p-SCHATTEN NORM INEQUALITIES FOR ČEBYŠEV'S FUNCTIONAL

SILVESTRU SEVER DRAGOMIR[1,2]

Abstract. An operator $A\in\mathcal{B}(H)$ is said to belong to the *von Neumann-Schatten class* $\mathcal{B}_p(H)$, $1\leq p<\infty$ if the *p-Schatten norm* is finite

$$\|A\|_p:=[\operatorname{tr}(|A|^p)]^{1/p}<\infty.$$

For two continuous functions $A,\ B:[a,b]\to\mathcal{B}(H)$ we define the *noncommutative Čebyšev functional*

$$D(A,B):=(b-a)\int_a^b A(t)B(t)\,dt-\int_a^b A(t)\,dt\int_a^b B(t)\,dt.$$

In this paper we show among others that, if $p,\ q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, and $A:[a,b]\to\mathcal{B}_p(H)$, $B:[a,b]\to\mathcal{B}_q(H)$ are strongly differentiable functions on the interval (a,b), then

$$\|D(A,B)\|_1\leq\sup_{u\in(a,b)}\|A'(u)\|_p\,D\left(\ell,\int_a^{\cdot}\|B'(u)\|_q\,du\right)$$
$$\leq\frac{1}{8}(b-a)^2\sup_{u\in(a,b)}\|A'(u)\|_p\int_a^b\|B'(u)\|_q\,du.$$

Some examples of interest for the operator monotone functions are also given.

1. Introduction

For two Lebesgue integrable functions $f,\ g:[a,b]\to\mathbb{C}$, in order to compare the integral mean of the product with the product of the integral means, we consider the *Čebyšev functional* defined by

$$D(f,g):=(b-a)\int_a^b f(t)g(t)\,dt-\int_a^b f(t)\,dt\int_a^b g(t)\,dt.$$

In 1934, G. Grüss [11] showed that

$$|D(f,g)|\leq\frac{1}{4}(b-a)^2(M-m)(N-n), \tag{1.1}$$

provided $m,\ M,\ n,\ N$ are real numbers with the property that

$$-\infty<m\leq f\leq M<\infty,\quad -\infty<n\leq g\leq N<\infty\quad\text{a.e. on}\quad [a,b]. \tag{1.2}$$

The constant $\frac{1}{4}$ is best possible in (1.1) in the sense that it cannot be replaced by a smaller one.

1991 *Mathematics Subject Classification.* 47A63, 47A60.

Key words and phrases. p-Schatten norms, Grüss' inequality, Čebyšev's inequality, Norm inequalities.

RGMIA Res. Rep. Coll. 24 (2021), Art. 116, 16 pp. Received 12/09/21

Another lesser known inequality for $D(f,g)$ was derived in 1882 by Čebyšev [8] under the assumption that f', g' exist and are continuous on $[a,b]$, and is given by

$$|D(f,g)| \le \frac{1}{12}\|f'\|_\infty \|g'\|_\infty (b-a)^4, \tag{1.3}$$

where $\|f'\|_\infty := \sup_{t\in[a,b]} |f'(t)| < \infty$.

The constant $\frac{1}{12}$ cannot be improved in general in (1.3).

Čebyšev's inequality (1.3) also holds if f, $g : [a,b] \to \mathbb{R}$ are assumed to be absolutely continuous and f', $g' \in L_\infty[a,b]$.

In 1970, A. M. Ostrowski [15] proved, amongst others, the following result that is in a sense a combination of the Čebyšev and Grüss results:

$$|D(f,g)| \le \frac{1}{8}(b-a)^3 (M-m)\|g'\|_\infty, \tag{1.4}$$

provided f is Lebesgue integrable on $[a,b]$ and satisfying (1.2) while $g : [a,b] \to \mathbb{R}$ is absolutely continuous and $g' \in L_\infty[a,b]$. Here the constant $\frac{1}{8}$ is also sharp.

