p-SCHATTEN NORM INEQUALITIES OF OPIAL'S TYPE

SILVESTRU SEVER DRAGOMIR[1,2]

Abstract. A bounded linear operator on the Hilbert space H, $A \in \mathcal{B}(H)$, is said to belong to the *von Neumann-Schatten class* $\mathcal{B}_p(H)$, $1 \leq p < \infty$ if the *p-Schatten norm* is finite

$$\|A\|_p := [\operatorname{tr}(|A|^p)]^{1/p} < \infty,$$

where $\operatorname{tr}(\cdot)$ is the operator trace functional. Assume that $A : [a,b] \to \mathcal{B}_2(H)$ is strongly differentiable on (a,b) and $A' \in L_2([a,b], \mathcal{B}_2(H))$. If $A(a) = A(b) = 0$, then

$$\begin{aligned}&\int_a^b \|A'(t) A(t)\|_1 dt \\ &\leq \left[\int_a^b K(t) \|A'(t)\|_2^2 dt\right]^{1/2} \left[\int_a^b \left|\frac{a+b}{2} - t\right| \|A'(t)\|_2^2 dt\right]^{1/2} \\ &\leq \frac{1}{4}(b-a) \int_a^b \|A'(t)\|_2^2 dt,\end{aligned}$$

where

$$K(t) := \begin{cases} t-a \text{ if } a \leq t \leq \frac{a+b}{2}, \\ b-t \text{ if } \frac{a+b}{2} < t \leq b. \end{cases}$$

A weighted version and some examples for particular weights are also given.

1. Introduction

We recall the following Opial type inequalities:

Theorem 1. *Assume that $u : [a,b] \subset \mathbb{R} \to \mathbb{R}$ is an absolutely continuous function on the interval $[a,b]$ and such that $u' \in L_2[a,b]$.*

(i) *If $u(a) = u(b) = 0$, then*

$$\int_a^b |u(t) u'(t)| dt \leq \frac{1}{4}(b-a) \int_a^b |u'(t)|^2 dt, \tag{1.1}$$

with equality if and only if

$$u(t) = \begin{cases} c(t-a) & \text{if } a \leq t \leq \frac{a+b}{2}, \\ c(b-t) & \text{if } \frac{a+b}{2} < t \leq b, \end{cases}$$

where c is an arbitrary constant;

(ii) *If $u(a) = 0$, then*

$$\int_a^b |u(t) u'(t)| dt \leq \frac{1}{2}(b-a) \int_a^b |u'(t)|^2 dt, \tag{1.2}$$

1991 *Mathematics Subject Classification.* 47A63, 47A60.

Key words and phrases. p-Schatten norms, Grüss' inequality, Čebyšev's inequality, Norm inequalities.

RGMIA Res. Rep. Coll. 24 (2021), Art. 113, 13 pp. Received 30/08/21

with equality if and only if $u(t) = c(t-a)$ *for some constant* c.

The inequality (1.1) was obtained by Olech in [10] in which he gave a simplified proof of an inequality originally due in a slightly less general form to Zdzislaw Opial [11].

Embedded in Olech's proof is the half-interval form of Opial's inequality, also discovered by Beesack [3], which is satisfied by those u vanishing only at a.

For various proofs of the above inequalities, see [7]-[8] and [12]. For some recent result related to Opial's inequality see [1], [2], [13] and [14].

In 1975, G. G. Vrânceanu extended Opial's inequality (1.2) for functions with values in Hilbert spaces $(H; \langle \cdot, \cdot \rangle)$ as follows:

Theorem 2. *Assume that the function* $f : [a, b] \to H$ *has a continuous derivative and* $f(a) = 0$, *then*

$$\int_a^b |\langle f(t), f'(t) \rangle| \, dt \leq \frac{1}{2}(b-a) \int_a^b \|f'(t)\|^2 \, dt. \tag{1.3}$$

In order to extend these results for p-Schatten norms of bounded linear operators we need the following preparations.

Let $(H; \langle ., . \rangle)$ be a complex Hilbert space and $\mathcal{B}(H)$ the Banach algebra of all bounded linear operators on H. If $\{e_i\}_{i \in I}$ an orthonormal basis of H, we say that $A \in \mathcal{B}(H)$ is of *trace class* if

$$\|A\|_1 := \sum_{i \in I} \langle |A| e_i, e_i \rangle < \infty. \tag{1.4}$$

The definition of $\|A\|_1$ does not depend on the choice of the orthonormal basis $\{e_i\}_{i \in I}$. We denote by $\mathcal{B}_1(H)$ the set of trace class operators in $\mathcal{B}(H)$.

We define the *trace* of a trace class operator $A \in \mathcal{B}_1(H)$ to be

$$\operatorname{tr}(A) := \sum_{i \in I} \langle A e_i, e_i \rangle, \tag{1.5}$$

where $\{e_i\}_{i \in I}$ an orthonormal basis of H. Note that this coincides with the usual definition of the trace if H is finite-dimensional. We observe that the series (1.5) converges absolutely and it is independent from the choice of basis.

