p-SCHATTEN NORM INEQUALITIES OF OPIAL-HÖLDER TYPE

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Abstract. A bounded linear operator on the Hilbert space H, $A \in \mathcal{B}(H)$, is said to belong to the *von Neumann-Schatten class* $\mathcal{B}_p(H)$, $1 \leq p < \infty$ if the *p-Schatten norm* is finite

$$\|A\|_p := [\operatorname{tr}(|A|^p)]^{1/p} < \infty,$$

where $\operatorname{tr}(\cdot)$ is the operator trace functional. Assume that $A : [a,b] \to \mathcal{B}_p(H)$, $B : [a,b] \to \mathcal{B}_q(H)$, with $p, q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, are strongly differentiable on (a,b) with $A(a) = B(b) = 0$ and $A' \in L_{\beta,(b-\ell)^2}([a,b], \mathcal{B}_p(H))$, $B' \in L_{\alpha,(\ell-a)^2}([a,b], \mathcal{B}_q(H))$, where $\ell(t) = t$, $\alpha, \beta > 1$ with $\frac{1}{\alpha} + \frac{1}{\beta} = 1$, then

$$\begin{aligned} &\int_a^b \|A(t) B(t)\|_1 dt \\ &\leq \frac{1}{2} \left(\int_a^b (b-t)^2 \|A'(t)\|_p^\beta dt \right)^{1/\beta} \left(\int_a^b (t-a)^2 \|B'(t)\|_q^\alpha dt \right)^{1/\alpha} \\ &\leq \frac{1}{4} \int_a^b \left[(b-t)^2 \|A'(t)\|_p^\beta + (t-a)^2 \|B'(t)\|_q^\alpha \right] dt. \end{aligned}$$

1. Introduction

We recall the following Opial type inequalities:

Theorem 1. *Assume that $u : [a,b] \subset \mathbb{R} \to \mathbb{R}$ is an absolutely continuous function on the interval $[a,b]$ and such that $u' \in L_2[a,b]$.*

(i) *If $u(a) = u(b) = 0$, then*

$$\int_a^b |u(t) u'(t)| dt \leq \frac{1}{4}(b-a) \int_a^b |u'(t)|^2 dt, \tag{1.1}$$

with equality if and only if

$$u(t) = \begin{cases} c(t-a) & \text{if } a \leq t \leq \frac{a+b}{2}, \\ c(b-t) & \text{if } \frac{a+b}{2} < t \leq b, \end{cases}$$

where c is an arbitrary constant;

(ii) *If $u(a) = 0$, then*

$$\int_a^b |u(t) u'(t)| dt \leq \frac{1}{2}(b-a) \int_a^b |u'(t)|^2 dt, \tag{1.2}$$

with equality if and only if $u(t) = c(t-a)$ for some constant c.

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The inequality (1.1) was obtained by Olech in [12] in which he gave a simplified proof of an inequality originally due in a slightly less general form to Zdzislaw Opial [13].

Embedded in Olech's proof is the half-interval form of Opial's inequality, also discovered by Beesack [4], which is satisfied by those u vanishing only at a.

For various proofs of the above inequalities, see [8]-[10] and [14].

In the recent paper [7] we obtained the following results:

Theorem 2. *Assume that $f,\ g:[a,b]\to\mathbb{C}$ are absolutely continuous on $[a,b]$ with $f'\in L_{\alpha}[a,b]$ and $g'\in L_{\beta}[a,b]$ for $\alpha,\ \beta>1$ with $\frac{1}{\alpha}+\frac{1}{\beta}=1$.*

(i) *If $g(a)=0$, then*

$$\int_a^b |f'(t)g(t)|\,dt \le \left(\int_a^b (t-a)|f'(t)|^{\alpha}dt\right)^{1/\alpha}\left(\int_a^b (b-t)|g'(t)|^{\beta}dt\right)^{1/\beta} \le \frac{1}{\alpha}\int_a^b (t-a)|f'(t)|^{\alpha}dt+\frac{1}{\beta}\int_a^b (b-t)|g'(t)|^{\beta}dt. \tag{1.3}$$

(ii) *If $g(b)=0$, then*

$$\int_a^b |f'(t)g(t)|\,dt \le \left(\int_a^b (b-t)|f'(t)|^{\alpha}dt\right)^{1/\alpha}\left(\int_a^b (t-a)|g'(t)|^{\beta}dt\right)^{1/\beta} \le \frac{1}{\alpha}\int_a^b (b-t)|f'(t)|^{\alpha}dt+\frac{1}{\beta}\int_a^b (t-a)|g'(t)|^{\beta}dt. \tag{1.4}$$

(iii) *If $g(a)=g(b)=0$, then*

$$\begin{aligned}\int_a^b |f'(t)g(t)|\,dt &\le \left(\int_a^b K(t)|f'(t)|^{\alpha}dt\right)^{1/\alpha}\left(\int_a^b \left|\frac{a+b}{2}-t\right||g'(t)|^{\beta}dt\right)^{1/\beta}\\ &\le \frac{1}{\alpha}\int_a^b K(t)|f'(t)|^{\alpha}dt+\frac{1}{\beta}\int_a^b \left|\frac{a+b}{2}-t\right||g'(t)|^{\beta}dt,\end{aligned} \tag{1.5}$$

where K is defined by

$$K(t):=\begin{cases} t-a & \text{if } a\le t\le \frac{a+b}{2},\\ b-t & \text{if } \frac{a+b}{2}<t\le b.\end{cases}$$

In particular, we have the Opial type inequalities [7]:

Corollary 1. *Assume that $f:[a,b]\to\mathbb{C}$ are absolutely continuous on $[a,b]$ and $f'\in L_{\alpha}[a,b]\cap L_{\beta}[a,b]$ for $\alpha,\ \beta>1$ with $\frac{1}{\alpha}+\frac{1}{\beta}=1$.*

(i) *If $f(a)=0$, then*

$$\int_a^b |f'(t)f(t)|\,dt \le \left(\int_a^b (t-a)|f'(t)|^{\alpha}dt\right)^{1/\alpha}\left(\int_a^b (b-t)|f'(t)|^{\beta}dt\right)^{1/\beta} \le \frac{1}{\alpha}\int_a^b (t-a)|f'(t)|^{\alpha}dt+\frac{1}{\beta}\int_a^b (b-t)|f'(t)|^{\beta}dt. \tag{1.6}$$