In 1973, A. Lupaş [12] (see also [13, p. 210]) obtained the following result as well:

$$|D(f,g)| \le \frac{1}{\pi^2}\|f'\|_2 \|g'\|_2 (b-a)^3, \tag{1.5}$$

provided f, g are absolutely continuous and f', $g' \in L_2[a,b]$.

Here the constant $\frac{1}{\pi^2}$ is the best possible as well.

In [4], P. Cerone and S. S. Dragomir proved the following inequalities:

$$|D(f,g)| \le (b-a)\times\begin{cases} \inf\limits_{\gamma\in\mathbb{R}} \|g-\gamma\|_\infty \int_a^b \left| f(t) - \frac{1}{b-a}\int_a^b f(s)\,ds\right| dt, \\[2ex] \inf\limits_{\gamma\in\mathbb{R}} \|g-\gamma\|_q \left(\int_a^b \left| f(t) - \frac{1}{b-a}\int_a^b f(s)\,ds\right|^p dt\right)^{\frac{1}{p}} \\ \qquad\qquad \text{where } p>1,\ 1/p+1/q=1. \end{cases} \tag{1.6}$$

For $\gamma = 0$, we get from the first inequality in (1.6)

$$|D(f,g)| \le (b-a)\|g\|_\infty \int_a^b \left| f(t) - \frac{1}{b-a}\int_a^b f(s)\,ds\right| dt \tag{1.7}$$

for which the constant 1 cannot be replaced by a smaller constant.

If $m \le g \le M$ for a.e. $x \in [a,b]$, then $\left\| g - \frac{m+M}{2}\right\|_\infty \le \frac{1}{2}(M-m)$ and by the first inequality in (1.6) we can deduce the following result obtained by Cheng and Sun [9]

$$|D(f,g)| \le \frac{1}{2}(b-a)(M-m)\int_a^b \left| f(t) - \frac{1}{b-a}\int_a^b f(s)\,ds\right| dt. \tag{1.8}$$

The constant $\frac{1}{2}$ is best in (1.8) as shown by Cerone and Dragomir in [5].

The following result holds [10].

Theorem 1. *Let $f : [a,b] \to \mathbb{C}$ be of bounded variation on $[a,b]$ and $g : [a,b] \to \mathbb{C}$ a Lebesgue integrable function on $[a,b]$. Then*

$$|D(f,g)| \le \frac{1}{2}(b-a)\bigvee_a^b (f) \int_a^b \left| g(t) - \frac{1}{b-a}\int_a^b g(s)\,ds\right| dt, \tag{1.9}$$

where $\bigvee_a^b (f)$ *denotes the total variation of* f *on the interval* $[a,b]$. *The constant* $\frac{1}{2}$ *is best possible in (1.9).*

In order to extend some of the above results for the p-Schatten norm of bounded linear operators in complex Hilbert spaces we need the following preparations.

2. Some Preliminary Facts on p-Schatten Norm

Let $(H;\langle .,.\rangle)$ be a complex Hilbert space and $\mathcal{B}(H)$ the Banach algebra of all bounded linear operators on H. If $\{e_i\}_{i\in I}$ an orthonormal basis of H, we say that $A\in\mathcal{B}(H)$ is of *trace class* if

$$\|A\|_1 := \sum_{i\in I}\langle |A| e_i, e_i\rangle < \infty. \tag{2.1}$$

The definition of $\|A\|_1$ does not depend on the choice of the orthonormal basis $\{e_i\}_{i\in I}$. We denote by $\mathcal{B}_1(H)$ the set of trace class operators in $\mathcal{B}(H)$.

We define the *trace* of a trace class operator $A\in\mathcal{B}_1(H)$ to be

$$\operatorname{tr}(A) := \sum_{i\in I}\langle Ae_i, e_i\rangle, \tag{2.2}$$

where $\{e_i\}_{i\in I}$ an orthonormal basis of H. Note that this coincides with the usual definition of the trace if H is finite-dimensional. We observe that the series (2.2) converges absolutely and it is independent from the choice of basis.