The following result collects some properties of the trace:

Theorem 3. *We have:*

(i) If $A \in \mathcal{B}_1(H)$ *then* $A^* \in \mathcal{B}_1(H)$ *and*

$$\operatorname{tr}(A^*) = \overline{\operatorname{tr}(A)}; \tag{1.6}$$

(ii) If $A \in \mathcal{B}_1(H)$ *and* $T \in \mathcal{B}(H)$, *then* $AT, TA \in \mathcal{B}_1(H)$ *and*

$$\operatorname{tr}(AT) = \operatorname{tr}(TA) \text{ and } |\operatorname{tr}(AT)| \leq \|A\|_1 \|T\|; \tag{1.7}$$

(iii) $\operatorname{tr}(\cdot)$ *is a bounded linear functional on* $\mathcal{B}_1(H)$ *with* $\|\operatorname{tr}\| = 1$;

(iv) If $A, B \in \mathcal{B}_2(H)$ *then* $AB, BA \in \mathcal{B}_1(H)$ *and* $\operatorname{tr}(AB) = \operatorname{tr}(BA)$;

(v) $\mathcal{B}_{fin}(H)$ *is a dense subspace of* $\mathcal{B}_1(H)$.

An operator $A \in \mathcal{B}(H)$ is said to belong to the *von Neumann-Schatten class* $\mathcal{B}_p(H)$, $1 \leq p < \infty$ if the *p-Schatten norm* is finite [17, p. 60-64]

$$\|A\|_p := [\operatorname{tr}(|A|^p)]^{1/p} < \infty.$$

For $1 < p < q < \infty$ we have that

$$\mathcal{B}_1(H) \subset \mathcal{B}_p(H) \subset \mathcal{B}_q(H) \subset \mathcal{B}(H) \tag{1.8}$$

and

$$\|A\|_1 \geq \|A\|_p \geq \|A\|_q \geq \|A\| . \tag{1.9}$$

For $p \geq 1$ the functional $\|\cdot\|_p$ is a *norm* on the $*$-ideal $\mathcal{B}_p(H)$ and $\left(\mathcal{B}_p(H), \|\cdot\|_p\right)$ is a Banach space.

Also, see for instance [17, p. 60-64], for $p \geq 1$,

$$\|A\|_p = \|A^*\|_p, \ A \in \mathcal{B}_p(H) \tag{1.10}$$

$$\|AB\|_p \leq \|A\|_p \|B\|_p, \ A, B \in \mathcal{B}_p(H) \tag{1.11}$$

and

$$\|AB\|_p \leq \|A\|_p \|B\|, \ \|BA\|_p \leq \|B\| \|A\|_p, \ A \in \mathcal{B}_p(H), \ B \in \mathcal{B}(H). \tag{1.12}$$

This implies that

$$\|CAB\|_p \leq \|C\| \|A\|_p \|B\|, \ A \in \mathcal{B}_p(H), \ B, \ C \in \mathcal{B}(H). \tag{1.13}$$

In terms of *p-Schatten norm* we have the *Hölder inequality* for $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$

$$(|\operatorname{tr}(AB)| \leq) \|AB\|_1 \leq \|A\|_p \|B\|_q, \ A \in \mathcal{B}_p(H), \ B \in \mathcal{B}_q(H). \tag{1.14}$$

For the theory of trace functionals and their applications the reader is referred to [15] and [17].

For some classical trace inequalities see [5], [6], and [9], which are continuations of the work of Bellman [4].

2. 1-Schatten Norm Inequalities

The following extension of Opial result holds:

Theorem 4. *Assume that $A : [a,b] \to \mathcal{B}_2(H)$ is strongly differentiable on (a,b) and $A' \in L_2([a,b], \mathcal{B}_2(H))$.*

(i) *If either $A(a) = 0$ or $A(b) = 0$, then*

$$\begin{aligned} \int_a^b \|A'(t) A(t)\|_1 dt &\leq \left(\int_a^b (t-a) \|A'(t)\|_2^2 dt\right)^{1/2} \left(\int_a^b (b-t) \|A'(t)\|_2^2 dt\right)^{1/2} \\ &\leq \frac{1}{2}(b-a) \int_a^b \|A'(t)\|_2^2 dt. \end{aligned} \tag{2.1}$$

(ii) *If* $A(a)=A(b)=0$, *then*

$$\begin{aligned}(2.2)\quad &\int_a^b \|A'(t)A(t)\|_1\,dt\\ &\le \left[\int_a^b K(t)\|A'(t)\|_2^2\,dt\right]^{1/2}\left[\int_a^b \left|\frac{a+b}{2}-t\right|\|A'(t)\|_2^2\,dt\right]^{1/2}\\ &\le \frac{1}{4}(b-a)\int_a^b \|A'(t)\|_2^2\,dt,\end{aligned}$$

where

$$K(t):=\begin{cases} t-a \text{ if } a\le t\le \frac{a+b}{2},\\ \\ b-t \text{ if } \frac{a+b}{2}<t\le b.\end{cases}\tag{2.3}$$