(ii) *If* $f(b)=0$, *then*

$$\int_a^b |f'(t)g(t)|\,dt \le \left(\int_a^b (b-t)|f'(t)|^{\alpha}dt\right)^{1/\alpha}\left(\int_a^b (t-a)|f'(t)|^{\beta}dt\right)^{1/\beta} \le \frac{1}{\alpha}\int_a^b (b-t)|f'(t)|^{\alpha}dt+\frac{1}{\beta}\int_a^b (t-a)|f'(t)|^{\beta}dt. \tag{1.7}$$

(iii) *If* $f(a)=f(b)=0$, *then*

$$\int_a^b |f'(t)f(t)|\,dt \le \left(\int_a^b K(t)|f'(t)|^{\alpha}dt\right)^{1/\alpha}\left(\int_a^b \left|\frac{a+b}{2}-t\right||f'(t)|^{\beta}dt\right)^{1/\beta} \le \frac{1}{\alpha}\int_a^b K(t)|f'(t)|^{\alpha}dt+\frac{1}{\beta}\int_a^b \left|\frac{a+b}{2}-t\right||f'(t)|^{\beta}dt. \tag{1.8}$$

In order to extend these results for p-Schatten norms of bounded linear operators we need the following preparations.

Let $(H;\langle .,.\rangle)$ be a complex Hilbert space and $\mathcal{B}(H)$ the Banach algebra of all bounded linear operators on H. If $\{e_i\}_{i\in I}$ an orthonormal basis of H, we say that $A\in\mathcal{B}(H)$ is of *trace class* if

$$\|A\|_1 := \sum_{i\in I}\langle |A| e_i, e_i\rangle < \infty. \tag{1.9}$$

The definition of $\|A\|_1$ does not depend on the choice of the orthonormal basis $\{e_i\}_{i\in I}$. We denote by $\mathcal{B}_1(H)$ the set of trace class operators in $\mathcal{B}(H)$.

We define the *trace* of a trace class operator $A\in\mathcal{B}_1(H)$ to be

$$\operatorname{tr}(A) := \sum_{i\in I}\langle Ae_i, e_i\rangle, \tag{1.10}$$

where $\{e_i\}_{i\in I}$ an orthonormal basis of H. Note that this coincides with the usual definition of the trace if H is finite-dimensional. We observe that the series (1.10) converges absolutely and it is independent from the choice of basis.

The following result collects some properties of the trace:

Theorem 3. *We have:*

(i) If $A\in\mathcal{B}_1(H)$ *then* $A^*\in\mathcal{B}_1(H)$ *and*

$$\operatorname{tr}(A^*) = \overline{\operatorname{tr}(A)}; \tag{1.11}$$

(ii) If $A\in\mathcal{B}_1(H)$ *and* $T\in\mathcal{B}(H)$, *then* $AT,\ TA\in\mathcal{B}_1(H)$ *and*

$$\operatorname{tr}(AT)=\operatorname{tr}(TA) \text{ and } |\operatorname{tr}(AT)|\le \|A\|_1\|T\|; \tag{1.12}$$

(iii) $\operatorname{tr}(\cdot)$ *is a bounded linear functional on* $\mathcal{B}_1(H)$ *with* $\|\operatorname{tr}\|=1$;

(iv) If $A,\ B\in\mathcal{B}_2(H)$ *then* $AB,\ BA\in\mathcal{B}_1(H)$ *and* $\operatorname{tr}(AB)=\operatorname{tr}(BA)$;

(v) $\mathcal{B}_{fin}(H)$ *is a dense subspace of* $\mathcal{B}_1(H)$.

An operator $A\in\mathcal{B}(H)$ is said to belong to the *von Neumann-Schatten class* $\mathcal{B}_p(H)$, $1\le p<\infty$ if the *p-Schatten norm* is finite [17, p. 60-64]

$$\|A\|_p := \left[\operatorname{tr}(|A|^p)\right]^{1/p} < \infty.$$

For $1<p<q<\infty$ we have that

$$\mathcal{B}_1(H)\subset\mathcal{B}_p(H)\subset\mathcal{B}_q(H)\subset\mathcal{B}(H) \tag{1.13}$$

and

$$\|A\|_1\geq\|A\|_p\geq\|A\|_q\geq\|A\|. \tag{1.14}$$

For $p\geq1$ the functional $\|\cdot\|_p$ is a *norm* on the $*$-ideal $\mathcal{B}_p(H)$ and $\left(\mathcal{B}_p(H),\|\cdot\|_p\right)$ is a Banach space.

Also, see for instance [17, p. 60-64], for $p\geq1$,

$$\|A\|_p=\|A^*\|_p,\ A\in\mathcal{B}_p(H) \tag{1.15}$$

$$\|AB\|_p\leq\|A\|_p\|B\|_p,\ A,B\in\mathcal{B}_p(H) \tag{1.16}$$

and

$$\|AB\|_p\leq\|A\|_p\|B\|,\ \ \|BA\|_p\leq\|B\|\|A\|_p,\ A\in\mathcal{B}_p(H),\ B\in\mathcal{B}(H). \tag{1.17}$$

This implies that

$$\|CAB\|_p\leq\|C\|\|A\|_p\|B\|,\ A\in\mathcal{B}_p(H),\ B,\ C\in\mathcal{B}(H). \tag{1.18}$$

In terms of p-*Schatten norm* we have the *Hölder inequality* for $p,\ q>1$ with $\frac{1}{p}+\frac{1}{q}=1$

$$(|\operatorname{tr}(AB)|\leq)\,\|AB\|_1\leq\|A\|_p\|B\|_q,\ A\in\mathcal{B}_p(H),\ B\in\mathcal{B}_q(H). \tag{1.19}$$

For the theory of trace functionals and their applications the reader is referred to [16] and [17].

For some classical trace inequalities see [5], [6], and [11], which are continuations of the work of Bellman [4].

2. 1-Schatten Norm Inequalities

We consider the positive weight

$$w_a(t;a,b):=\frac{1}{2}\left[(b-a)^2-(t-a)^2\right]=(b-t)\left(\frac{b+t}{2}-a\right), \tag{2.1}$$

for $t\in[a,b]$.

