p-SCHATTEN NORM INEQUALITIES OF GRÜSS' TYPE

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Abstract. An operator $A \in \mathcal{B}(H)$ is said to belong to the *von Neumann-Schatten class* $\mathcal{B}_p(H)$, $1 \leq p < \infty$ if the *p-Schatten norm* is finite

$$\|A\|_p := [\operatorname{tr}(|A|^p)]^{1/p} < \infty.$$

For two continuous functions $A, B : [a,b] \to \mathcal{B}(H)$ we define the *noncommutative Čebyšev functional*

$$D(A,B) := (b-a)\int_a^b A(t) B(t)\,dt - \int_a^b A(t)\,dt \int_a^b B(t)\,dt.$$

If $p, q > 1$ with $\frac{1}{p}+\frac{1}{q}=1$, let $A : [a,b] \to \mathcal{B}_p(H)$, $B : [a,b] \to \mathcal{B}_q(H)$ be strongly differentiable functions on the interval (a,b), then

$$\|D(A,B)\|_1 \leq D\left(\int_a^{\cdot} \|A'(u)\|_p\,du, \int_a^{\cdot} \|B'(u)\|_q\,du\right)$$
$$\leq \frac{1}{4}(b-a)^2 \int_a^b \|A'(u)\|_p\,du \int_a^b \|B'(u)\|_q\,du.$$

Some examples of interest for the operator exponential are also given.

1. Introduction

For two Lebesgue integrable functions $f, g : [a,b] \to \mathbb{C}$, in order to compare the integral mean of the product with the product of the integral means, we consider the *Čebyšev functional* defined by

$$D(f,g) := (b-a)\int_a^b f(t) g(t)\,dt - \int_a^b f(t)\,dt \int_a^b g(t)\,dt.$$

In 1934, G. Grüss [10] showed that

$$|D(f,g)| \leq \frac{1}{4}(b-a)^2 (M-m)(N-n), \tag{1.1}$$

provided m, M, n, N are real numbers with the property that

$$-\infty < m \leq f \leq M < \infty, \quad -\infty < n \leq g \leq N < \infty \quad \text{a.e. on} \quad [a,b]. \tag{1.2}$$

The constant $\frac{1}{4}$ is best possible in (1.1) in the sense that it cannot be replaced by a smaller one.

Another lesser known inequality for $D(f,g)$ was derived in 1882 by Čebyšev [7] under the assumption that f', g' exist and are continuous on $[a,b]$, and is given by

$$|D(f,g)| \leq \frac{1}{12}\|f'\|_\infty \|g'\|_\infty (b-a)^4, \tag{1.3}$$

where $\|f'\|_\infty := \sup_{t\in[a,b]} |f'(t)| < \infty$.

1991 *Mathematics Subject Classification.* 47A63, 47A60.

Key words and phrases. p-Schatten norms, Grüss' inequality, Čebyšev's inequality, Norm inequalities.

RGMIA Res. Rep. Coll. 24 (2021), Art. 115, 16 pp. Received 12/09/21

The constant $\frac{1}{12}$ cannot be improved in general in (1.3).

Čebyšev's inequality (1.3) also holds if $f,\ g : [a,b] \to \mathbb{R}$ are assumed to be absolutely continuous and $f',\ g' \in L_\infty [a,b]$.

In 1970, A. M. Ostrowski [14] proved, amongst others, the following result that is in a sense a combination of the Čebyšev and Grüss results:

$$|D(f,g)| \leq \frac{1}{8}(b-a)^3 (M-m) \|g'\|_\infty , \tag{1.4}$$

provided f is Lebesgue integrable on $[a,b]$ and satisfying (1.2) while $g : [a,b] \to \mathbb{R}$ is absolutely continuous and $g' \in L_\infty [a,b]$. Here the constant $\frac{1}{8}$ is also sharp.

In 1973, A. Lupaş [11] (see also [12, p. 210]) obtained the following result as well:

$$|D(f,g)| \leq \frac{1}{\pi^2} \|f'\|_2 \|g'\|_2 (b-a)^3 , \tag{1.5}$$

provided $f,\ g$ are absolutely continuous and $f',\ g' \in L_2 [a,b]$.

Here the constant $\frac{1}{\pi^2}$ is the best possible as well.

In [3], P. Cerone and S. S. Dragomir proved the following inequalities:

$$|D(f,g)| \leq (b-a) \times \begin{cases} \inf\limits_{\gamma \in \mathbb{R}} \|g-\gamma\|_\infty \int_a^b \left| f(t) - \frac{1}{b-a}\int_a^b f(s)\,ds \right| dt, \\ \\ \inf\limits_{\gamma \in \mathbb{R}} \|g-\gamma\|_q \left(\int_a^b \left| f(t) - \frac{1}{b-a}\int_a^b f(s)\,ds \right|^p dt \right)^{\frac{1}{p}} \\ \qquad \text{where } p>1,\ 1/p+1/q=1. \end{cases} \tag{1.6}$$

For $\gamma = 0$, we get from the first inequality in (1.6)

$$|D(f,g)| \leq (b-a) \|g\|_\infty \int_a^b \left| f(t) - \frac{1}{b-a}\int_a^b f(s)\,ds \right| dt \tag{1.7}$$

for which the constant 1 cannot be replaced by a smaller constant.

If $m \leq g \leq M$ for a.e. $x \in [a,b]$, then $\left\| g - \frac{m+M}{2} \right\|_\infty \leq \frac{1}{2}(M-m)$ and by the first inequality in (1.6) we can deduce the following result obtained by Cheng and Sun [8]

$$|D(f,g)| \leq \frac{1}{2}(b-a)(M-m) \int_a^b \left| f(t) - \frac{1}{b-a}\int_a^b f(s)\,ds \right| dt. \tag{1.8}$$

The constant $\frac{1}{2}$ is best in (1.8) as shown by Cerone and Dragomir in [4].

The following result holds [9].