Proof. (i). Since $A(a)=0$, then $A(t)=\int_a^t A'(s)\,ds$ for $t\in[a,b]$. We have, by utilising (1.14) for $p=q=2$, that

$$\begin{aligned}(2.4)\quad &\int_a^b \|A'(t)A(t)\|_1\,dt\\ &\le \int_a^b \|A'(t)\|_2\|A(t)\|_2\,dt=\int_a^b (t-a)^{1/2}\|A'(t)\|_2(t-a)^{-1/2}\|A(t)\|_2\,dt\\ &=\int_a^b (t-a)^{1/2}\|A'(t)\|_2(t-a)^{-1/2}\left\|\int_a^t A'(s)\,ds\right\|_2 dt=:I.\end{aligned}$$

Using Cauchy-Bunyakovsky-Schwarz (CBS) integral inequality, we have

$$\begin{aligned}(2.5)\quad I&\le \left(\int_a^b \left[(t-a)^{1/2}\|A'(t)\|_2\right]^2 dt\right)^{1/2}\\ &\times\left(\int_a^b \left[(t-a)^{-1/2}\left\|\int_a^t A'(s)\,ds\right\|_2\right]^2 dt\right)^{1/2}\\ &=\left(\int_a^b (t-a)\|A'(t)\|_2^2\,dt\right)^{1/2}\left(\int_a^b (t-a)^{-1}\left\|\int_a^t A'(s)\,ds\right\|_2^2 dt\right)^{1/2}\\ &=:J.\end{aligned}$$

By (CBS) integral inequality we also have

$$(t-a)^{-1}\left\|\int_a^t A'(s)\,ds\right\|_2^2\le \int_a^t \|A'(s)\|_2^2\,ds,$$

which gives

$$J\le \left(\int_a^b (t-a)\|A'(t)\|_2^2\,dt\right)^{1/2}\left(\int_a^b\left(\int_a^t \|A'(s)\|_2^2\,ds\right)dt\right)^{1/2}.\tag{2.6}$$

Using integration by parts, we have

$$\int_a^b \left(\int_a^t \|A'(s)\|_2^2\, ds\right) dt = b\int_a^b \|A'(s)\|_2^2\, ds - \int_a^b t\,\|A'(t)\|_2^2\, dt$$
$$= \int_a^b (b-t)\,\|A'(t)\|_2^2$$

and by (2.5) we get the first inequality in (2.1).

The last part follows by the elementary inequality

$$\sqrt{\alpha\beta} \le \frac{1}{2}(\alpha+\beta),\ \ \alpha, \beta \ge 0. \tag{2.7}$$

The case $A(b)=0$ can be proved in a similar way and the details are omitted.

(ii). If we write the inequality (2.1) on the interval $\left[a, \frac{a+b}{2}\right]$, we have

$$\int_a^{\frac{a+b}{2}} \|A'(t)\,A(t)\|_1\, dt \le \left(\int_a^{\frac{a+b}{2}} (t-a)\,\|A'(t)\|_2^2\, dt\right)^{1/2} \left(\int_a^{\frac{a+b}{2}} \left(\frac{a+b}{2}-t\right)\|A'(t)\|_2^2\, dt\right)^{1/2} \tag{2.8}$$

and if we write the inequality (2.1) on the interval $\left[\frac{a+b}{2}, b\right]$, we have

$$\int_{\frac{a+b}{2}}^b \|A'(t)\,A(t)\|_1\, dt \le \left(\int_{\frac{a+b}{2}}^b (b-t)\,\|A'(t)\|_2^2\, dt\right)^{1/2} \left(\int_{\frac{a+b}{2}}^b \left(t-\frac{a+b}{2}\right)\|A'(t)\|_2^2\, dt\right)^{1/2}. \tag{2.9}$$

If we add the inequalities (2.8) and (2.9) we get

$$\begin{aligned}
&\int_a^b \|A'(t)\,A(t)\|_1\, dt \\
&\le \left(\int_a^{\frac{a+b}{2}} (t-a)\,\|A'(t)\|_2^2\, dt\right)^{1/2} \left(\int_a^{\frac{a+b}{2}} \left(\frac{a+b}{2}-t\right)\|A'(t)\|_2^2\, dt\right)^{1/2} \\
&+ \left(\int_{\frac{a+b}{2}}^b (b-t)\,\|A'(t)\|_2^2\, dt\right)^{1/2} \left(\int_{\frac{a+b}{2}}^b \left(t-\frac{a+b}{2}\right)\|A'(t)\|_2^2\, dt\right)^{1/2} \\
&\le \left[\int_a^{\frac{a+b}{2}} (t-a)\,\|A'(t)\|_2^2\, dt + \int_{\frac{a+b}{2}}^b (b-t)\,\|A'(t)\|_2^2\, dt\right]^{1/2} \\
&\times \left[\int_a^{\frac{a+b}{2}} \left(\frac{a+b}{2}-t\right)\|A'(t)\|_2^2\, dt + \int_{\frac{a+b}{2}}^b \left(t-\frac{a+b}{2}\right)\|A'(t)\|_2^2\, dt\right]^{1/2} \\
&= \left[\int_a^b K(t)\,\|A'(t)\|_2^2\, dt\right]^{1/2} \left[\int_a^b \left|\frac{a+b}{2}-t\right| \|A'(t)\|_2^2\, dt\right]^{1/2},
\end{aligned}$$