Theorem 4. *Assume that* $A:[a,b]\to\mathcal{B}_p(H)$, $B:[a,b]\to\mathcal{B}_q(H)$, *with* p, $q>1$, $\frac{1}{p}+\frac{1}{q}=1$, *are strongly differentiable on* (a,b) *with* $A(a)=B(a)=0$ *and* $A'\in L_{\beta,w_a}([a,b],\mathcal{B}_p(H))$, $B'\in L_{\alpha,w_a}([a,b],\mathcal{B}_q(H))$, *where* α, $\beta>1$ *with* $\frac{1}{\alpha}+\frac{1}{\beta}=1$, *then*

$$\begin{aligned}&\int_a^b\|A(t)B(t)\|_1dt\\&\leq\left(\int_a^b w_a(t;a,b)\|A'(t)\|_p^\beta dt\right)^{1/\beta}\left(\int_a^b w_a(t;a,b)\|B'(t)\|_q^\alpha dt\right)^{1/\alpha}\\&\leq\int_a^b w_a(t;a,b)\left[\frac{1}{\beta}\|A'(t)\|_p^\beta+\frac{1}{\alpha}\|B'(t)\|_q^\alpha\right]dt.\end{aligned} \tag{2.2}$$

Proof. Since $A(a) = B(a) = 0$, hence $A(t) = \int_a^t A'(s)\,ds$ and $B(t) = \int_a^t B'(s)\,ds$ and we have by Hölder inequality (1.19) for $p,\ q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, that

$$\begin{aligned} &\int_a^b \|A(t)\,B(t)\|_1\,dt \\ &\le \int_a^b \|A(t)\|_p \|B(t)\|_q\,dt = \int_a^b \left\|\int_a^t A'(s)\,ds\right\|_p \left\|\int_a^t B'(s)\,ds\right\|_q dt \\ &\le \int_a^b \left(\int_a^t \|A'(s)\|_p\,ds\right)\left(\int_a^t \|B'(s)\|_q\,ds\right) dt =: J. \end{aligned} \tag{2.3}$$

Applying twice Hölder's inequality, we get for $\alpha,\ \beta > 1$ with $\frac{1}{\alpha} + \frac{1}{\beta} = 1$, that

$$\int_a^t \|A'(s)\|_p\,ds \le (t-a)^{1/\alpha}\left(\int_a^t \|A'(s)\|_p^\beta\,ds\right)^{1/\beta}$$

and

$$\int_a^t \|B'(s)\|_q\,ds \le (t-a)^{1/\beta}\left(\int_a^t \|B'(s)\|_q^\alpha\,ds\right)^{1/\alpha}.$$

Therefore

$$J \le \int_a^b (t-a)\left(\int_a^t \|A'(s)\|_p^\beta\,ds\right)^{1/\beta}\left(\int_a^t \|B'(s)\|_q^\alpha\,ds\right)^{1/\alpha} dt.$$

By applying again Hölder's weighted inequality, we get

$$\begin{aligned} &\int_a^b (t-a)\left(\int_a^t \|A'(s)\|_p^\beta\,ds\right)^{1/\beta}\left(\int_a^t \|B'(s)\|_q^\alpha\,ds\right)^{1/\alpha} dt \\ &\le \left[\int_a^b (t-a)\left(\left(\int_a^t \|A'(s)\|_p^\beta\,ds\right)^{1/\beta}\right)^\beta dt\right]^{1/\beta} \\ &\times \left[\int_a^b (t-a)\left(\left(\int_a^t \|B'(s)\|_q^\alpha\,ds\right)^{1/\alpha}\right)^\alpha dt\right]^{1/\alpha} \\ &= \left[\int_a^b (t-a)\left(\int_a^t \|A'(s)\|_p^\beta\,ds\right) dt\right]^{1/\beta} \\ &\times \left[\int_a^b (t-a)\left(\int_a^t \|B'(s)\|_q^\alpha\,ds\right) dt\right]^{1/\alpha} \\ &=: L. \end{aligned} \tag{2.4}$$

Using integration by parts, we have

$$\begin{aligned}
&\int_a^b (t-a)\left(\int_a^t \|A'(s)\|_p^\beta ds\right) dt \\
&= \int_a^b \left(\int_a^t \|A'(s)\|_p^\beta ds\right) d\left(\frac{(t-a)^2}{2}\right) \\
&= \left(\int_a^t \|A'(s)\|_p^\beta ds\right)\left(\frac{(t-a)^2}{2}\right)\Bigg|_a^b - \int_a^b \frac{(t-a)^2}{2} \|A'(t)\|_p^\beta \beta dt \\
&= \frac{(b-a)^2}{2}\left(\int_a^b \|A'(s)\|_p^\beta ds\right) - \int_a^b \frac{(t-a)^2}{2} \|A'(t)\|_p^\beta dt \\
&= \int_a^b \left[\frac{(b-a)^2}{2} - \frac{(t-a)^2}{2}\right] \|A'(t)\|_p^\beta dt
\end{aligned}$$

and

$$\int_a^b (t-a)\left(\int_a^t \|B'(s)\|_q^\alpha ds\right) dt = \int_a^b \left[\frac{(b-a)^2}{2} - \frac{(t-a)^2}{2}\right] \|B'(t)\|_q^\alpha dt.$$

Therefore

$$\begin{aligned}
L &\le \left(\int_a^b \left[\frac{(b-a)^2}{2} - \frac{(t-a)^2}{2}\right] \|A'(t)\|_p^\beta dt\right)^{1/\beta} \\
&\quad\times \left(\int_a^b \left[\frac{(b-a)^2}{2} - \frac{(t-a)^2}{2}\right] \|B'(t)\|_q^\alpha dt\right)^{1/\alpha} \\
&= \left(\int_a^b w_a(t;a,b) \|A'(t)\|_p^\beta dt\right)^{1/\beta} \left(\int_a^b w_a(t;a,b) \|B'(t)\|_q^\alpha dt\right)^{1/\alpha}.
\end{aligned} \tag{2.5}$$

This proves the first part of (2.2).

By utilising Young's inequality

$$\alpha^{1/\beta}\beta^{1/\alpha} \le \frac{1}{\beta}\alpha + \frac{1}{\alpha}\beta, \text{ for } \alpha,\ \beta \ge 0, \tag{2.6}$$

where $\alpha,\ \beta > 1$ with $\frac{1}{\alpha} + \frac{1}{\beta} = 1$, we get

$$\begin{aligned}
&\left(\int_a^b w_a(t;a,b) \|A'(t)\|_p^\beta dt\right)^{1/\beta} \left(\int_a^b w_a(t;a,b) \|B'(t)\|_q^\alpha dt\right)^{1/\alpha} \\
&\le \frac{1}{\beta}\int_a^b w_a(t;a,b) \|A'(t)\|_p^\beta dt + \frac{1}{\alpha}\int_a^b w_a(t;a,b) \|B'(t)\|_q^\alpha dt \\
&= \int_a^b w_a(t;a,b) \left[\frac{1}{\beta}\|A'(t)\|_p^\beta + \frac{1}{\alpha}\|B'(t)\|_q^\alpha\right] dt,
\end{aligned}$$

which proves the last part of (2.2). □

Remark 1. *Assume that A' is differentiable on (a,b). If $A(a) = A'(a) = 0$ and $A' \in L_{\beta,w_a}([a,b], \mathcal{B}_p(H))$, $A'' \in L_{\alpha,w_a}([a,b], \mathcal{B}_q(H))$, where $\alpha,\ \beta > 1$ with $\frac{1}{\alpha} +$*