The following result collects some properties of the trace:

Theorem 2. *We have:*

(i) If $A\in\mathcal{B}_1(H)$ *then* $A^*\in\mathcal{B}_1(H)$ *and*

$$\operatorname{tr}(A^*) = \overline{\operatorname{tr}(A)}; \tag{2.3}$$

(ii) If $A\in\mathcal{B}_1(H)$ *and* $T\in\mathcal{B}(H)$, *then* AT, $TA\in\mathcal{B}_1(H)$ *and*

$$\operatorname{tr}(AT) = \operatorname{tr}(TA) \ \textit{ and } \ |\operatorname{tr}(AT)| \le \|A\|_1\|T\|; \tag{2.4}$$

(iii) $\operatorname{tr}(\cdot)$ *is a bounded linear functional on* $\mathcal{B}_1(H)$ *with* $\|\operatorname{tr}\| = 1$;

(iv) If A, $B\in\mathcal{B}_2(H)$ *then* AB, $BA\in\mathcal{B}_1(H)$ *and* $\operatorname{tr}(AB) = \operatorname{tr}(BA)$;

(v) $\mathcal{B}_{fin}(H)$ *is a dense subspace of* $\mathcal{B}_1(H)$.

An operator $A\in\mathcal{B}(H)$ is said to belong to the *von Neumann-Schatten class* $\mathcal{B}_p(H)$, $1\le p<\infty$ if the *p-Schatten norm* is finite [17, p. 60-64]

$$\|A\|_p := \left[\operatorname{tr}(|A|^p)\right]^{1/p} < \infty.$$

For $1<p<q<\infty$ we have that

$$\mathcal{B}_1(H)\subset\mathcal{B}_p(H)\subset\mathcal{B}_q(H)\subset\mathcal{B}(H) \tag{2.5}$$

and

$$\|A\|_1 \ge \|A\|_p \ge \|A\|_q \ge \|A\|. \tag{2.6}$$

For $p\ge 1$ the functional $\|\cdot\|_p$ is a *norm* on the $*$-ideal $\mathcal{B}_p(H)$ and $\left(\mathcal{B}_p(H), \|\cdot\|_p\right)$ is a Banach space.

Also, see for instance [17, p. 60-64],

$$\|A\|_p = \|A^*\|_p, \ A\in\mathcal{B}_p(H) \tag{2.7}$$

$$\|AB\|_p \le \|A\|_p \|B\|_p, \ A, B \in \mathcal{B}_p(H) \tag{2.8}$$

and

$$\|AB\|_p \le \|A\|_p \|B\|, \ \|BA\|_p \le \|B\| \|A\|_p, \ A \in \mathcal{B}_p(H), \ B \in \mathcal{B}(H). \tag{2.9}$$

This implies that

$$\|CAB\|_p \le \|C\| \|A\|_p \|B\|, \ A \in \mathcal{B}_p(H), \ B, \ C \in \mathcal{B}(H). \tag{2.10}$$

In terms of *p-Schatten norm* we have the *Hölder inequality* for $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$

$$(|\operatorname{tr}(AB)| \le) \|AB\|_1 \le \|A\|_p \|B\|_q, \ A \in \mathcal{B}_p(H), \ B \in \mathcal{B}_q(H). \tag{2.11}$$

For the theory of trace functionals and their applications the reader is referred to [16] and [17].

For some classical trace inequalities see [6], [7], and [14], which are continuations of the work of Bellman [1].

3. p-Schatten Norm Inequalities

For two continuous functions $A, B : [a,b] \to \mathcal{B}(H)$ we define the *noncommutative Čebyšev functional*

$$D(A,B) := (b-a) \int_a^b A(t) B(t) dt - \int_a^b A(t) dt \int_a^b B(t) dt.$$

We have the following result of interest:

Theorem 3. *For $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, let $A : [a,b] \to \mathcal{B}_p(H)$, $B : [a,b] \to \mathcal{B}_q(H)$ be strongly differentiable functions on the interval (a,b). Then*

$$\begin{aligned} \|D(A,B)\|_1 &\le \sup_{u \in (a,b)} \|A'(u)\|_p D\left(\ell, \int_a^{\cdot} \|B'(u)\|_q du\right) \\ &\le \frac{1}{8}(b-a)^2 \sup_{u \in (a,b)} \|A'(u)\|_p \int_a^b \|B'(u)\|_q du. \end{aligned} \tag{3.1}$$

Proof. Observe that

$$\begin{aligned} &\int_a^b \int_a^b [A(t) - A(s)][B(t) - B(s)] dtds \\ &= \int_a^b \int_a^b (A(t)B(t) - A(s)B(t) - A(t)B(s) + A(s)B(s)) dtds \\ &= (b-a)\int_a^b A(t)B(t)dt - \int_a^b A(s)ds \int_a^b B(t)dt \\ &- \int_a^b A(t)dt \int_a^b B(s)ds + (b-a)\int_a^b A(s)B(s)ds \\ &= 2(b-a)\int_a^b A(t)B(t)dt - 2\int_a^b A(t)dt \int_a^b B(t)dt = 2D(A,B), \end{aligned}$$

which give the Korkine's noncommutative identity for functions with values in Banach algebra $\mathcal{B}(H)$

$$D(A,B) = \frac{1}{2}\int_a^b \int_a^b [A(t) - A(s)][B(t) - B(s)] dtds. \tag{3.2}$$

For Korkine's classical identity for real-valued functions, see [13, p. 242].

If we take the 1-Schatten norm in (3.2), use the integral's properties and employ Hölder's inequality (2.11) for p, $q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, then we get

$$\begin{aligned} \|D(A,B)\|_1 &\le \frac{1}{2}\int_a^b\int_a^b \|[A(t)-A(s)][B(t)-B(s)]\|_1\,dtds \\ &\le \frac{1}{2}\int_a^b\int_a^b \|A(t)-A(s)\|_p\|B(t)-B(s)\|_q\,dtds. \end{aligned} \tag{3.3}$$

Observe that for s, $t\in[a,b]$

$$A(t)-A(s)=\int_s^t A'(u)\,du,\ \ B(t)-B(s)=\int_s^t B'(u)\,du,$$

which implies that

$$\begin{aligned} \|A(t)-A(s)\|_p\|B(t)-B(s)\|_q &= \left\|\int_s^t A'(u)\,du\right\|_p\left\|\int_s^t B'(u)\,du\right\|_q \\ &\le \left|\int_s^t \|A'(u)\|_p\,du\right|\left|\int_s^t \|B'(u)\|_q\,du\right| \\ &\le \sup_{u\in(a,b)}\|A'(u)\|_p\,|t-s|\left|\int_s^t \|B'(u)\|_q\,du\right| \\ &= \sup_{u\in(a,b)}\|A'(u)\|_p\,(t-s)\int_s^t \|B'(u)\|_q\,du, \end{aligned}$$

for all s, $t\in[a,b]$.

By (3.3) we get

$$\|D(A,B)\|_1\le\frac{1}{2}\sup_{u\in(a,b)}\|A'(u)\|_p\int_a^b\int_a^b(t-s)\left(\int_s^t\|B'(u)\|_q\,du\right)dtds \tag{3.4}$$

for p, $q>1$ with $\frac{1}{p}+\frac{1}{q}=1$.

Since

$$(t-s)\int_s^t\|B'(u)\|_q\,du=(t-s)\left(\int_a^t\|B'(u)\|_q\,du-\int_a^s\|B'(u)\|_q\,du\right),$$

hence by Korkine's identity for real valued functions $f(t)=\ell(t)$ and $g(t)=\int_a^t\|B'(u)\|_q\,du$, we have

$$\begin{aligned} &\frac{1}{2}\int_a^b\int_a^b(\ell(t)-\ell(s))\left(\int_a^t\|B'(u)\|_q\,du-\int_a^s\|B'(u)\|_q\,du\right)dtds \\ &=(b-a)\int_a^b\ell(t)\left(\int_a^t\|B'(u)\|_q\,du\right)dt \\ &-\int_a^b\ell(t)\,dt\int_a^b\left(\int_a^t\|B'(u)\|_q\,du\right)dt \\ &=D\left(\ell,\int_a^{\cdot}\|B'(u)\|_q\,du\right). \end{aligned} \tag{3.5}$$

By utilising (3.4) and (3.5), we deduce the first inequality in (3.1).