Theorem 1. *Let $f : [a,b] \to \mathbb{C}$ be of bounded variation on $[a,b]$ and $g : [a,b] \to \mathbb{C}$ a Lebesgue integrable function on $[a,b]$. Then*

$$|D(f,g)| \leq \frac{1}{2}(b-a) \bigvee_a^b (f) \int_a^b \left| g(t) - \frac{1}{b-a}\int_a^b g(s)\,ds \right| dt \tag{1.9}$$

where $\bigvee_a^b (f)$ denotes the total variation of f on the interval $[a,b]$. The constant $\frac{1}{2}$ is best possible in (1.9).

In order to extend some of the above results for the p-Schatten norm of bounded linear operators in complex Hilbert spaces we need the following preparations.

2. Some Preliminary Facts on p-Schatten Norm

Let $(H;\langle .,.\rangle)$ be a complex Hilbert space and $\mathcal{B}(H)$ the Banach algebra of all bounded linear operators on H. If $\{e_i\}_{i\in I}$ an orthonormal basis of H, we say that $A\in\mathcal{B}(H)$ is of *trace class* if

$$\|A\|_1 := \sum_{i\in I} \langle |A| e_i, e_i\rangle < \infty. \tag{2.1}$$

The definition of $\|A\|_1$ does not depend on the choice of the orthonormal basis $\{e_i\}_{i\in I}$. We denote by $\mathcal{B}_1(H)$ the set of trace class operators in $\mathcal{B}(H)$.

We define the *trace* of a trace class operator $A\in\mathcal{B}_1(H)$ to be

$$\operatorname{tr}(A) := \sum_{i\in I} \langle Ae_i, e_i\rangle, \tag{2.2}$$

where $\{e_i\}_{i\in I}$ an orthonormal basis of H. Note that this coincides with the usual definition of the trace if H is finite-dimensional. We observe that the series (2.2) converges absolutely and it is independent from the choice of basis.

The following result collects some properties of the trace:

Theorem 2. *We have:*

(i) If $A\in\mathcal{B}_1(H)$ *then* $A^*\in\mathcal{B}_1(H)$ *and*

$$\operatorname{tr}(A^*) = \overline{\operatorname{tr}(A)}; \tag{2.3}$$

(ii) If $A\in\mathcal{B}_1(H)$ *and* $T\in\mathcal{B}(H)$, *then* $AT,\ TA\in\mathcal{B}_1(H)$ *and*

$$\operatorname{tr}(AT) = \operatorname{tr}(TA) \text{ and } |\operatorname{tr}(AT)| \le \|A\|_1 \|T\|; \tag{2.4}$$

(iii) $\operatorname{tr}(\cdot)$ *is a bounded linear functional on* $\mathcal{B}_1(H)$ *with* $\|\operatorname{tr}\| = 1$;

(iv) If $A,\ B\in\mathcal{B}_2(H)$ *then* $AB,\ BA\in\mathcal{B}_1(H)$ *and* $\operatorname{tr}(AB) = \operatorname{tr}(BA)$;

(v) $\mathcal{B}_{fin}(H)$ *is a dense subspace of* $\mathcal{B}_1(H)$.

An operator $A\in\mathcal{B}(H)$ is said to belong to the *von Neumann-Schatten class* $\mathcal{B}_p(H)$, $1\le p<\infty$ if the *p-Schatten norm* is finite [16, p. 60-64]

$$\|A\|_p := [\operatorname{tr}(|A|^p)]^{1/p} < \infty.$$

For $1<p<q<\infty$ we have that

$$\mathcal{B}_1(H) \subset \mathcal{B}_p(H) \subset \mathcal{B}_q(H) \subset \mathcal{B}(H) \tag{2.5}$$

and

$$\|A\|_1 \ge \|A\|_p \ge \|A\|_q \ge \|A\|. \tag{2.6}$$

For $p\ge 1$ the functional $\|\cdot\|_p$ is a *norm* on the $*$-ideal $\mathcal{B}_p(H)$ and $\left(\mathcal{B}_p(H), \|\cdot\|_p\right)$ is a Banach space.

Also, see for instance [16, p. 60-64],

$$\|A\|_p = \|A^*\|_p,\ A\in\mathcal{B}_p(H) \tag{2.7}$$

$$\|AB\|_p \le \|A\|_p \|B\|_p,\ A,B\in\mathcal{B}_p(H) \tag{2.8}$$

and

$$\|AB\|_p \le \|A\|_p \|B\|,\ \|BA\|_p \le \|B\| \|A\|_p,\ A\in\mathcal{B}_p(H),\ B\in\mathcal{B}(H). \tag{2.9}$$

This implies that

$$\|CAB\|_p \leq \|C\| \|A\|_p \|B\|, \ A \in \mathcal{B}_p(H), \ B, \ C \in \mathcal{B}(H). \tag{2.10}$$

In terms of *p-Schatten norm* we have the *Hölder inequality* for $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$

$$(|\text{tr}(AB)| \leq) \|AB\|_1 \leq \|A\|_p \|B\|_q, \ A \in \mathcal{B}_p(H), \ B \in \mathcal{B}_q(H). \tag{2.11}$$

For the theory of trace functionals and their applications the reader is referred to [15] and [16].

For some classical trace inequalities see [5], [6], and [13], which are continuations of the work of Bellman [1].