$\frac{1}{\beta}=1,$ *then*

$$\begin{aligned}(2.7)\qquad &\int_a^b \|A(t)A'(t)\|_1\,dt\\ &\leq\left(\int_a^b w_a(t;a,b)\|A'(t)\|_p^\beta\,dt\right)^{1/\beta}\left(\int_a^b w_a(t;a,b)\|A''(t)\|_q^\alpha\,dt\right)^{1/\alpha}\\ &\leq\int_a^b w_a(t;a,b)\left[\frac{1}{\beta}\|A'(t)\|_p^\beta+\frac{1}{\alpha}\|A''(t)\|_q^\alpha\right]dt.\end{aligned}$$

Corollary 2. *Assume that A is differentiable on (a,b) with $A(a)=0$ and $A'\in L_{\beta,w_a}([a,b],\mathcal{B}_p(H))\cap L_{\alpha,w_a}([a,b],\mathcal{B}_q(H)),$ where $\alpha,\ \beta>1$ with $\frac{1}{\alpha}+\frac{1}{\beta}=1,$ then*

$$\begin{aligned}(2.8)\qquad &\int_a^b \left\|A^2(t)\right\|_1\,dt\\ &\leq\left(\int_a^b w_a(t;a,b)\|A'(t)\|_p^\beta\,dt\right)^{1/\beta}\left(\int_a^b w_a(t;a,b)\|A'(t)\|_q^\alpha\,dt\right)^{1/\alpha}\\ &\leq\int_a^b w_a(t;a,b)\left[\frac{1}{\beta}\|A'(t)\|_p^\beta+\frac{1}{\alpha}\|A'(t)\|_q^\alpha\right]dt.\end{aligned}$$

Remark 2. *We observe that for $\alpha=\beta=2,$ we get for $A(a)=0$ that*

$$(2.9)\qquad \int_a^b \left\|A^2(t)\right\|_1 dt\leq\int_a^b w_a(t;a,b)\left[\frac{1}{2}\|A'(t)\|_p^2+\frac{1}{2}\|A'(t)\|_q^2\right]dt.$$

Moreover, for $p=q=2$ we derive

$$(2.10)\qquad \int_a^b \left\|A^2(t)\right\|_1 dt\leq\int_a^b w_a(t;a,b)\|A'(t)\|_2^2\,dt.$$

Corollary 3. *Assume that $C:[a,b]\to\mathcal{B}_p(H)\cap\mathcal{B}_q(H),$ with $p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1,$ are strongly differentiable on (a,b) with $C(a)=0$ and $C'\in L_{\beta,w_a}([a,b],\mathcal{B}_p(H))\cap L_{\alpha,w_a}([a,b],\mathcal{B}_q(H)),$ where $\alpha,\ \beta>1$ with $\frac{1}{\alpha}+\frac{1}{\beta}=1,$ then*

$$\begin{aligned}(2.11)\qquad &\int_a^b \|C(t)\|_2^2\,dt\\ &\leq\left(\int_a^b w_a(t;a,b)\|C'(t)\|_p^\beta\,dt\right)^{1/\beta}\left(\int_a^b w_a(t;a,b)\|C'(t)\|_q^\alpha\,dt\right)^{1/\alpha}\\ &\leq\int_a^b w_a(t;a,b)\left[\frac{1}{\beta}\|C'(t)\|_p^\beta+\frac{1}{\alpha}\|C'(t)\|_q^\alpha\right]dt.\end{aligned}$$

In particular, for $\alpha=\beta=2$ we get

$$\begin{aligned}(2.12)\qquad &\int_a^b \|C(t)\|_2^2\,dt\\ &\leq\left(\int_a^b w_a(t;a,b)\|C'(t)\|_p^2\,dt\right)^{1/2}\left(\int_a^b w_a(t;a,b)\|C'(t)\|_q^2\,dt\right)^{1/2}\\ &\leq\int_a^b w_a(t;a,b)\left[\frac{1}{2}\|C'(t)\|_p^2+\frac{1}{2}\|C'(t)\|_q^2\right]dt.\end{aligned}$$

Moreover, for $p = q = 2$ *we derive*

$$\int_a^b \|C(t)\|_2^2 \, dt \le \int_a^b w_a(t; a, b) \|C'(t)\|_2^2 \, dt. \tag{2.13}$$

Proof. We have from (2.2) for $A(t) = C^*(t)$ and $B(t) = C(t)$, $t \in [a, b]$ that

$$\begin{aligned} &\int_a^b \|C^*(t) C(t)\|_1 \, dt \\ &\le \left(\int_a^b w_a(t; a, b) \left\| (C^*)'(t) \right\|_p^\beta dt \right)^{1/\beta} \left(\int_a^b w_a(t; a, b) \|C'(t)\|_q^\alpha \, dt \right)^{1/\alpha} \\ &\le \int_a^b w_a(t; a, b) \left[\frac{1}{\beta} \left\| (C^*)'(t) \right\|_p^\beta + \frac{1}{\alpha} \|C'(t)(t)\|_q^\alpha \right] dt. \end{aligned} \tag{2.14}$$

Observe that

$$\int_a^b \|C^*(t) C(t)\|_1 \, dt = \int_a^b \operatorname{tr} |C^*(t) C(t)| \, dt = \int_a^b \operatorname{tr} |C(t)|^2 \, dt = \int_a^b \|C(t)\|_2^2 \, dt$$

and, by (2.24)

$$\left\| (C^*)'(t) \right\|_p^\beta = \left\| (C')^*(t) \right\|_p^\beta = \|C'(t)\|_p^\beta ,$$

and by making use of (2.14) we derive □

Now consider the dual weight

$$w_b(t; a, b) := \frac{1}{2} \left[(b-a)^2 - (b-t)^2 \right] = (t-a) \left(b - \frac{a+t}{2} \right), \tag{2.15}$$

for $t \in [a, b]$.