Observe that $a \le \ell(t) \le b$ and

$$0 \le \int_a^t \|B'(u)\|_q \, du \le \int_a^b \|B'(u)\|_q \, du$$

for all $t \in [a,b]$, then by (1.8) for the functions $f(t) = \ell(t)$ and $g(t) = \int_a^t \|B'(u)\|_q \, du$, $t \in [a,b]$, we get

$$\begin{aligned}\left|D\left(\ell, \int_a^{\cdot} \|B'(u)\|_q \, du\right)\right| &\le \frac{1}{2}(b-a) \int_a^b \|B'(u)\|_q \, du \int_a^b \left|t - \frac{1}{b-a}\int_a^b s ds\right| dt \\ &= \frac{1}{2}(b-a) \int_a^b \|B'(u)\|_q \, du \int_a^b \left|t - \frac{a+b}{2}\right| dt \\ &= \frac{1}{8}(b-a)^3 \int_a^b \|B'(u)\|_q \, du,\end{aligned}$$

which proves the last part of (3.1). □

Remark 1. *If we apply the same inequality (1.8) for the functions* $f(t) = \int_a^t \|B'(u)\|_q \, du$ *and* $g(t) = \ell(t)$, $t \in [a,b]$, *then we get*

$$\begin{aligned}&\left|D\left(\ell, \int_a^{\cdot} \|B'(u)\|_q \, du\right)\right| \\ &\le \frac{1}{2}(b-a)^2 \int_a^b \left|\int_a^t \|B'(u)\|_q \, du - \frac{1}{b-a}\int_a^b \left(\int_a^s \|B'(u)\|_q \, du\right) ds\right| dt.\end{aligned} \tag{3.6}$$

Observe that

$$\begin{aligned}&\int_a^b \left|\int_a^t \|B'(u)\|_q \, du - \frac{1}{b-a}\int_a^b \left(\int_a^s \|B'(u)\|_q \, du\right) ds\right| dt \\ &= \int_a^b \left|\int_a^t \|B'(u)\|_q \, du - \frac{1}{b-a}\left(\left(\int_a^b \|B'(u)\|_q \, du\right) b - \int_a^b \|B'(s)\|_q \, s ds\right)\right| dt \\ &= \int_a^b \left|\int_a^t \|B'(u)\|_q \, du - \frac{1}{b-a}\left(\int_a^b (b-u) \|B'(u)\|_q \, du\right)\right| dt \\ &= \frac{1}{b-a}\int_a^b \left|(b-a)\int_a^t \|B'(u)\|_q \, du - \int_a^b (b-u) \|B'(u)\|_q \, du\right| dt \\ &= \frac{1}{b-a}\int_a^b \left|\int_a^t (u-a) \|B'(u)\|_q \, du - \int_t^b (b-u) \|B'(u)\|_q \, du\right| dt.\end{aligned}$$

Then by (3.1) and (3.6) we obtain

$$\begin{aligned}\|D(A,B)\|_1 &\le \sup_{u \in (a,b)} \|A'(u)\|_p \, D\left(\ell, \int_a^{\cdot} \|B'(u)\|_q \, du\right) \\ &\le \frac{1}{2}(b-a) \sup_{u \in (a,b)} \|A'(u)\|_p \\ &\quad \times \int_a^b \left|\int_a^t (u-a) \|B'(u)\|_q \, du - \int_t^b (b-u) \|B'(u)\|_q \, du\right| dt.\end{aligned} \tag{3.7}$$