3. 1-Schatten Norm Inequalities

For two continuous functions $A, B : [a, b] \to \mathcal{B}(H)$ we define the *noncommutative Čebyšev functional*

$$D(A, B) := (b-a) \int_a^b A(t) B(t) dt - \int_a^b A(t) dt \int_a^b B(t) dt.$$

We have the following result of interest:

Theorem 3. *For $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, let $A : [a, b] \to \mathcal{B}_p(H)$, $B : [a, b] \to \mathcal{B}_q(H)$ be strongly differentiable functions on the interval (a, b). Then*

$$\|D(A, B)\|_1 \leq D\left(\int_a^{\cdot} \|A'(u)\|_p du, \int_a^{\cdot} \|B'(u)\|_q du\right) \tag{3.1}$$
$$\leq \frac{1}{4}(b-a)^2 \int_a^b \|A'(u)\|_p du \int_a^b \|B'(u)\|_q du.$$

Proof. Observe that

$$\begin{aligned}
&\int_a^b \int_a^b [A(t) - A(s)][B(t) - B(s)] dtds \\
&= \int_a^b \int_a^b (A(t) B(t) - A(s) B(t) - A(t) B(s) + A(s) B(s)) dtds \\
&= (b-a) \int_a^b A(t) B(t) dt - \int_a^b A(s) ds \int_a^b B(t) dt \\
&- \int_a^b A(t) dt \int_a^b B(s) ds + (b-a) \int_a^b A(s) B(s) ds \\
&= 2(b-a) \int_a^b A(t) B(t) dt - 2 \int_a^b A(t) dt \int_a^b B(t) dt = 2D(A, B),
\end{aligned}$$

which give the Korkine's noncommutative identity for functions with values in Banach algebra $\mathcal{B}(H)$

$$D(A, B) = \frac{1}{2} \int_a^b \int_a^b [A(t) - A(s)][B(t) - B(s)] dtds. \tag{3.2}$$

For Korkine's classical identity for real-valued functions, see [12, p. 242].

If we take the 1-Schatten norm, use the integral's properties and employ Hölder's inequality (2.11) for $p,\ q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, then we get

$$\begin{aligned} \|D(A,B)\|_1 &\leq \frac{1}{2}\int_a^b\int_a^b \|[A(t)-A(s)][B(t)-B(s)]\|_1\, dtds \\ &\leq \frac{1}{2}\int_a^b\int_a^b \|A(t)-A(s)\|_p \|B(t)-B(s)\|_q\, dtds. \end{aligned} \tag{3.3}$$

Observe that for $s,\ t\in[a,b]$

$$A(t)-A(s)=\int_s^t A'(u)\,du,\ \ B(t)-B(s)=\int_s^t B'(u)\,du,$$

which implies that

$$\begin{aligned} \|A(t)-A(s)\|_p \|B(t)-B(s)\|_q &= \left\|\int_s^t A'(u)\,du\right\|_p \left\|\int_s^t B'(u)\,du\right\|_q \\ &\leq \left|\int_s^t \|A'(u)\|_p\,du\right| \left|\int_s^t \|B'(u)\|_q\,du\right| \\ &= \left|\int_s^t \|A'(u)\|_p\,du \int_s^t \|B'(u)\|_q\,du\right| \\ &= \int_s^t \|A'(u)\|_p\,du \int_s^t \|B'(u)\|_q\,du, \end{aligned}$$

for all $s,\ t\in[a,b]$.

By (3.3) we get

$$\|D(A,B)\|_1 \leq \frac{1}{2}\int_a^b\int_a^b \left(\int_s^t \|A'(u)\|_p\,du\right)\left(\int_s^t \|B'(u)\|_q\,du\right) dtds \tag{3.4}$$

for $p,\ q>1$ with $\frac{1}{p}+\frac{1}{q}=1$.

Since

$$\begin{aligned} &\int_s^t \|A'(u)\|_p\,du \int_s^t \|B'(u)\|_q\,du \\ &= \left(\int_a^t \|A'(u)\|_p\,du - \int_a^s \|A'(u)\|_p\,du\right)\left(\int_a^t \|B'(u)\|_q\,du - \int_a^s \|B'(u)\|_q\,du\right), \end{aligned}$$

hence by Korkine's identity for real valued functions $f(t) = \int_a^t \|A'(u)\|_p du$ and $g(t) = \int_a^t \|B'(u)\|_q du$, we have

$$\begin{aligned}
(3.5)\qquad & \frac{1}{2}\int_a^b \int_a^b \left(\int_a^t \|A'(u)\|_p du - \int_a^s \|A'(u)\|_p du \right) \\
& \times \left(\int_a^t \|B'(u)\|_q du - \int_a^s \|B'(u)\|_q du \right) dtds \\
& = (b-a) \int_a^b \left(\int_a^t \|A'(u)\|_p du \right) \left(\int_a^t \|B'(u)\|_q du \right) dt \\
& - \int_a^b \left(\int_a^t \|A'(u)\|_p du \right) dt \int_a^b \left(\int_a^t \|B'(u)\|_q du \right) dt \\
& = D\left(\int_a^{\cdot} \|A'(u)\|_p du, \int_a^{\cdot} \|B'(u)\|_q du \right).
\end{aligned}$$

By utilising (3.4) and (3.5), we deduce the first inequality in (3.1).

Observe that

$$0 \le \int_a^t \|A'(u)\|_p du \le \int_a^b \|A'(u)\|_p du$$

and

$$0 \le \int_a^t \|B'(u)\|_q du \le \int_a^b \|B'(u)\|_q du$$

for all $t \in [a,b]$, then by Grüss's inequality for the functions $f(t) = \int_a^t \|A'(u)\|_p du$ and $g(t) = \int_a^t \|B'(u)\|_p du$, $t \in [a,b]$, we get the last part of (3.1). □

We have the noncommutative Čebyšev's inequality:

Corollary 1. *Let $A : [a,b] \to \mathcal{B}_p(H)$, $B : [a,b] \to \mathcal{B}_q(H)$ for $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, be strongly differentiable functions on the interval (a,b). Then*

$$\|D(A,B)\|_1 \le \frac{1}{12}(b-a)^4 \sup_{u\in(a,b)} \|A'(u)\|_p \sup_{u\in(a,b)} \|B'(u)\|_q, \tag{3.6}$$

provided that the right hand side in (3.6) is finite.