Theorem 5. *Assume that* $A : [a, b] \to \mathcal{B}_p(H)$, $B : [a, b] \to \mathcal{B}_q(H)$, *with* p, $q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, *are strongly differentiable on* (a, b) *with* $A(b) = B(b) = 0$ *and* $A' \in L_{\beta, w_b}([a, b], \mathcal{B}_p(H))$, $B' \in L_{\alpha, w_b}([a, b], \mathcal{B}_q(H))$, *where* α, $\beta > 1$ *with* $\frac{1}{\alpha} + \frac{1}{\beta} = 1$, *then*

$$\begin{aligned} &\int_a^b \|A(t) B(t)\|_1 \, dt \\ &\le \left(\int_a^b w_b(t; a, b) \|A'(t)\|_p^\beta \, dt \right)^{1/\beta} \left(\int_a^b w_b(t; a, b) \|B'(t)\|_q^\alpha \, dt \right)^{1/\alpha} \\ &\le \int_a^b w_b(t; a, b) \left[\frac{1}{\beta} \|A'(t)\|_p^\beta + \frac{1}{\alpha} \|B'(t)\|_q^\alpha \right] dt. \end{aligned} \tag{2.16}$$

Proof. Since $A(b)=B(b)=0$, hence $A(t)=-\int_b^t A'(s)\,ds$ and $B(t)=-\int_b^t B'(s)\,ds$ and we have by Hölder inequality (1.19) for $p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1,$

$$\begin{aligned}(2.17)\qquad &\int_a^b \|A(t)\,B(t)\|_1\,dt\\ &\le \int_a^b \|A(t)\|_p\,\|B(t)\|_q\,dt=\int_a^b \left\|\int_t^b A'(s)\,ds\right\|_p \left\|\int_t^b B'(s)\,ds\right\|_q dt\\ &\le \int_a^b \left(\int_t^b \|A'(s)\|_p\,ds\right)\left(\int_t^b \|B'(s)\|_q\,ds\right)dt.\end{aligned}$$

Now, by using a similar argument as in the proof of Theorem 4 we deduce the desired inequality (2.16). □

Remark 3. *Assume that A' is strongly differentiable on (a,b). If $A(b)=A'(b)=0$ and $A'\in L_{\beta,w_b}([a,b],\mathcal{B}_p(H))$, $A''\in L(_{\alpha,w_b}[a,b],\mathcal{B}_q(H))$ where $\alpha,\ \beta>1$ with $\frac{1}{\alpha}+\frac{1}{\beta}=1$, then*

$$\begin{aligned}(2.18)\qquad &\int_a^b \|A(t)\,A'(t)\|_1\,dt\\ &\le \left(\int_a^b w_b(t;a,b)\,\|A'(t)\|_p^\beta\,dt\right)^{1/\beta}\left(\int_a^b w_b(t;a,b)\,\|A''(t)\|_q^\alpha\,dt\right)^{1/\alpha}\\ &\le \int_a^b w_b(t;a,b)\left[\frac{1}{\beta}\|A'(t)\|_p^\beta+\frac{1}{\alpha}\|A''(t)\|_q^\alpha\right]dt.\end{aligned}$$

Corollary 4. *Assume that A is strongly differentiable on (a,b) with $A(b)=0$, then*

$$\begin{aligned}(2.19)\qquad &\int_a^b \left\|A^2(t)\right\|_1\,dt\\ &\le \left(\int_a^b w_b(t;a,b)\,\|A'(t)\|_p^\beta\,dt\right)^{1/\beta}\left(\int_a^b w_b(t;a,b)\,\|A'(t)\|_q^\alpha\,dt\right)^{1/\alpha}\\ &\le \int_a^b w_b(t;a,b)\left[\frac{1}{\beta}\|A'(t)\|_p^\beta+\frac{1}{\alpha}\|A'(t)\|_q^\alpha\right]dt,\end{aligned}$$

where $\alpha,\ \beta>1$ with $\frac{1}{\alpha}+\frac{1}{\beta}=1$.

Remark 4. *We observe that for $\alpha=\beta=2$, we get*

$$(2.20)\qquad \int_a^b \left\|A^2(t)\right\|_1 dt\le \int_a^b w_b(t;a,b)\left[\frac{1}{2}\|A'(t)\|_p^2+\frac{1}{2}\|A'(t)\|_q^2\right]dt,$$

with the assumption that $A(b)=0$. For $p=q=2$ we get

$$(2.21)\qquad \int_a^b \left\|A^2(t)\right\|_1 dt\le \int_a^b w_b(t;a,b)\,\|A'(t)\|_2^2\,dt,$$

with the assumption that $A(b)=0$.

We also have:

Theorem 6. *Assume that* $A:[a,b]\rightarrow\mathcal{B}_p(H)$, $B:[a,b]\rightarrow\mathcal{B}_q(H)$, *with* p, $q>1$, $\frac{1}{p}+\frac{1}{q}=1$, *are strongly differentiable on* (a,b) *with* $A(a)=B(b)=0$ *and* $A'\in L_{\beta,(b-\ell)^2}([a,b],\mathcal{B}_p(H))$, $B'\in L_{\alpha,(\ell-a)^2}([a,b],\mathcal{B}_q(H))$, *where* $\ell(t)=t$, α, $\beta>1$ *with* $\frac{1}{\alpha}+\frac{1}{\beta}=1$, *then*

$$\begin{aligned}\int_a^b\|A(t)B(t)\|_1\,dt & \\ \leq\frac{1}{2}\left(\int_a^b(b-t)^2\|A'(t)\|_p^{\beta}\,dt\right)^{1/\beta}&\left(\int_a^b(t-a)^2\|B'(t)\|_q^{\alpha}\,dt\right)^{1/\alpha}\\ \leq\frac{1}{4}\int_a^b\left[(b-t)^2\|A'(t)\|_p^{\beta}+(t-a)^2\|B'(t)\|_q^{\alpha}\right]dt.&\end{aligned}\tag{2.22}$$

Proof. Since $A(a)=B(b)=0$, hence $A(t)=\int_a^tA'(s)\,ds$ and $B(t)=-\int_t^bB'(s)\,ds$. Therefore, by Hölder inequality (1.19) for p, $q>1$, $\frac{1}{p}+\frac{1}{q}=1$,

$$\begin{aligned}&\int_a^b\|A(t)B(t)\|_1\,dt\\ &\leq\int_a^b\|A(t)\|_p\|B(t)\|_q\,dt=\int_a^b\left\|\int_a^tA'(s)\,ds\right\|_p\left\|\int_t^bB'(s)\,ds\right\|_q dt\\ &=\int_a^b(t-a)^{1/\alpha}(b-t)^{1/\beta}(t-a)^{-1/\alpha}\\ &\times\left\|\int_a^tA'(s)\,ds\right\|_p(b-t)^{-1/\beta}\left\|\int_t^bB'(s)\,ds\right\|_q dt\\ &=:E.\end{aligned}\tag{2.23}$$