where for the last inequality we used the elementary (CBS) inequality

$$\alpha\beta+\gamma\delta \le \left(\alpha^2+\gamma^2\right)^{1/2}\left(\beta^2+\delta^2\right)^{1/2},\ \ \alpha,\ \beta,\ \gamma,\ \delta \ge 0.$$

The last part follows by (2.7), namely

$$\left[\int_a^b K(t)\|A'(t)\|_2^2\,dt\right]^{1/2}\left[\int_a^b \left|\frac{a+b}{2}-t\right|\|A'(t)\|_2^2\,dt\right]^{1/2}$$
$$\le \frac{1}{2}\left[\int_a^b K(t)\|A'(t)\|_2^2\,dt+\int_a^b \left|\frac{a+b}{2}-t\right|\|A'(t)\|_2^2\,dt\right]$$
$$=\frac{1}{2}\int_a^b\left[K(t)+\left|\frac{a+b}{2}-t\right|\right]\|A'(t)\|_2^2\,dt=\frac{1}{4}\int_a^b\|A'(t)\|_2^2\,dt,$$

since

$$K(t)+\left|\frac{a+b}{2}-t\right|=\frac{1}{2}(b-a)\ \text{ for } t\in[a,b].$$

□

Remark 1. *The inequality (2.2) can also be written as*

$$\begin{aligned}(2.10)\qquad &\int_a^b\|A'(t)A(t)\|_1\,dt\\ &\le\left[\frac{1}{2}(b-a)\int_a^b\|A'(t)\|_2^2\,dt-\int_a^b\left|\frac{a+b}{2}-t\right|\|A'(t)\|_2^2\,dt\right]^{1/2}\\ &\times\left[\int_a^b\left|\frac{a+b}{2}-t\right|\|A'(t)\|_2^2\,dt\right]^{1/2}\\ &\le\frac{1}{4}(b-a)\int_a^b\|A'(t)\|_2^2\,dt.\end{aligned}$$

We also have the following composite inequality:

Theorem 5. *Let $h:[a,b]\to[h(a),h(b)]$ be a continuous strictly increasing function that is of class C^1 on (a,b). Assume that $A:[a,b]\subset\mathbb{R}\to\mathcal{B}_2(H)$ is strongly differentiable on the interval (a,b) and such that $\frac{A'}{[h']^{1/2}}\in L_2([a,b],\mathcal{B}_2(H))$.*

(i) *If $A(a)=0$ or $A(b)=0$, then*

$$\begin{aligned}(2.11)\qquad &\int_a^b\|A'(t)A(t)\|_1\,dt\\ &\le\left(\int_a^b\frac{[h(t)-h(a)]\|A'(t)\|_2^2}{h'(t)}dt\right)^{1/2}\left(\int_a^b\frac{[h(b)-h(t)]\|A'(t)\|_2^2}{h'(t)}dt\right)^{1/2}\\ &\le\frac{1}{2}[h(b)-h(a)]\int_a^b\frac{\|A'(t)\|_2^2}{h'(t)}dt.\end{aligned}$$

(ii) *If* $A(a) = A(b) = 0$, *then*

$$\int_a^b \|A'(t)A(t)\|_1 dt \le \left[\frac{1}{2}[h(b)-h(a)]\int_a^b \frac{\|A'(t)\|_2^2}{h'(t)}dt - \int_a^b \left|\frac{h(a)+h(b)}{2} - h(t)\right| \frac{\|A'(t)\|_2^2}{h'(t)}dt\right]^{1/2} \times \left[\int_a^b \left|\frac{h(a)+h(b)}{2} - h(t)\right| \frac{\|A'(t)\|_2^2}{h'(t)}dt\right]^{1/2} \le \frac{1}{4}[h(b)-h(a)]\int_a^b \frac{\|A'(t)\|_2^2}{h'(t)}dt. \tag{2.12}$$

Proof. (i). Consider the function $B := A \circ h^{-1} : [h(a), h(b)] \to \mathcal{B}_2(H)$. The function B is absolutely continuous on $[h(a), h(b)]$, $B(h(a)) = A \circ h^{-1}(h(a)) = A(a) = 0$ or $B(h(b)) = A \circ h^{-1}(h(b)) = A(b) = 0$.

Using the chain rule and the derivative of inverse functions we have

$$\left(A \circ h^{-1}\right)'(z) = \left(A' \circ h^{-1}\right)(z)\left(h^{-1}\right)'(z) = \frac{\left(A' \circ h^{-1}\right)(z)}{\left(h' \circ h^{-1}\right)(z)} \tag{2.13}$$

for almost every (a.e.) $z \in [h(a), h(b)]$.