Proof. If we use Čebyšev's inequality (1.3) for $f(t) = \int_a^t \|A'(u)\|_p du$ and $g(t) = \int_a^t \|B'(u)\|_q du$, $t \in [a,b]$, then we get

$$\begin{aligned}
0 & \le D\left(\int_a^{\cdot} \|A'(u)\|_p du, \int_a^{\cdot} \|B'(u)\|_q du \right) \\
& \le \frac{1}{12}(b-a)^4 \|f'\|_\infty \|g'\|_\infty \\
& = \frac{1}{12}(b-a)^4 \sup_{u\in(a,b)} \|A'(u)\|_p \sup_{u\in(a,b)} \|B'(u)\|_q,
\end{aligned}$$

which by the first inequality in (3.1) gives the desired result (3.6). □

By the use of Ostrowski's inequality (1.4) we derive:

Corollary 2. *Let* $A : [a,b] \to \mathcal{B}_p(H)$, $B : [a,b] \to \mathcal{B}_q(H)$ *for* $p,\ q > 1$ *with* $\frac{1}{p} + \frac{1}{q} = 1$, *be strongly differentiable functions on the interval* (a,b), *then*

$$\|D(A,B)\|_1 \leq \frac{1}{8}(b-a)^3 \sup_{u \in (a,b)} \|B'(u)\|_q \int_a^b \|A'(u)\|_p \, du. \tag{3.7}$$

By the use of Lupaş inequality for $f(t) = \int_a^t \|A'(u)\|_p \, du$ and $g(t) = \int_a^t \|B'(u)\|_q \, du$, $t \in [a,b]$, we get:

Corollary 3. *Let* $A : [a,b] \to \mathcal{B}_p(H)$, $B : [a,b] \to \mathcal{B}_q(H)$ *for* $p,\ q > 1$ *with* $\frac{1}{p} + \frac{1}{q} = 1$, *be strongly differentiable functions on the interval* (a,b), *then*

$$\|D(A,B)\|_1 \leq \frac{1}{\pi^2}(b-a)^3 \int_a^b \|A'(u)\|_p^2 \, du \int_a^b \|B'(u)\|_q^2 \, du. \tag{3.8}$$

We also have:

Corollary 4. *Let* $A : [a,b] \to \mathcal{B}_p(H)$, $B : [a,b] \to \mathcal{B}_q(H)$ *for* $p,\ q > 1$ *with* $\frac{1}{p} + \frac{1}{q} = 1$, *be strongly differentiable functions on the interval* (a,b), *then*

$$\begin{aligned} \|D(A,B)\|_1 &\leq \frac{1}{2} \int_a^b \|B'(u)\|_q \, du \\ &\quad \times \int_a^b \left| \int_a^t (u-a) \|A'(u)\|_p \, du - \int_t^b (b-u) \|A'(u)\|_p \, du \right| dt. \end{aligned} \tag{3.9}$$

Proof. Observe that for $f(t) = \int_a^t \|A'(u)\|_p \, du$, we get, integrating by parts, that

$$\begin{aligned} &\int_a^b \left| f(t) - \frac{1}{b-a} \int_a^b f(s) \, ds \right| dt \\ &= \int_a^b \left| \int_a^t \|A'(u)\|_p \, du - \frac{1}{b-a} \int_a^b \left(\int_a^s \|A'(u)\|_p \, du \right) ds \right| dt \\ &= \int_a^b \left| \int_a^t \|A'(u)\|_p \, du - \frac{1}{b-a} \left(\left(\int_a^b \|A'(u)\|_p \, du \right) b - \int_a^b \|A'(s)\|_p \, s ds \right) \right| dt \\ &\qquad = \int_a^b \left| \int_a^t \|A'(u)\|_p \, du - \frac{1}{b-a} \left(\int_a^b (b-u) \|A'(u)\|_p \, du \right) \right| dt \\ &\qquad = \frac{1}{b-a} \int_a^b \left| (b-a) \int_a^t \|A'(u)\|_p \, du - \int_a^b (b-u) \|A'(u)\|_p \, du \right| dt \\ &\qquad = \frac{1}{b-a} \int_a^b \left| \int_a^t (u-a) \|A'(u)\|_p \, du - \int_t^b (b-u) \|A'(u)\|_p \, du \right| dt. \end{aligned}$$

By utilising (1.8) for $f(t) = \int_a^t \|A'(u)\|_p \, du$ and $g(t) = \int_a^t \|B'(u)\|_q \, du$, $t \in [a,b]$, we then get (3.9). □

We also have:

Corollary 5. *With the assumptions of Corollary 4,*

$$\|D(A,B)\|_1 \le \int_a^b \|B'(u)\|_q\,du \int_a^b (b-t)(t-a)\|A'(t)\|_p\,dt \le \frac{1}{4}(b-a)^2 \int_a^b \|B'(u)\|_q\,du \int_a^b \|A'(t)\|_p\,dt. \tag{3.10}$$

Proof. We observe that

$$\begin{aligned}
&\int_a^b \left|\int_a^t (u-a)\|A'(u)\|_p\,du - \int_t^b (b-u)\|A'(u)\|_p\,du\right| dt \\
&\le \int_a^b \left[\left|\int_a^t (u-a)\|A'(u)\|_p\,du\right| + \left|\int_t^b (b-u)\|A'(u)\|_p\,du\right|\right] dt \\
&\le \int_a^b \left[\int_a^t (u-a)\|A'(u)\|_p\,du + \int_t^b (b-u)\|A'(u)\|_p\,du\right] dt \\
&= \left[\int_a^t (u-a)\|A'(u)\|_p\,du + \int_t^b (b-u)\|A'(u)\|_p\,du\right] t\Big|_a^b \\
&- \int_a^b t\left((t-a)\|A'(t)\|_p - (b-t)\|A'(t)\|_p\right) dt \\
&= b\int_a^b (u-a)\|A'(u)\|_p\,du - a\int_a^b (b-u)\|A'(u)\|_p\,du \\
&- \int_a^b t\left((t-a)\|A'(t)\|_p - (b-t)\|A'(t)\|_p\right) dt \\
&= 2\int_a^b (b-t)(t-a)\|A'(t)\|_p\,dt
\end{aligned}$$

and by (3.9) we get (3.10). □

We also have:

Theorem 4. *Let* $A:[a,b]\to\mathcal{B}_p(H)$, $B:[a,b]\to\mathcal{B}_q(H)$ *for* $p,\ q>1$ *with* $\frac{1}{p}+\frac{1}{q}=1$, *be continuous functions on the interval* $[a,b]$, *then*

$$\|D(A,B)\|_1 \le \begin{cases} \sup_{t\in[a,b]}\|A(t)-V\|_p \int_a^b \left\|(b-a)B(t)-\int_a^b B(s)\,ds\right\|_q dt, \\ \left(\int_a^b \|A(t)-V\|_p^\alpha\,dt\right)^{1/\alpha}\left(\int_a^b \left\|(b-a)B(t)-\int_a^b B(s)\,ds\right\|_q^\beta dt\right)^{1/\beta}, \\ \text{if } \alpha,\ \beta>1 \text{ with } \frac{1}{\alpha}+\frac{1}{\beta}=1, \\ \sup_{t\in[a,b]}\left\|(b-a)B(t)-\int_a^b B(s)\,ds\right\|_q \int_a^b \|A(t)-V\|_p\,dt \end{cases} \tag{3.11}$$

for all $V\in\mathcal{B}_p(H)$.

Proof. For all $V\in\mathcal{B}(H)$ we have

$$\begin{aligned}&\int_a^b [A(t)-V]\left[(b-a)B(t)-\int_a^b B(s)\,ds\right]dt\\&=\int_a^b A(t)\left[(b-a)B(t)-\int_a^b B(s)\,ds\right]dt\\&-V\int_a^b\left[(b-a)B(t)-\int_a^b B(s)\,ds\right]dt\\&=(b-a)\int_a^b A(t)B(t)\,dt-\int_a^b A(t)\,dt\int_a^b B(s)\,ds\\&-V\left[(b-a)\int_a^b B(t)\,dt-(b-a)\int_a^b B(s)\,ds\right]=D(A,B).\end{aligned}$$

If we take the 1-Schatten norm, use the integral's properties and employ Hölder's inequality (2.11) for $p,\ q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, then we get

$$\|D(A,B)\|_1\le\int_a^b\left\|[A(t)-V]\left[(b-a)B(t)-\int_a^b B(s)\,ds\right]\right\|_1 dt$$

$$\le\int_a^b\|A(t)-V\|_p\left\|(b-a)B(t)-\int_a^b B(s)\,ds\right\|_q dt$$

$$\le\begin{cases}\sup_{t\in[a,b]}\|A(t)-V\|_p\int_a^b\left\|(b-a)B(t)-\int_a^b B(s)\,ds\right\|_q dt,\\ \\ \left(\int_a^b\|A(t)-V\|_p^\alpha dt\right)^{1/\alpha}\left(\int_a^b\left\|(b-a)B(t)-\int_a^b B(s)\,ds\right\|_q^\beta dt\right)^{1/\beta},\\ \\ \int_a^b\|A(t)-V\|_p\,dt\sup_{t\in[a,b]}\left\|(b-a)B(t)-\int_a^b B(s)\,ds\right\|_q\end{cases}$$

for all $V\in\mathcal{B}(H)$. □

Corollary 6. *With the assumptions of Theorem 4 and if there exists $V\in\mathcal{B}_p(H)$ and $M>0$ such that*

$$\|A(t)-V\|_p\le M \text{ for all } t\in[a,b],$$

then

$$\|D(A,B)\|_1\le M\int_a^b\left\|(b-a)B(t)-\int_a^b B(s)\,ds\right\|_q dt. \tag{3.12}$$

The proof is obvious from the first branch of (3.11).

Corollary 7. *Let $A:[a,b]\to\mathcal{B}_p(H)$, $B:[a,b]\to\mathcal{B}_q(H)$ for $p,\ q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, be continuous functions on the interval $[a,b]$ and A strongly differentiable* *on* (a,b), *then for* $p,\ q>1$ *with* $\frac{1}{p}+\frac{1}{q}=1,$

$$\begin{aligned}
&\left\|D(A,B)\right\|_1 \\
&\leq \sup_{t\in[a,b]}\left\|(b-a)B(t)-\int_a^b B(s)\,ds\right\|_q \\
&\times\left[\int_a^{\frac{a+b}{2}}(t-a)\left\|A'(t)\right\|_p dt+\int_{\frac{a+b}{2}}^b(b-t)\left\|A'(t)\right\|_p dt\right] \\
&\leq \sup_{t\in[a,b]}\left\|(b-a)B(t)-\int_a^b B(s)\,ds\right\|_q \\
&\times\begin{cases}\frac{1}{8}(b-a)^2\left[\sup_{t\in\left[a,\frac{a+b}{2}\right]}\left\|A'(t)\right\|_p+\sup_{t\in\left[\frac{a+b}{2},b\right]}\left\|A'(t)\right\|_p\right], \\ \\ \frac{(b-a)^{1+1/\beta}}{(\beta+1)^{1/\beta}2^{1+1/\beta}} \\ \times\left[\left(\int_a^{\frac{a+b}{2}}\left\|A'(t)\right\|_p^\alpha dt\right)^{1/\alpha}+\left(\int_{\frac{a+b}{2}}^b\left\|A'(t)\right\|_p^\alpha dt\right)^{1/\alpha}\right], \\ \textit{if } \alpha,\ \beta>1,\ \frac{1}{\alpha}+\frac{1}{\beta}=1, \\ \\ \frac{1}{2}(b-a)\int_a^b\left\|A'(t)\right\|_p dt.\end{cases}
\end{aligned} \tag{3.13}$$