By Hölder's integral inequality, we have

$$(t-a)^{-1/\alpha}\left\|\int_a^tA'(s)\,ds\right\|_p\leq\left(\int_a^t\|A'(s)\|_p^{\beta}\,ds\right)^{1/\beta}$$

and

$$(b-t)^{-1/\beta}\left\|\int_t^bB'(s)\,ds\right\|_q\leq\left(\int_t^b\|B'(s)\|_q^{\alpha}\,ds\right)^{1/\alpha},$$

which imply that

$$\begin{aligned}E&\leq\int_a^b(t-a)^{1/\alpha}(b-t)^{1/\beta}\left(\int_a^t\|A'(s)\|_p^{\beta}\,ds\right)^{1/\beta}\left(\int_t^b\|B'(s)\|_q^{\alpha}\,ds\right)^{1/\alpha}\\ &=\int_a^b(b-t)^{1/\beta}\left(\int_a^t\|A'(s)\|_p^{\beta}\,ds\right)^{1/\beta}(t-a)^{1/\alpha}\left(\int_t^b\|B'(s)\|_q^{\alpha}\,ds\right)^{1/\alpha}.\end{aligned}\tag{2.24}$$

By Hölder's integral inequality, we also have

$$\int_a^b (b-t)^{1/\beta}\left(\int_a^t \|A'(s)\|_p^\beta ds\right)^{1/\beta}(t-a)^{1/\alpha}\left(\int_t^b \|B'(s)\|_q^\alpha ds\right)^{1/\alpha} \tag{2.25}$$
$$\leq \left(\int_a^b \left[(b-t)^{1/\beta}\left(\int_a^t \|A'(s)\|_p^\beta ds\right)^{1/\beta}\right]^\beta dt\right)^{1/\beta}$$
$$\times \left(\int_a^b \left[(t-a)^{1/\alpha}\left(\int_t^b \|B'(s)\|_q^\alpha ds\right)^{1/\alpha}\right]^\alpha dt\right)^{1/\alpha}$$
$$= \left(\int_a^b (b-t)\left(\int_a^t \|A'(s)\|_p^\beta ds\right) dt\right)^{1/\beta}$$
$$\times \left(\int_a^b (t-a)\left(\int_t^b \|B'(s)\|_q^\alpha ds\right) dt\right)^{1/\alpha}.$$

Using integration by parts, we have

$$\int_a^b (b-t)\left(\int_a^t \|A'(s)\|_p^\beta ds\right) dt \tag{2.26}$$
$$= -\int_a^b \left(\int_a^t \|A'(s)\|_p^\beta ds\right) d\left(\frac{(b-t)^2}{2}\right)$$
$$= -\left[\left(\int_a^t \|A'(s)\|_p^\beta ds\right)\left(\frac{(b-t)^2}{2}\right)\Bigg|_a^b - \int_a^b \frac{(b-t)^2}{2}\|A'(t)\|_p^\beta dt\right]$$
$$= \frac{1}{2}\int_a^b (b-t)^2 \|A'(t)\|_p^\beta dt$$

and

$$\int_a^b (t-a)\left(\int_t^b \|B'(s)\|_q^\alpha ds\right) dt \tag{2.27}$$
$$= \int_a^b \left(\int_t^b \|B'(s)\|_q^\alpha ds\right) dt\left(\frac{(t-a)^2}{2}\right)$$
$$= \left(\int_t^b \|B'(s)\|_q^\alpha ds\right)\frac{(t-a)^2}{2}\Bigg|_a^b + \int_a^b \frac{(t-a)^2}{2}\|B'(t)\|_q^\alpha dt$$
$$= \frac{1}{2}\int_a^b (t-a)^2 \|B'(t)\|_q^\alpha dt,$$

which proves the first inequality in (2.18).

The last part follows by Young's inequality (2.6). □

Corollary 5. *Assume that A is strongly differentiable with $A(a) = A(b) = 0$ and $A' \in L_{\beta,(b-\ell)^2}([a,b], \mathcal{B}_p(H)) \cap L_{\alpha,(\ell-a)^2}([a,b], \mathcal{B}_q(H))$, where $\ell(t) = t$, α, $\beta > 1$*

with $\frac{1}{\alpha}+\frac{1}{\beta}=1,$ *then*

$$\begin{aligned}
\int_a^b \left\|A^2(t)\right\|_1 dt \tag{2.28}
&\le \frac{1}{2}\left(\int_a^b (b-t)^2 \left\|A'(t)\right\|_p^\beta dt\right)^{1/\beta}\left(\int_a^b (t-a)^2 \left\|A'(t)\right\|_q^\alpha dt\right)^{1/\alpha} \\
&\le \frac{1}{4}\int_a^b \left[(b-t)^2 \left\|A'(t)\right\|_p^\beta + (t-a)^2 \left\|A'(t)\right\|_q^\alpha\right] dt.
\end{aligned}$$

Remark 5. *Assume that* A, B *are strongly differentiable with* $A(a)=B(b)=0$ *and* $A' \in L_{2,(b-\ell)^2}([a,b], \mathcal{B}_p(H))$, $B' \in L_{2,(\ell-a)^2}([a,b], \mathcal{B}_q(H))$ *where* $\ell(t)=t$, *then*

$$\begin{aligned}
\int_a^b \left\|A(t)B(t)\right\|_1 dt \tag{2.29}
&\le \frac{1}{2}\left(\int_a^b (b-t)^2 \left\|A'(t)\right\|_p^2 dt\right)^{1/2}\left(\int_a^b (t-a)^2 \left\|B'(t)\right\|_q^2 dt\right)^{1/2} \\
&\le \frac{1}{4}\int_a^b \left[(b-t)^2 \left\|A'(t)\right\|_p^2 + (t-a)^2 \left\|B'(t)\right\|_q^2\right] dt.
\end{aligned}$$

Also, if $A(a)=A(b)=0$, *then for* $\alpha=\beta=2$

$$\begin{aligned}
\int_a^b \left\|A^2(t)\right\|_1 dt \tag{2.30}
&\le \frac{1}{2}\left(\int_a^b (b-t)^2 \left\|A'(t)\right\|_p^2 dt\right)^{1/2}\left(\int_a^b (t-a)^2 \left\|A'(t)\right\|_q^2 dt\right)^{1/2} \\
&\le \frac{1}{4}\int_a^b \left[(b-t)^2 \left\|A'(t)\right\|_p^2 + (t-a)^2 \left\|A'(t)\right\|_q^2\right] dt.
\end{aligned}$$