If we apply the inequality (2.1) for the function $B = A \circ h^{-1}$ on the interval $[h(a), h(b)]$, then we get

$$\int_{h(a)}^{h(b)} \left\|\frac{\left(A' \circ h^{-1}\right)(z)}{\left(h' \circ h^{-1}\right)(z)} A \circ h^{-1}(z)\right\|_1 dz \le \left(\int_{h(a)}^{h(b)} (z - h(a)) \left\|\frac{\left(A' \circ h^{-1}\right)(z)}{\left(h' \circ h^{-1}\right)(z)}\right\|_2^2 dz\right)^{1/2} \times \left(\int_{h(a)}^{h(b)} (h(b) - z) \left\|\frac{\left(A' \circ h^{-1}\right)(z)}{\left(h' \circ h^{-1}\right)(z)}\right\|_2^2 dz\right)^{1/2} \le \frac{1}{2}[h(b)-h(a)]\int_{h(a)}^{h(b)} \left\|\frac{\left(A' \circ h^{-1}\right)(z)}{\left(h' \circ h^{-1}\right)(z)}\right\|_2^2 dz. \tag{2.14}$$

If we make the change of variable $t = h^{-1}(z)$, $z \in [h(a), h(b)]$, then $z = h(t)$, $dz = h'(t)\,dt$,

$$\int_{h(a)}^{h(b)} \left\|\frac{\left(A' \circ h^{-1}\right)(z)}{\left(h' \circ h^{-1}\right)(z)} A \circ h^{-1}(z)\right\|_1 dz = \int_a^b \left\|\frac{A'(t)}{h'(t)} A(t)\right\|_1 h'(t)\,dt = \int_a^b \|A'(t)A(t)\|_1 dt,$$

$$\begin{aligned}\int_{h(a)}^{h(b)}(z-h(a))\left\|\frac{\left(A'\circ h^{-1}\right)(z)}{\left(h'\circ h^{-1}\right)(z)}\right\|_2^2dz &= \int_a^b[h(t)-h(a)]\left\|\frac{A'(t)}{h'(t)}\right\|_2^2h'(t)\,dt\\ &= \int_a^b[h(t)-h(a)]\frac{\|A'(t)\|_2^2}{h'(t)}dt\end{aligned}$$

$$\begin{aligned}\int_{h(a)}^{h(b)}(h(b)-z)\left\|\frac{\left(A'\circ h^{-1}\right)(z)}{\left(h'\circ h^{-1}\right)(z)}\right\|_2^2dz &= \int_{h(a)}^{h(b)}[h(b)-h(t)]\left\|\frac{A'(t)}{h'(t)}\right\|_2^2h'(t)\,dt\\ &= \int_a^b[h(b)-h(t)]\frac{\|A'(t)\|_2^2}{h'(t)}dt\end{aligned}$$

and

$$\int_{h(a)}^{h(b)}\left\|\frac{\left(A'\circ h^{-1}\right)(z)}{\left(h'\circ h^{-1}\right)(z)}\right\|_2^2dz=\int_a^b\left\|\frac{A'(t)}{h'(t)}\right\|_2^2h'(t)\,dt=\int_a^b\frac{\|A'(t)\|_2{}^2}{h'(t)}dt.$$

By utilising (2.14), we then get the desired inequality (2.11).

(ii). By using the inequality (2.2) for the function $u=A\circ h^{-1}$ on the interval $[h(a),h(b)]$, then we get

$$\begin{aligned}&\int_{h(a)}^{h(b)}\left\|\frac{\left(A'\circ h^{-1}\right)(z)}{\left(h'\circ h^{-1}\right)(z)}A\circ h^{-1}(z)\right\|_1dz \qquad (2.15)\\ &\le\left[\int_{h(a)}^{h(b)}\left(\frac{1}{2}(h(b)-h(a))-\left|\frac{h(a)+h(b)}{2}-z\right|\right)\right.\\ &\quad\left.\times\left\|\frac{\left(A'\circ h^{-1}\right)(z)}{\left(h'\circ h^{-1}\right)(z)}\right\|_2^2dz\right]^{1/2}\\ &\quad\times\left[\int_a^b\left|\frac{h(a)+h(b)}{2}-z\right|\left\|\frac{\left(A'\circ h^{-1}\right)(z)}{\left(h'\circ h^{-1}\right)(z)}\right\|_2^2dz\right]^{1/2}\\ &\le\frac{1}{4}[h(b)-h(a)]\int_{h(a)}^{h(b)}\left\|\frac{\left(A'\circ h^{-1}\right)(z)}{\left(h'\circ h^{-1}\right)(z)}\right\|_2^2dz.\end{aligned}$$

If we make the change of variable $t=h^{-1}(z)$, $z\in[h(a),h(b)]$, then by (2.15) we get the desired result (2.12). □

If $w:[a,b]\to\mathbb{R}$ is continuous and positive on the interval $[a,b]$, then the function $W:[a,b]\to[0,\infty)$, $W(x):=\int_a^x w(s)\,ds$ is strictly increasing and differentiable on (a,b). We have $W'(x)=w(x)$ for any $x\in(a,b)$.