Proof. We have for $V=A\left(\frac{a+b}{2}\right)$ that

$$\begin{aligned}
&\int_a^b\left\|A(t)-A\left(\frac{a+b}{2}\right)\right\|dt \\
&=\int_a^b\left\|\int_{\frac{a+b}{2}}^t A'(s)\,ds\right\|dt\leq\int_a^b\left|\int_{\frac{a+b}{2}}^t\left\|A'(s)\right\|ds\right|dt \\
&=\int_a^{\frac{a+b}{2}}\left(\int_t^{\frac{a+b}{2}}\left\|A'(s)\right\|ds\right)dt+\int_{\frac{a+b}{2}}^b\left(\int_{\frac{a+b}{2}}^t\left\|A'(s)\right\|ds\right)dt
\end{aligned}$$

$$\begin{aligned}
&=\left(\int_t^{\frac{a+b}{2}}\left\|A'(s)\right\|ds\right)t\Big|_a^{\frac{a+b}{2}}+\int_a^{\frac{a+b}{2}}t\left\|A'(t)\right\|dt \\
&+\left(\int_{\frac{a+b}{2}}^t\left\|A'(s)\right\|ds\right)t\Big|_{\frac{a+b}{2}}^b-\int_{\frac{a+b}{2}}^b t\left\|A'(t)\right\|dt \\
&=\int_a^{\frac{a+b}{2}}t\left\|A'(t)\right\|dt-a\int_a^{\frac{a+b}{2}}\left\|A'(s)\right\|ds \\
&+b\int_{\frac{a+b}{2}}^b\left\|A'(s)\right\|ds-\int_{\frac{a+b}{2}}^b t\left\|A'(t)\right\|dt \\
&=\int_a^{\frac{a+b}{2}}(t-a)\left\|A'(t)\right\|dt+\int_{\frac{a+b}{2}}^b(b-t)\left\|A'(t)\right\|dt,
\end{aligned}$$

which, by the third branch of (3.11), gives the first part of (3.13).

The last part follows by Hölder's inequality for the integrals

$$\int_a^{\frac{a+b}{2}} (t-a)\|A'(t)\|\,dt \text{ and } \int_{\frac{a+b}{2}}^b (b-t)\|A'(t)\|\,dt.$$

□

4. p-Schatten Norm Inequalities

We also have the following inequalities in terms of p-Schatten norm:

Theorem 5. *Let $A,\ B:[a,b]\to\mathcal{B}_p(H)$ for $p>1$, be strongly differentiable functions on the interval (a,b). Then*

$$\|D(A,B)\|_p \le D\left(\int_a^\cdot \|A'(u)\|_p\,du, \int_a^\cdot \|B'(u)\|_p\,du\right) \le \frac{1}{4}\int_a^b \|A'(u)\|_p\,du \int_a^b \|B'(u)\|_p\,du. \tag{4.1}$$

Proof. If we take the p-Schatten norm in the Korkine's identity (3.2), use the integral's properties and the property (2.8), then we get

$$\|D(A,B)\|_p \le \frac{1}{2}\int_a^b\int_a^b \|[A(t)-A(s)][B(t)-B(s)]\|_p\,dtds \le \frac{1}{2}\int_a^b\int_a^b \|A(t)-A(s)\|_p\|B(t)-B(s)\|_p\,dtds. \tag{4.2}$$

Observe that for $s,\ t\in[a,b]$

$$A(t)-A(s)=\int_s^t A'(u)\,du,\ B(t)-B(s)=\int_s^t B'(u)\,du,$$

which implies that

$$\begin{aligned}\|A(t)-A(s)\|_p\|B(t)-B(s)\|_p &= \left\|\int_s^t A'(u)\,du\right\|_p\left\|\int_s^t B'(u)\,du\right\|_p\\ &\le \left|\int_s^t \|A'(u)\|_p\,du\right|\left|\int_s^t \|B'(u)\|_p\,du\right|\\ &= \left|\int_s^t \|A'(u)\|_p\,du\int_s^t \|B'(u)\|_p\,du\right|\\ &= \int_s^t \|A'(u)\|_p\,du\int_s^t \|B'(u)\|_p\,du,\end{aligned}$$

for all $s,\ t\in[a,b]$.

By (4.2) we get

$$\|D(A,B)\|_p \le \frac{1}{2}\int_a^b\int_a^b \left(\int_s^t \|A'(u)\|_p\,du\right)\left(\int_s^t \|B'(u)\|_p\,du\right)dtds \tag{4.3}$$

for $p>1$.

Since

$$\int_s^t \|A'(u)\|_p \, du \int_s^t \|B'(u)\|_p \, du$$
$$= \left(\int_a^t \|A'(u)\|_p \, du - \int_a^s \|A'(u)\|_p \, du \right) \left(\int_a^t \|B'(u)\|_p \, du - \int_a^s \|B'(u)\|_p \, du \right),$$

hence by Korkine's identity for real valued functions $f(t) = \int_a^t \|A'(u)\|_p \, du$ and $g(t) = \int_a^t \|B'(u)\|_p \, du$, we have

$$\begin{aligned}
&\frac{1}{2} \int_a^b \int_a^b \left(\int_a^t \|A'(u)\|_p \, du - \int_a^s \|A'(u)\|_p \, du \right) \\
&\times \left(\int_a^t \|B'(u)\|_p \, du - \int_a^s \|B'(u)\|_p \, du \right) dtds \\
&= (b-a) \int_a^b \left(\int_a^t \|A'(u)\|_p \, du \right) \left(\int_a^t \|B'(u)\|_p \, du \right) dt \\
&- \int_a^b \left(\int_a^t \|A'(u)\|_p \, du \right) dt \int_a^b \left(\int_a^t \|B'(u)\|_p \, du \right) dt \\
&= D\left(\int_a^{\cdot} \|A'(u)\|_p \, du, \int_a^{\cdot} \|B'(u)\|_p \, du \right).
\end{aligned} \tag{4.4}$$

By utilising (4.3) and (4.4), we deduce the first inequality in (4.1).