Remark 2. *Using (1.3) we have*

$$0 \leq D\left(\ell, \int_a^{\cdot} \|B'(u)\|_q \, du\right) \leq \frac{1}{12} \sup_{t\in(a,b)} \|B'(u)\|_q (b-a)^4 ,$$

and by (3.1) we derive

$$\begin{aligned} \text{(3.8)} \qquad \|D(A,B)\|_1 &\leq \sup_{u\in(a,b)} \|A'(u)\|_p \, D\left(\ell, \int_a^{\cdot} \|B'(u)\|_q \, du\right) \\ &\leq \frac{1}{12} \sup_{u\in(a,b)} \|A'(u)\|_p \sup_{u\in(a,b)} \|B'(u)\|_q (b-a)^4 \end{aligned}$$

provided that $\sup_{u\in(a,b)} \|A'(u)\|_p , \sup_{u\in(a,b)} \|B'(u)\|_q < \infty$.

Using (1.4) we have

$$0 \leq D\left(\ell, \int_a^{\cdot} \|B'(u)\|_q \, du\right) \leq \frac{1}{8} (b-a)^3 \int_a^b \|B'(u)\|_q \, du,$$

and by (3.1) we obtain

$$\begin{aligned} \text{(3.9)} \qquad \|D(A,B)\|_1 &\leq \sup_{u\in(a,b)} \|A'(u)\|_p \, D\left(\ell, \int_a^{\cdot} \|B'(u)\|_q \, du\right) \\ &\leq \frac{1}{8} (b-a)^3 \sup_{u\in(a,b)} \|A'(u)\|_p \int_a^b \|B'(u)\|_q \, du, \end{aligned}$$

provided that $\sup_{u\in(a,b)} \|A'(u)\|_p < \infty$.

From (1.5) we also have

$$0 \leq D\left(\ell, \int_a^{\cdot} \|B'(u)\|_q \, du\right) \leq \frac{1}{\pi^2} (b-a)^{3+1/2} \left(\int_a^b \|B'(u)\|_q^2 \, du\right)^{1/2} ,$$

and by (3.1) we obtain

$$\begin{aligned} \text{(3.10)} \qquad \|D(A,B)\|_1 &\leq \sup_{u\in(a,b)} \|A'(u)\|_p \, D\left(\ell, \int_a^{\cdot} \|B'(u)\|_q \, du\right) \\ &\leq \frac{1}{\pi^2} (b-a)^{3+1/2} \sup_{u\in(a,b)} \|A'(u)\|_p \left(\int_a^b \|B'(u)\|_q^2 \, du\right)^{1/2} . \end{aligned}$$

Corollary 1. *For $p,\ q>1$ with $\frac{1}{p}+\frac{1}{q}=1,$ let $A:[a,b]\to\mathcal{B}_p(H)\,,\ B:[a,b]\to\mathcal{B}_q(H)$ be strongly differentiable on the interval $(a,b)\,.$ Then*

$$
\begin{aligned}
&\left\|D(A,B)\right\|_1 \\
&\leq \frac{1}{2}(b-a)\sup_{u\in(a,b)}\left\|A'(u)\right\|_p\int_a^b(b-t)(t-a)\left\|B'(u)\right\|_q dt \\
&\leq \frac{1}{2}(b-a)^3\sup_{u\in(a,b)}\left\|A'(u)\right\|_p \\
&\quad\times\begin{cases}\frac{1}{4}\int_a^b\left\|B'(u)\right\|_q du, \\[2mm] (b-a)^{1/\beta}\left[B(\beta+1,\beta+1)\right]^{1/\beta}\left(\int_a^b\left\|B'(u)\right\|_q^\alpha du\right)^{1/\alpha}, \\[2mm] \frac{1}{6}(b-a)\sup_{u\in(a,b)}\left\|B'(u)\right\|_q,\end{cases}
\end{aligned}
\tag{3.11}
$$

where $B(\cdot,\cdot)$ is Beta function and $\alpha,\ \beta>1$ with $\frac{1}{\alpha}+\frac{1}{\beta}=1.$