Corollary 1. *Assume that $w:[a,b]\to(0,\infty)$ is continuous on $[a,b]$ and that $A:[a,b]\subset\mathbb{R}\to\mathcal{B}_2(H)$ is strongly differentiable on the interval (a,b) and such that $\frac{A'}{w^{1/2}}\in L_2([a,b],\mathcal{B}_2(H))$.*

(i) *If* $A(a)=0$ *or* $A(b)=0$, *then*

$$\int_a^b \|A'(t)A(t)\|_1 dt \le \left(\int_a^b \left(\int_a^t w(s)\,ds\right)\frac{\|A'(t)\|_2^2}{w(t)}dt\right)^{1/2} \times \left(\int_a^b \left(\int_t^b w(s)\,ds\right)\frac{\|A'(t)\|_2^2}{w(t)}dt\right)^{1/2} \le \frac{1}{2}\int_a^b w(s)\,ds\int_a^b \frac{\|A'(t)\|_2^2}{w(t)}dt. \tag{2.16}$$

(ii) *If* $A(a)=A(b)=0$, *then*

$$\int_a^b \|A'(t)A(t)\|_1 dt \le \frac{1}{2}\left[\int_a^b w(s)\,ds\int_a^b \frac{\|A'(t)\|_2^2}{w(t)}dt - \int_a^b \left|\int_t^b w(s)\,ds - \int_a^t w(s)\,ds\right|\frac{\|A'(t)\|_2^2}{w(t)}dt\right]^{1/2} \times \left[\int_a^b \left|\int_t^b w(s)\,ds - \int_a^t w(s)\,ds\right|\frac{\|A'(t)\|_2^2}{w(t)}dt\right]^{1/2} \le \frac{1}{4}\int_a^b w(s)\,ds\int_a^b \frac{\|A'(t)\|_2^2}{w(t)}dt. \tag{2.17}$$

3. p-Schatten Norm Inequalities

The following extension of Opial result for p-Schatten norm holds:

Theorem 6. *Assume that* $A:[a,b]\to\mathcal{B}_p(H)$, $p\ge 1$ *is strongly differentiable on* (a,b) *and* $A'\in L_2([a,b],\mathcal{B}_p(H))$.

(i) *If either* $A(a)=0$ *or* $A(b)=0$, *then*

$$\int_a^b \|A'(t)A(t)\|_p dt \le \left(\int_a^b (t-a)\|A'(t)\|_p^2 dt\right)^{1/2}\left(\int_a^b (b-t)\|A'(t)\|_p^2 dt\right)^{1/2} \le \frac{1}{2}(b-a)\int_a^b \|A'(t)\|_p^2 dt. \tag{3.1}$$

(ii) *If* $A(a)=A(b)=0$, *then*

$$\int_a^b \|A'(t)A(t)\|_p dt \le \left[\int_a^b K(t)\|A'(t)\|_p^2 dt\right]^{1/2}\left[\int_a^b \left|\frac{a+b}{2}-t\right|\|A'(t)\|_p^2 dt\right]^{1/2} \le \frac{1}{4}(b-a)\int_a^b \|A'(t)\|_p^2 dt, \tag{3.2}$$

where $K(t)$ *is defined by (2.3).*

Proof. We have, by utilising (1.11) for $p \geq 1$, that

$$\begin{aligned}
&\int_a^b \left\|A'(t)\,A(t)\right\|_p dt \\
&\leq \int_a^b \left\|A'(t)\right\|_p \left\|A(t)\right\|_p dt = \int_a^b (t-a)^{1/2} \left\|A'(t)\right\|_p (t-a)^{-1/2} \left\|A(t)\right\|_p dt \\
&= \int_a^b (t-a)^{1/2} \left\|A'(t)\right\|_p (t-a)^{-1/2} \left\|\int_a^t A'(s)\,ds\right\|_p dt.
\end{aligned}$$

By utilising the same argument to the one in Theorem 4 we derive the desired result. □

We also have the following composite inequality:

Theorem 7. *Let $h:[a,b]\to[h(a),h(b)]$ be a continuous strictly increasing function that is of class C^1 on (a,b). Assume that $A:[a,b]\subset\mathbb{R}\to\mathcal{B}_p(H)$ is strongly differentiable on the interval (a,b) and such that $\frac{A'}{[h']^{1/2}}\in L_2([a,b],\mathcal{B}_p(H))$, $p\geq 1$.*