Observe that

$$0 \le \int_a^t \|A'(u)\|_p \, du \le \int_a^b \|A'(u)\|_p \, du$$

and

$$0 \le \int_a^t \|B'(u)\|_p \, du \le \int_a^b \|B'(u)\|_p \, du$$

for all $t \in [a,b]$, then by Grüss's inequality for the functions $f(t) = \int_a^t \|A'(u)\|_p \, du$ and $g(t) = \int_a^t \|B'(u)\|_p \, du$, $t \in [a,b]$, we get the last part of (4.1). □

We have the noncommutative Čebyšev's inequality:

Corollary 8. *Let A, $B : [a,b] \to \mathcal{B}_p(H)$ for $p > 1$, be strongly differentiable functions on the interval (a,b). Then*

$$\|D(A,B)\|_p \le \frac{1}{12} (b-a)^4 \sup_{u \in (a,b)} \|A'(u)\|_p \sup_{u \in (a,b)} \|B'(u)\|_p, \tag{4.5}$$

provided that the right hand side in (4.5) is finite.

Proof. If we use Čebyšev's inequality (1.3) for $f(t) = \int_a^t \|A'(u)\|_p \, du$ and $g(t) = \int_a^t \|B'(u)\|_p \, du$, $t \in [a,b]$, then we get

$$\begin{aligned}
0 &\le D\left(\int_a^{\cdot} \|A'(u)\|_p \, du, \int_a^{\cdot} \|B'(u)\|_p \, du \right) \\
&\le \frac{1}{12} (b-a)^4 \|f'\|_\infty \|g'\|_\infty \\
&= \frac{1}{12} (b-a)^4 \sup_{u \in (a,b)} \|A'(u)\|_p \sup_{u \in (a,b)} \|B'(u)\|_p,
\end{aligned}$$

which by the first inequality in (4.1) gives the desired result (4.5). □

By the use of Ostrowski's inequality (1.4) we derive:

Corollary 9. *Let A, $B : [a,b] \to \mathcal{B}_p(H)$ for $p > 1$, be strongly differentiable functions on the interval (a,b), then*

$$\|D(A,B)\|_p \leq \frac{1}{8}(b-a)^3 \sup_{u\in(a,b)} \|B'(u)\|_p \int_a^b \|A'(u)\|_p \, du. \tag{4.6}$$

By the use of Lupaş inequality for $f(t) = \int_a^t \|A'(u)\|_p \, du$ and $g(t) = \int_a^t \|B'(u)\|_p \, du$, $t \in [a,b]$, we get:

Corollary 10. *Let A, $B : [a,b] \to \mathcal{B}_p(H)$ for $p > 1$, be strongly differentiable functions on the interval (a,b), then*

$$\|D(A,B)\|_p \leq \frac{1}{\pi^2}(b-a)^3 \int_a^b \|A'(u)\|_p^2 \, du \int_a^b \|B'(u)\|_p^2 \, du. \tag{4.7}$$

We also have:

Corollary 11. *Let A, $B : [a,b] \to \mathcal{B}_p(H)$ for $p > 1$, be strongly differentiable functions on the interval (a,b), then*

$$\begin{aligned} \|D(A,B)\|_p &\leq \frac{1}{2}\int_a^b \|B'(u)\|_p \, du \\ &\quad \times \int_a^b \left| \int_a^t (u-a)\|A'(u)\|_p \, du - \int_t^b (b-u)\|A'(u)\|_p \, du \right| dt. \end{aligned} \tag{4.8}$$

Proof. Observe that for $f(t) = \int_a^t \|A'(u)\|_p \, du$, we get, integrating by parts, that

$$\begin{aligned} &\int_a^b \left| f(t) - \frac{1}{b-a}\int_a^b f(s)\, ds \right| dt \\ &= \int_a^b \left| \int_a^t \|A'(u)\|_p \, du - \frac{1}{b-a}\int_a^b \left(\int_a^s \|A'(u)\|_p \, du \right) ds \right| dt \\ &= \int_a^b \left| \int_a^t \|A'(u)\|_p \, du - \frac{1}{b-a}\left(\left(\int_a^b \|A'(u)\|_p \, du \right) b - \int_a^b \|A'(s)\|_p \, s ds \right) \right| dt \\ &\qquad = \int_a^b \left| \int_a^t \|A'(u)\|_p \, du - \frac{1}{b-a}\left(\int_a^b (b-u)\|A'(u)\|_p \, du \right) \right| dt \\ &\qquad = \frac{1}{b-a}\int_a^b \left| (b-a)\int_a^t \|A'(u)\|_p \, du - \int_a^b (b-u)\|A'(u)\|_p \, du \right| dt \\ &\qquad = \frac{1}{b-a}\int_a^b \left| \int_a^t (u-a)\|A'(u)\|_p \, du - \int_t^b (b-u)\|A'(u)\|_p \, du \right| dt. \end{aligned}$$

By utilising (1.8) for $f(t) = \int_a^t \|A'(u)\|_p \, du$ and $g(t) = \int_a^t \|B'(u)\|_p \, du$, $t \in [a,b]$, we then get (4.8). □

We also have:

Corollary 12. *With the assumptions of Corollary 4,*

$$\|D(A,B)\|_p \leq \int_a^b \|B'(u)\|_p\,du \int_a^b (b-t)(t-a)\|A'(t)\|_p\,dt \tag{4.9}$$
$$\leq \frac{1}{4}(b-a)^2 \int_a^b \|B'(u)\|_p\,du \int_a^b \|A'(u)\|_p\,du.$$

