)X + sY\right\|\right)^{-2} ds \\
&\leq \frac{1}{2} R I_0(R)\left\|Y - X\right\|_p \\
&\times \left\{\frac{1}{2}\left[\left(R - \|Y\|\right)^{-2} + \left(R - \|X\|\right)^{-2}\right] + \left(R - \left\|\frac{X+Y}{2}\right\|\right)^{-2}\right\} \\
&\leq \frac{1}{2} R I_0(R)\left\|Y - X\right\|_p \left[\left(R - \|Y\|\right)^{-2} + \left(R - \|X\|\right)^{-2}\right].
\end{aligned}$$

Let f be an analytic functions on the open disk $D(0,\rho)$ given by the *power series* $f(\lambda) := \sum_{j=0}^{\infty} \alpha_j \lambda^j$ $(|\lambda| < \rho)$. If $\nu(A) < \rho$, then the series $\sum_{j=0}^{\infty} \alpha_j A^j$ converges in the Banach algebra $\mathrm{B}(H)$ because $\sum_{j=0}^{\infty} |\alpha_j| \left\|A^j\right\| < \infty$, and we can define $f(A)$ to be its sum. Clearly $f(A)$ is well defined and there are many examples of important functions on a Banach algebra $\mathrm{B}(H)$ that can be constructed in this way. We define the associated function

$$\lambda \mapsto f_A(\lambda) : D(0,\rho) \to \mathbb{C},\, f_A(\lambda) := \sum_{n=0}^{\infty} |\alpha_n| \lambda^n.$$

As some natural examples that are useful for applications, we can point out that, if

$$f(\lambda) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \lambda^n = \ln \frac{1}{1+\lambda}, \ \lambda \in D(0,1); \tag{3.4}$$
$$g(\lambda) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \lambda^{2n} = \cos \lambda, \ \lambda \in \mathbb{C};$$
$$h(\lambda) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \lambda^{2n+1} = \sin \lambda, \ \lambda \in \mathbb{C};$$
$$l(\lambda) = \sum_{n=0}^{\infty} (-1)^n \lambda^n = \frac{1}{1+\lambda}, \ \lambda \in D(0,1);$$

then the corresponding functions constructed by the use of the absolute values of the coefficients are

$$f_A(\lambda) = \sum_{n=1}^{\infty} \frac{1}{n} \lambda^n = \ln \frac{1}{1-\lambda}, \ \lambda \in D(0,1); \tag{3.5}$$
$$g_A(\lambda) = \sum_{n=0}^{\infty} \frac{1}{(2n)!} \lambda^{2n} = \cosh \lambda, \ \lambda \in \mathbb{C};$$

$$h_A(\lambda) = \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} \lambda^{2n+1} = \sinh \lambda, \ \lambda \in \mathbb{C};$$
$$l_A(\lambda) = \sum_{n=0}^{\infty} \lambda^n = \frac{1}{1-\lambda}, \ \lambda \in D(0,1).$$

Other important examples of functions as power series representations with non-negative coefficients are:

$$\exp(\lambda) = \sum_{n=0}^{\infty} \frac{1}{n!} \lambda^n \qquad \lambda \in \mathbb{C}, \tag{3.6}$$
$$\frac{1}{2} \ln\left(\frac{1+\lambda}{1-\lambda}\right) = \sum_{n=1}^{\infty} \frac{1}{2n-1} \lambda^{2n-1}, \qquad \lambda \in D(0,1);$$
$$\sin^{-1}(\lambda) = \sum_{n=0}^{\infty} \frac{\Gamma\left(n+\frac{1}{2}\right)}{\sqrt{\pi}(2n+1)n!} \lambda^{2n+1}, \qquad \lambda \in D(0,1);$$
$$\tanh^{-1}(\lambda) = \sum_{n=1}^{\infty} \frac{1}{2n-1} \lambda^{2n-1}, \qquad \lambda \in D(0,1)$$
$${}_2F_1(\alpha,\beta,\gamma,\lambda) = \sum_{n=0}^{\infty} \frac{\Gamma(n+\alpha)\Gamma(n+\beta)\Gamma(\gamma)}{n!\Gamma(\alpha)\Gamma(\beta)\Gamma(n+\gamma)} \lambda^n, \alpha,\beta,\gamma > 0,$$
$$\lambda \in D(0,1);$$

where Γ is *Gamma function*.

Lemma 2. *Let f be an analytic functions on the open disk $D(0,\rho)$ given by the power series $f(\lambda) := \sum_{j=0}^{\infty} \alpha_j \lambda^j$. We have*

$$|f(\lambda)| \le f_A(|\lambda|) \text{ for } \lambda \in D(0,\rho). \tag{3.7}$$

Proof. For $\lambda \in D(0,\rho)$ we have

$$\begin{aligned} |f(\lambda)| &= \left|\sum_{j=0}^{\infty} \alpha_j \lambda^j\right| = \lim_{n\to\infty} \left|\sum_{j=0}^{n} \alpha_j \lambda^j\right| \le \lim_{n\to\infty} \sum_{j=0}^{n} \left|\alpha_j \lambda^j\right| \\ &= \lim_{n\to\infty} \sum_{j=0}^{n} |\alpha_j| |\lambda|^j = \sum_{j=0}^{\infty} |\alpha_j| |\lambda|^j = f_A(|\lambda|), \end{aligned}$$

which proves the statement. □

Proposition 1. *Let f be an analytic function on the open disk $D(0,\rho)$ given by the power series $f(\lambda) := \sum_{j=0}^{\infty} \alpha_j \lambda^j$. If $X, Y \in \mathcal{B}_p(H)$, $p \ge 1$ with $\sigma(X)$, $\sigma(Y) \subset D(0,R)$ and $0 < R < \rho$, then*

$$\begin{aligned} &\|f(Y) - f(X)\|_p \\ &\le R\|Y - X\|_p f_A(R) \int_0^1 \left(\int_0^1 \left\| \left(Re^{2\pi i t} - (1-s)X - sY\right)^{-1} \right\|^2 ds \right) dt \\ &\le R\|Y - X\|_p f_A(R) \int_0^1 (R - \|(1-s)X + sY\|)^{-2} ds \\ &\le \frac{R f_A(R) \|Y - X\|_p}{(R - \|Y\|)(R - \|X\|)} \end{aligned} \tag{3.8}$$

and

$$\begin{aligned} &\|f(Y) - f(X)\|_p \\ &\le R\|Y - X\|_p f_A(R) \int_0^1 \left(\int_0^1 \left\| \left(Re^{2\pi i t} - (1-s)X - sY\right)^{-1} \right\|^2 ds \right) dt \\ &\le R\|Y - X\|_p f_A(R) \int_0^1 (R - \|(1-s)X + sY\|)^{-2} ds \\ &\le \frac{1}{2} R\|Y - X\|_p f_A(R) \\ &\times \left\{ \frac{1}{2} \left[(R - \|Y\|)^{-2} + (R - \|X\|)^{-2} \right] + \left(R - \left\| \frac{X+Y}{2} \right\| \right)^{-2} \right\}. \end{aligned} \tag{3.9}$$

The proof follows by Remarks 2, 4 and Lemma 2. As examples, one can consider the functions f and f_A listed above.

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