(i) *If $A(a)=0$ or $A(b)=0$, then*

$$\begin{aligned}
&\int_a^b \left\|A'(t)\,A(t)\right\|_p dt \\
&\leq \left(\int_a^b \frac{[h(t)-h(a)]\left\|A'(t)\right\|_p^2}{h'(t)}dt\right)^{1/2} \left(\int_a^b \frac{[h(b)-h(t)]\left\|A'(t)\right\|_p^2}{h'(t)}dt\right)^{1/2} \\
&\leq \frac{1}{2}[h(b)-h(a)]\int_a^b \frac{\left\|A'(t)\right\|_p^2}{h'(t)}dt.
\end{aligned} \tag{3.3}$$

(ii) *If $A(a)=A(b)=0$, then*

$$\begin{aligned}
\int_a^b \left\|A'(t)\,A(t)\right\|_p dt &\leq \left[\frac{1}{2}[h(b)-h(a)]\int_a^b \frac{\left\|A'(t)\right\|_p^2}{h'(t)}dt \right.\\
&\quad \left. - \int_a^b \left|\frac{h(a)+h(b)}{2}-h(t)\right| \frac{\left\|A'(t)\right\|_p^2}{h'(t)}dt\right]^{1/2} \\
&\quad \times \left[\int_a^b \left|\frac{h(a)+h(b)}{2}-h(t)\right| \frac{\left\|A'(t)\right\|_p^2}{h'(t)}dt\right]^{1/2} \\
&\leq \frac{1}{4}[h(b)-h(a)]\int_a^b \frac{\left\|A'(t)\right\|_p^2}{h'(t)}dt.
\end{aligned} \tag{3.4}$$

The proof follows by Theorem 6 in a similar way to the one from the proof of Theorem 5.

Corollary 2. *Assume that $w:[a,b]\to(0,\infty)$ is continuous on $[a,b]$ and that $A:[a,b]\subset\mathbb{R}\to\mathcal{B}_p(H)$ is strongly differentiable on the interval (a,b) and such that $\frac{A'}{w^{1/2}}\in L_2([a,b],\mathcal{B}_p(H))$, $p\geq 1$.*

(i) *If* $A(a) = 0$ *or* $A(b) = 0$, *then*

$$\int_a^b \|A'(t)A(t)\|_p dt \le \left(\int_a^b \left(\int_a^t w(s)ds\right)\frac{\|A'(t)\|_p^2}{w(t)}dt\right)^{1/2} \times \left(\int_a^b \left(\int_t^b w(s)ds\right)\frac{\|A'(t)\|_p^2}{w(t)}dt\right)^{1/2} \le \frac{1}{2}\int_a^b w(s)ds\int_a^b \frac{\|A'(t)\|_p^2}{w(t)}dt. \tag{3.5}$$

(ii) *If* $A(a) = A(b) = 0$, *then*

$$\int_a^b \|A'(t)A(t)\|_p dt \le \frac{1}{2}\left[\int_a^b w(s)ds\int_a^b \frac{\|A'(t)\|_p^2}{w(t)}dt - \int_a^b \left|\int_t^b w(s)ds - \int_a^t w(s)ds\right|\frac{\|A'(t)\|_p^2}{w(t)}dt\right]^{1/2} \times \left[\int_a^b \left|\int_t^b w(s)ds - \int_a^t w(s)ds\right|\frac{\|A'(t)\|_p^2}{w(t)}dt\right]^{1/2} \le \frac{1}{4}\int_a^b w(s)ds\int_a^b \frac{\|A'(t)\|_p^2}{w(t)}dt. \tag{3.6}$$

4. Examples

We give now some examples.

Consider the function $h(t) = \ln t$, $t \in [a,b] \subset (0,\infty)$. Assume that $A : [a,b] \subset \mathbb{R} \to \mathcal{B}_2(H)$ is strongly differentiable on (a,b) and such that $\ell^{1/2}A' \in L_2([a,b], \mathcal{B}_2(H))$, where $\ell(t) = t$.

If $A(a) = 0$ or $A(b) = 0$, then by (2.11) we get

$$\int_a^b \|A'(t)A(t)\|_1 dt \le \left(\int_a^b t\ln\left(\frac{t}{a}\right)\|A'(t)\|_2^2 dt\right)^{1/2}\left(\int_a^b t\ln\left(\frac{b}{t}\right)\|A'(t)\|_2^2 dt\right)^{1/2} \le \frac{1}{2}\ln\left(\frac{b}{a}\right)\int_a^b t\|A'(t)\|_2^2 dt. \tag{4.1}$$

If $A(a)=A(b)=0$, then by (2.12) we derive

$$\begin{aligned}
&\int_a^b \|A'(t)A(t)\|_1\,dt \\
&\leq \left[\ln\left(\sqrt{\frac{b}{a}}\right)\int_a^b t\|A'(t)\|_2^2\,dt-\int_a^b\left|\ln\left(\frac{\sqrt{ab}}{t}\right)\right| t\|A'(t)\|_2^2\,dt\right]^{1/2} \\
&\times\left[\int_a^b\left|\ln\left(\frac{\sqrt{ab}}{t}\right)\right| t\|A'(t)\|_2^2\,dt\right]^{1/2} \\
&\leq \frac{1}{4}\ln\left(\frac{b}{a}\right)\int_a^b t\|A'(t)\|_2^2\,dt.
\end{aligned}\tag{4.2}$$

Consider the function $h(t)=\frac{t^2}{2}$, $t\in[a,b]\subset(0,\infty)$. Assume that $A:[a,b]\subset\mathbb{R}\to\mathcal{B}_2(H)$ is strongly differentiable on (a,b) and such that $\frac{A'}{\ell^{1/2}}\in L_2([a,b],\mathcal{B}_2(H))$, where $\ell(t)=t$.