Moreover, for $p=q=2$ *we derive*

$$\begin{aligned}
\int_a^b \left\|A^2(t)\right\|_1 dt \tag{2.31}
&\le \frac{1}{2}\left(\int_a^b (b-t)^2 \left\|A'(t)\right\|_2^2 dt\right)^{1/2}\left(\int_a^b (t-a)^2 \left\|A'(t)\right\|_2^2 dt\right)^{1/2} \\
&\le \frac{1}{4}\int_a^b \left[(b-t)^2 + (t-a)^2\right] \left\|A'(t)\right\|_2^2 dt.
\end{aligned}$$

We also have:

Corollary 6. *Assume that* $C:[a,b]\to \mathcal{B}_p(H)\cap\mathcal{B}_q(H)$, *with* p, $q>1$, $\frac{1}{p}+\frac{1}{q}=1$, *is strongly differentiable on* (a,b) *with* $C(a)=C(b)=0$ *and* $C' \in L_{\beta,(b-\ell)^2}([a,b], \mathcal{B}_p(H)) \cap L_{\alpha,(\ell-a)^2}([a,b], \mathcal{B}_q(H))$, *where* α, $\beta>1$ *with* $\frac{1}{\alpha}+\frac{1}{\beta}=1$, *then*

$$\begin{aligned}(2.32)\qquad &\int_a^b \|C(t)\|_2^2\,dt \\ &\le \frac{1}{2}\left(\int_a^b (b-t)^2\|C'(t)\|_p^\beta\,dt\right)^{1/\beta}\left(\int_a^b (t-a)^2\|C'(t)\|_q^\alpha\,dt\right)^{1/\alpha} \\ &\le \frac{1}{4}\int_a^b \left[(b-t)^2\|C'(t)\|_p^\beta + (t-a)^2\|C'(t)\|_q^\alpha\right]dt.\end{aligned}$$

In particular, for $\alpha=\beta=2$ *we get*

$$\begin{aligned}(2.33)\qquad &\int_a^b \|C(t)\|_2^2\,dt \\ &\le \frac{1}{2}\left(\int_a^b (b-t)^2\|C'(t)\|_p^2\,dt\right)^{1/2}\left(\int_a^b (t-a)^2\|C'(t)\|_q^2\,dt\right)^{1/2} \\ &\le \frac{1}{4}\int_a^b \left[(b-t)^2\|C'(t)\|_p^2 + (t-a)^2\|C'(t)\|_q^2\right]dt.\end{aligned}$$

Moreover, for $p=q=2$ *we derive*

$$\begin{aligned}(2.34)\qquad &\int_a^b \|C(t)\|_2^2\,dt \\ &\le \frac{1}{2}\left(\int_a^b (b-t)^2\|C'(t)\|_2^2\,dt\right)^{1/2}\left(\int_a^b (t-a)^2\|C'(t)\|_2^2\,dt\right)^{1/2} \\ &\le \frac{1}{4}\int_a^b \left[(b-t)^2 + (t-a)^2\right]\|C'(t)\|_2^2\,dt.\end{aligned}$$

3. p-Schatten Norm Inequalities

We also have the following result for the p-Schatten norm:

Theorem 7. *Assume that* $A,\ B:[a,b]\to\mathcal{B}_p(H)$ *with* $p\ge 1$, *are strongly differentiable on* (a,b) *with* $A(a)=B(a)=0$ *and* $A'\in L_{\beta,w_a}([a,b],\mathcal{B}_p(H))$, $B'\in L_{\alpha,w_a}([a,b],\mathcal{B}_p(H))$, *where* $\alpha,\ \beta>1$ *with* $\frac{1}{\alpha}+\frac{1}{\beta}=1$, *then*

$$\begin{aligned}(3.1)\qquad &\int_a^b \|A(t)B(t)\|_p\,dt \\ &\le \left(\int_a^b w_a(t;a,b)\|A'(t)\|_p^\beta\,dt\right)^{1/\beta}\left(\int_a^b w_a(t;a,b)\|B'(t)\|_p^\alpha\,dt\right)^{1/\alpha} \\ &\le \int_a^b w_a(t;a,b)\left[\frac{1}{\beta}\|A'(t)\|_p^\beta + \frac{1}{\alpha}\|B'(t)\|_p^\alpha\right]dt,\end{aligned}$$

where $w_a(t;a,b)$ *is given by (2.1).*

In particular, for $\alpha = \beta = 2$ we get

$$\begin{aligned}(3.2)\quad &\int_a^b \|A(t)B(t)\|_p\,dt \\ &\le \left(\int_a^b w_a(t;a,b)\|A'(t)\|_p^2\,dt\right)^{1/2}\left(\int_a^b w_a(t;a,b)\|B'(t)\|_p^2\,dt\right)^{1/2} \\ &\le \int_a^b w_a(t;a,b)\left[\frac{1}{2}\|A'(t)\|_p^2+\frac{1}{2}\|B'(t)\|_p^2\right]dt.\end{aligned}$$

Proof. Since $A(a) = B(a) = 0$, hence $A(t) = \int_a^t A'(s)\,ds$ and $B(t) = \int_a^t B'(s)\,ds$ and we have by (1.16) for $p \ge 1$, that

$$\begin{aligned}&\int_a^b \|A(t)B(t)\|_p\,dt \\ &\le \int_a^b \|A(t)\|_p\|B(t)\|_p\,dt = \int_a^b \left\|\int_a^t A'(s)\,ds\right\|_p\left\|\int_a^t B'(s)\,ds\right\|_p dt \\ &\le \int_a^b \left(\int_a^t \|A'(s)\|_p\,ds\right)\left(\int_a^t \|B'(s)\|_p\,ds\right)dt.\end{aligned}$$

By utilising a similar argument to the one in the proof of Theorem 4 we deduce the desired result (3.1). □

Remark 6. *Assume that $A : [a,b] \to \mathcal{B}_p(H)$, $p \ge 1$ is strongly differentiable on (a,b) with $A(a) = 0$ and $A' \in L_{2,w_a}([a,b],\mathcal{B}_p(H))$, then by (3.2) we derive*

$$(3.3)\qquad \int_a^b \left\|A^2(t)\right\|_p dt \le \int_a^b w_a(t;a,b)\|A'(t)\|_p^2\,dt.$$