Proof. Observe that, integrating by parts, we have

$$
\begin{aligned}
&\frac{1}{2}\int_a^b(b-t)(t-a)\left\|B'(t)\right\|_q dt \\
&=\frac{1}{2}\int_a^b(b-t)(t-a)\,d\left(\int_a^t\left\|B'(u)\right\|_q du\right) \\
&=\frac{1}{2}\left[(b-t)(t-a)\int_a^t\left\|B'(u)\right\|_q du\Big|_a^b+\int_a^b(2t-a-b)\left(\int_a^t\left\|B'(u)\right\|_q du\right)dt\right] \\
&=\int_a^b\left(t-\frac{a+b}{2}\right)\left(\int_a^t\left\|B'(u)\right\|_q du\right)dt \\
&=\int_a^b t\left(\int_a^t\left\|B'(u)\right\|_q du\right)dt-\frac{a+b}{2}\int_a^b\left(\int_a^t\left\|B'(u)\right\|_q du\right)dt \\
&=\frac{1}{b-a}D\left(\ell,\int_a^{\cdot}\left\|B'(u)\right\|_q du\right),
\end{aligned}
$$

namely

$$
D\left(\ell,\int_a^{\cdot}\left\|B'(u)\right\|_q du\right)=\frac{1}{2}(b-a)\int_a^b(b-t)(t-a)\left\|B'(u)\right\|_q dt. \tag{3.12}
$$

By utilising the first inequality (3.1) we deduce the first inequality in (3.11).

By Hölder's integral inequality we have for α, $\beta > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$,

$$\int_a^b (b-t)(t-a) \|B'(u)\|_q \, dt$$

$$\leq \begin{cases} \sup_{t\in[a,b]} [(b-t)(t-a)] \int_a^b \|B'(u)\|_q \, dt, \\ \\ \left(\int_a^b [(b-t)(t-a)]^\beta \, dt\right)^{1/\beta} \left(\int_a^b \|B'(u)\|_q^\alpha \, dt\right)^{1/\alpha}, \\ \\ \int_a^b (b-t)(t-a) \, dt \sup_{u\in[a,b]} \|B'(u)\|_q , \end{cases}$$

$$= \begin{cases} \frac{1}{4}(b-a)^2 \int_a^b \|B'(t)\|_q \, dt, \\ \\ (b-a)^{2+1/\beta} [B(\beta+1,\beta+1)]^{1/\beta} \left(\int_a^b \|B'(u)\|_q^\alpha \, dt\right)^{1/\alpha}, \\ \\ \frac{1}{6}(b-a)^3 \sup_{t\in[a,b]} \|B'(t)\|_q , \end{cases}$$

which proves the last part of (3.11). □

We have the following result for p-Schatten norm:

Theorem 4. *For $p \geq 1$, let A, $B : [a,b] \to \mathcal{B}_p(H)$ be strongly differentiable functions on the interval (a,b). Then*

$$\|D(A,B)\|_p \leq \sup_{u\in(a,b)} \|A'(u)\|_p \, D\left(\ell, \int_a^{\cdot} \|B'(u)\|_p \, du\right) \tag{3.13}$$

$$\leq \frac{1}{8}(b-a)^2 \sup_{u\in(a,b)} \|A'(u)\|_p \int_a^b \|B'(u)\|_p \, du,$$

provided that $\sup_{u\in(a,b)} \|A'(u)\|_p < \infty$.

Proof. If we take the p-Schatten norm in (3.2), use the integral's properties and employ the inequality (2.8) for $p \geq 1$, then we get

$$\|D(A,B)\|_p \leq \frac{1}{2}\int_a^b \int_a^b \|[A(t)-A(s)][B(t)-B(s)]\|_p \, dtds \tag{3.14}$$

$$\leq \frac{1}{2}\int_a^b \int_a^b \|A(t)-A(s)\|_p \|B(t)-B(s)\|_p \, dtds.$$

Since

$$\|A(t)-A(s)\|_p \|B(t)-B(s)\|_p = \left\|\int_s^t A'(u)\, du\right\|_p \left\|\int_s^t B'(u)\, du\right\|_p$$