If $A(a)=0$ or $A(b)=0$, then by (2.11) we get

$$\begin{aligned}
&\int_a^b \|A'(t)A(t)\|_1\,dt \\
&\leq \frac{1}{2}\left(\int_a^b\frac{(t^2-a^2)\|A'(t)\|_2^2}{t}dt\right)^{1/2}\left(\int_a^b\frac{(b^2-t^2)\|A'(t)\|_2^2}{t}dt\right)^{1/2} \\
&\leq \frac{1}{4}(b^2-a^2)\int_a^b\frac{\|A'(t)\|_2^2}{t}dt.
\end{aligned}\tag{4.3}$$

If $A(a)=A(b)=0$, then by (2.12) we derive

$$\begin{aligned}
&\int_a^b \|A'(t)A(t)\|_1\,dt \\
&\leq \frac{1}{2}\left[\frac{b^2-a^2}{2}\int_a^b\frac{\|A'(t)\|_2^2}{t}dt-\int_a^b\left|\frac{a^2+b^2}{2}-t^2\right|\frac{\|A'(t)\|_2^2}{t}dt\right]^{1/2} \\
&\times\left[\int_a^b\left|\frac{a^2+b^2}{2}-t^2\right|\frac{\|A'(t)\|_2^2}{t}dt\right]^{1/2}\leq\frac{1}{8}(b^2-a^2)\int_a^b\frac{\|A'(t)\|_2^2}{t}dt.
\end{aligned}\tag{4.4}$$

References

[1] G. A. Anastassiou, Complex Opial type inequalities. *Rom. J. Math. Comput. Sci.* **9** (2019), no. 2, 93–97

[2] G.A. Anastassiou, Integer and fractional self adjoint operator Opial type inequalities. *J. Comput. Anal. Appl.* **23** (2017), no. 8, 1398–1411.

[3] P. R. Beesack, On an integral inequality of Z. Opial. *Trans. Am. Math. Soc.* **104** (1962), 470–475.

[4] R. Bellman, Some inequalities for positive definite matrices, in: E. F. Beckenbach (Ed.), *General Inequalities 2*, Proceedings of the 2nd International Conference on General Inequalities, Birkhäuser, Basel, 1980, pp. 89–90.

[5] D. Chang, A matrix trace inequality for products of Hermitian matrices, *J. Math. Anal. Appl.* **237** (1999) 721–725.

[6] I. D. Coop, On matrix trace inequalities and related topics for products of Hermitian matrix, *J. Math. Anal. Appl.* **188** (1994) 999–1001.

[7] L.-G. Hua, On an inequality of Opial. *Sci. Sinica* 14 (1965), 789–790.

[8] N. Levinson, On an inequality of Opial and Beesack. *Proc. Amer. Math. Soc.* **15** (1964), 565–566.

[9] H. Neudecker, A matrix trace inequality, *J. Math. Anal. Appl.* **166** (1992) 302–303.

[10] C. Olech, A simple proof of a certain result of Z. Opial. *Ann. Polon. Math.* **8** (1960), 61–63.

[11] Z. Opial, Sur une inégalité. *Ann. Polon. Math.* **8** (1960), 29–32.

[12] R. N. Pederson, On an inequality of Opial, Beesack and Levinson. *Proc. Amer. Math. Soc.* **16** (1965), 174.

[13] S. H. Saker, D. M. Abdou and I. Kubiaczyk, Opial and Pólya type inequalities via convexity. *Fasc. Math.* No. **60** (2018), 145–159.

[14] M. Z. Sarıkaya, On the generalization of Opial type inequality for convex function. *Konuralp J. Math.* **7** (2019), no. 2, 456–461.

[15] B. Simon, *Trace Ideals and Their Applications,* Cambridge University Press, Cambridge, 1979.

[16] G. G. Vrânceanu, On an inequality of Opial. *Bull. Math. Soc. Sci. Math. R.S. Roumanie* (N.S.) 17(65) (1973), 315-316 (1975).

[17] V. A. Zagrebnov, *Gibbs Semigroups,* Operator Theory: Advances and Applications, Volume 273, Birkhäuser, 2019

[1]Mathematics, College of Engineering & Science, Victoria University, PO Box 14428, Melbourne City, MC 8001, Australia.

E-mail address: sever.dragomir@vu.edu.au

URL: http://rgmia.org/dragomir

[2]DST-NRF Centre of Excellence in the Mathematical, and Statistical Sciences, School of Computer Science, & Applied Mathematics, University of the Witwatersrand,, Johannesburg, South Africa.