Corollary 7. *Assume that $C : [a,b] \to \mathcal{B}_p(H)$, with $p \ge 1$, is strongly differentiable on (a,b) with $C(a) = 0$ and $C' \in L_{\beta,w_a}([a,b],\mathcal{B}_p(H)) \cap L_{\alpha,w_a}([a,b],\mathcal{B}_p(H))$, where α, $\beta > 1$ with $\frac{1}{\alpha}+\frac{1}{\beta} = 1$, then*

$$\begin{aligned}(3.4)\quad &\int_a^b \|C(t)\|_{2p}^2\,dt \\ &\le \left(\int_a^b w_a(t;a,b)\|C'(t)\|_p^\beta\,dt\right)^{1/\beta}\left(\int_a^b w_a(t;a,b)\|C'(t)\|_p^\alpha\,dt\right)^{1/\alpha} \\ &\le \int_a^b w_a(t;a,b)\left[\frac{1}{\beta}\|C'(t)\|_p^\beta+\frac{1}{\alpha}\|C'(t)\|_p^\alpha\right]dt.\end{aligned}$$

In particular, for $\alpha = \beta = 2$ we get

$$(3.5)\qquad \int_a^b \|C(t)\|_{2p}^2\,dt \le \int_a^b w_a(t;a,b)\|C'(t)\|_p^2\,dt.$$

Also, we have:

Theorem 8. *Assume that A, $B : [a,b] \to \mathcal{B}_p(H)$, $p \ge 1$, are strongly differentiable on (a,b) with $A(b) = B(b) = 0$ and $A' \in L_{\beta,w_b}([a,b],\mathcal{B}_p(H))$, $B' \in$*

$L_{\alpha,w_b}([a,b],\mathcal{B}_p(H))$, *where* α, $\beta>1$ *with* $\frac{1}{\alpha}+\frac{1}{\beta}=1$, *then*

$$\begin{aligned}\int_a^b \|A(t)B(t)\|_p\,dt &\leq \left(\int_a^b w_b(t;a,b)\|A'(t)\|_p^\beta\,dt\right)^{1/\beta}\left(\int_a^b w_b(t;a,b)\|B'(t)\|_p^\alpha\,dt\right)^{1/\alpha}\\ &\leq \int_a^b w_b(t;a,b)\left[\frac{1}{\beta}\|A'(t)\|_p^\beta+\frac{1}{\alpha}\|B'(t)\|_p^\alpha\right]dt,\end{aligned}\tag{3.6}$$

where $w_b(t;a,b)$ *is given by (2.15).*

In particular, for $\alpha=\beta=2$ *we get*

$$\begin{aligned}\int_a^b \|A(t)B(t)\|_p\,dt &\leq \left(\int_a^b w_b(t;a,b)\|A'(t)\|_p^2\,dt\right)^{1/2}\left(\int_a^b w_b(t;a,b)\|B'(t)\|_p^2\,dt\right)^{1/2}\\ &\leq \int_a^b w_b(t;a,b)\left[\frac{1}{2}\|A'(t)\|_p^2+\frac{1}{2}\|B'(t)\|_p^2\right]dt.\end{aligned}\tag{3.7}$$

Remark 7. *Assume that* $A:[a,b]\to\mathcal{B}_p(H)$, $p\geq 1$ *is strongly differentiable on* (a,b) *with* $A(b)=0$ *and* $A'\in L_{2,w_b}([a,b],\mathcal{B}_p(H))$, *then by (3.2) we derive*

$$\int_a^b \left\|A^2(t)\right\|_p dt\leq\int_a^b w_b(t;a,b)\|A'(t)\|_p^2\,dt.\tag{3.8}$$

Finally, we can state:

Theorem 9. *Assume that* A, $B:[a,b]\to\mathcal{B}_p(H)$, *with* $p\geq 1$, *are strongly differentiable on* (a,b) *with* $A(a)=B(b)=0$ *and* $A'\in L_{\beta,(b-\ell)^2}([a,b],\mathcal{B}_p(H))$, $B'\in L_{\alpha,(\ell-a)^2}([a,b],\mathcal{B}_p(H))$, *where* $\ell(t)=t$, α, $\beta>1$ *with* $\frac{1}{\alpha}+\frac{1}{\beta}=1$, *then*

$$\begin{aligned}\int_a^b \|A(t)B(t)\|_p\,dt &\leq \frac{1}{2}\left(\int_a^b (b-t)^2\|A'(t)\|_p^\beta\,dt\right)^{1/\beta}\left(\int_a^b (t-a)^2\|B'(t)\|_p^\alpha\,dt\right)^{1/\alpha}\\ &\leq \frac{1}{4}\int_a^b\left[(b-t)^2\|A'(t)\|_p^\beta+(t-a)^2\|B'(t)\|_p^\alpha\right]dt.\end{aligned}\tag{3.9}$$

In particular, for $\alpha=\beta=2$ *we get*

$$\begin{aligned}\int_a^b \|A(t)B(t)\|_p\,dt &\leq \frac{1}{2}\left(\int_a^b (b-t)^2\|A'(t)\|_p^2\,dt\right)^{1/2}\left(\int_a^b (t-a)^2\|B'(t)\|_p^2\,dt\right)^{1/2}\\ &\leq \frac{1}{4}\int_a^b\left[(b-t)^2\|A'(t)\|_p^2+(t-a)^2\|B'(t)\|_p^2\right]dt.\end{aligned}\tag{3.10}$$

Remark 8. *Assume that* $A:[a,b]\rightarrow\mathcal{B}_p(H)$, $p\geq 1$ *is strongly differentiable on* (a,b) *with* $A(a)=A(b)=0$ *and* $A'\in L_{2,(b-\ell)^2}([a,b],\mathcal{B}_p(H))\cap L_{2,(\ell-a)^2}([a,b],\mathcal{B}_p(H))$, *then by (3.2) we derive*

$$\begin{aligned}(3.11)\qquad &\int_a^b \left\|A^2(t)\right\|_p dt\\ &\leq \frac{1}{2}\left(\int_a^b (b-t)^2\left\|A'(t)\right\|_p^2 dt\right)^{1/2}\left(\int_a^b (t-a)^2\left\|A'(t)\right\|_p^2 dt\right)^{1/2}\\ &\leq \frac{1}{4}\int_a^b\left[(b-t)^2+(t-a)^2\right]\left\|A'(t)\right\|_p^2 dt.\end{aligned}$$

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