Proof. We observe that

$$\int_a^b \left| \int_a^t (u-a)\|A'(u)\|_p\,du - \int_t^b (b-u)\|A'(u)\|_p\,du \right| dt$$
$$\leq \int_a^b \left[\left| \int_a^t (u-a)\|A'(u)\|_p\,du \right| + \left| \int_t^b (b-u)\|A'(u)\|_p\,du \right| \right] dt$$
$$\leq \int_a^b \left[\int_a^t (u-a)\|A'(u)\|_p\,du + \int_t^b (b-u)\|A'(u)\|_p\,du \right] dt$$
$$= \left[\int_a^t (u-a)\|A'(u)\|_p\,du + \int_t^b (b-u)\|A'(u)\|_p\,du \right] t \Big|_a^b$$
$$- \int_a^b t\left((t-a)\|A'(t)\|_p - (b-t)\|A'(t)\|_p \right) dt$$
$$= b\int_a^b (u-a)\|A'(u)\|_p\,du - a\int_a^b (b-u)\|A'(u)\|_p\,du$$
$$- \int_a^b t\left((t-a)\|A'(t)\|_p - (b-t)\|A'(t)\|_p \right) dt$$
$$= 2\int_a^b (b-t)(t-a)\|A'(t)\|_p\,dt$$

and by (4.8) we get (4.9). □

5. Some Examples

Consider the function $A(t) = \exp(tT)$, where $t \in \mathbb{R}$ and $T \in \mathcal{B}(H)$. Then $A'(t) = T\exp(tT)$, for $t \in \mathbb{R}$ and $T \in \mathcal{B}(H)$. If T is invertible, then [2]

$$\int_a^b \exp(tT)\,dt = T^{-1}\left[\exp(bT) - \exp(aT)\right]. \tag{5.1}$$

For T, V commutative, i.e. $TV = VT$ then

$$\exp(T+V) = \exp T \exp B.$$

Consider the functions

$$A_T(t) = \exp(tT),\; B_V(t) = \exp(tV),\; t \in [a,b].$$

For p, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, let $T \in \mathcal{B}_p(H)$, $V \in \mathcal{B}_q(H)$. Then

$$D(A_T, B_V) := (b-a)\int_a^b \exp[t(T+V)]\,dt - \int_a^b \exp(tT)\,dt \int_a^b \exp(tV)\,dt.$$

By (3.1) we derive

$$\begin{aligned} &\left\| D\left(A_T, B_V\right)\right\|_1 \\ &\le \frac{1}{4}(b-a)^2 \int_a^b \left\| T \exp(uT)\right\|_p du \int_a^b \left\| V \exp(uV)\right\|_q du. \end{aligned} \tag{5.2}$$

By the property (2.8) we derive

$$\left\| T \exp(uT)\right\|_p \le \|T\|_p \left\| \exp(uT)\right\|$$

and

$$\left\| V \exp(uV)\right\|_q \le \|V\|_q \left\| \exp(uV)\right\|_q$$

for $u \in [a, b]$.

Since $\left\| \exp(tT)\right\|_p \le \exp\left(|t| \|T\|_p\right)$, $t \in \mathbb{R}$, $T \in \mathcal{B}_p(H)$ then by (5.2) we derive

$$\begin{aligned} &\left\| D\left(A_T, B_V\right)\right\|_1 \\ &\le \frac{1}{4}(b-a)^2 \|T\|_p \|V\|_q \int_a^b \exp\left(|t| \|T\|_p\right) du \int_a^b \exp\left(|u| \|V\|_q\right) du. \end{aligned} \tag{5.3}$$

If $0 \le a \le b$, then

$$\int_a^b \exp\left(\|T\|_p |t|\right) dt = \int_a^b \exp\left(\|T\|_p t\right) dt = \frac{\exp\left(\|T\|_p b\right) - \exp\left(\|T\|_p a\right)}{\|T\|},$$

$$\int_a^b \exp\left(|u| \|V\|_q\right) du = \int_a^b \exp\left(\|V\|_q u\right) du = \frac{\exp\left(\|V\|_q b\right) - \exp\left(\|V\|_q a\right)}{\|V\|_q}$$

and by (5.3) we get

$$\begin{aligned} \left\| D\left(A_T, B_V\right)\right\|_1 \le \frac{1}{4}(b-a)^2 &\left[\exp\left(\|T\|_p b\right) - \exp\left(\|T\|_p a\right)\right] \\ \times &\left[\exp\left(\|V\|_q b\right) - \exp\left(\|V\|_q a\right)\right]. \end{aligned} \tag{5.4}$$

Moreover, if T, V and $T+V$ are invertible, then

$$\begin{aligned} D\left(A_T, B_V\right) = {} & (b-a)(T+V)^{-1}\left[\exp(b(T+V)) - \exp(a(T+V))\right] \\ & - T^{-1}\left[\exp(bT) - \exp(aT)\right] V^{-1}\left[\exp(bV) - \exp(aV)\right] \end{aligned}$$

and by (5.4) we derive

$$\begin{aligned} &\Big\| (b-a)(T+V)^{-1}\left[\exp(b(T+V)) - \exp(a(T+V))\right] \\ &-T^{-1}\left[\exp(bT) - \exp(aT)\right] V^{-1}\left[\exp(bV) - \exp(aV)\right]\Big\|_1 \\ &\le \frac{1}{4}(b-a)^2 \left[\exp\left(\|T\|_p b\right) - \exp\left(\|T\|_p a\right)\right] \\ &\times \left[\exp\left(\|V\|_q b\right) - \exp\left(\|V\|_q a\right)\right], \end{aligned} \tag{5.5}$$

provide that $T \in \mathcal{B}_p(H)$, $V \in \mathcal{B}_q(H)$ for $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, are commutative, T, V and $T+V$ are invertible while $0 \le a \le b$.

In particular, for $a=0$ and $b=1$ we derive from (5.5) the simpler inequality

$$\left\| (T+V)^{-1}\left[\exp\left((T+V)\right)-1\right]-T^{-1}\left[\exp(T)-1\right]V^{-1}\left[\exp(V)-1\right]\right\|_1 \leq \frac{1}{4}\left[\exp\left(\|T\|_p\right)-1\right]\left[\exp\left(\|V\|_q\right)-1\right], \tag{5.6}$$

provide that $T\in\mathcal{B}_p(H)$, $V\in\mathcal{B}_q(H)$ for $p,\ q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, are commutative, and T, V and $T+V$ are invertible.

Similar exponential inequalities may be derived by employing the inequalities (3.6)-(3.8) and (3.10) as well as, (4.1), (4.5)-(4.7) and (4.9).

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