$$\leq \left|\int_s^t \|A'(u)\|_p \, du\right| \left|\int_s^t \|B'(u)\|_p \, du\right|$$

$$\leq \sup_{u\in(a,b)} \|A'(u)\|_p |t-s| \left|\int_s^t \|B'(u)\|_p \, du\right|$$

$$= \sup_{u\in(a,b)} \|A'(u)\|_p (t-s) \int_s^t \|B'(u)\|_p \, du,$$

for all $s,\, t\in[a,b]$, hence by (3.14) we get

$$\|D(A,B)\|_p \le \frac{1}{2}\sup_{u\in(a,b)}\|A'(u)\|_p \int_a^b\int_a^b (t-s)\left(\int_s^t \|B'(u)\|_p\,du\right)dtds.$$

Now, by utilising a similar argument to the one in the proof of Theorem 3, we derive the desired result (3.13). □

Remark 3. *If we apply the same inequality (1.8) for the functions* $f(t)=\int_a^t\|B'(u)\|_p\,du$ *and* $g(t)=\ell(t)$, $t\in[a,b]$, *then we get*

$$\left|D\left(\ell,\int_a^{\cdot}\|B'(u)\|_p\,du\right)\right| \le \frac{1}{2}(b-a)^2\int_a^b\left|\int_a^t\|B'(u)\|_p\,du-\frac{1}{b-a}\int_a^b\left(\int_a^s\|B'(u)\|_p\,du\right)ds\right|dt. \tag{3.15}$$

Observe that

$$\int_a^b\left|\int_a^t\|B'(u)\|_p\,du-\frac{1}{b-a}\int_a^b\left(\int_a^s\|B'(u)\|_p\,du\right)ds\right|dt = \frac{1}{b-a}\int_a^b\left|\int_a^t(u-a)\|B'(u)\|_p\,du-\int_t^b(b-u)\|B'(u)\|_p\,du\right|dt.$$

Then by (3.13) and (3.15) we obtain

$$\begin{aligned}\|D(A,B)\|_p &\le \sup_{u\in(a,b)}\|A'(u)\|_p\, D\left(\ell,\int_a^{\cdot}\|B'(u)\|_p\,du\right)\\ &\le \frac{1}{2}(b-a)\sup_{u\in(a,b)}\|A'(u)\|_p\\ &\quad\times\int_a^b\left|\int_a^t(u-a)\|B'(u)\|_p\,du-\int_t^b(b-u)\|B'(u)\|_p\,du\right|dt.\end{aligned} \tag{3.16}$$

Remark 4. *Using (1.3) we have*

$$0\le D\left(\ell,\int_a^{\cdot}\|B'(u)\|_p\,du\right)\le\frac{1}{12}\sup_{t\in(a,b)}\|B'(u)\|_p(b-a)^4,$$

and by (3.13) we derive

$$\begin{aligned}\|D(A,B)\|_p &\le \sup_{u\in(a,b)}\|A'(u)\|_p\, D\left(\ell,\int_a^{\cdot}\|B'(u)\|_p\,du\right)\\ &\le\frac{1}{12}\sup_{u\in(a,b)}\|A'(u)\|_p\sup_{u\in(a,b)}\|B'(u)\|_p(b-a)^4\end{aligned} \tag{3.17}$$

provided that $\sup_{u\in(a,b)}\|A'(u)\|_p,\ \sup_{u\in(a,b)}\|B'(u)\|_p<\infty.$

Using (1.4) we have

$$0\le D\left(\ell,\int_a^{\cdot}\|B'(u)\|_p\,du\right)\le\frac{1}{8}(b-a)^3\int_a^b\|B'(u)\|_p\,du,$$

and by (3.13) we obtain

$$\|D(A,B)\|_p \le \sup_{u\in(a,b)} \|A'(u)\|_p D\left(\ell, \int_a^{\cdot} \|B'(u)\|_p du\right) \le \frac{1}{8}(b-a)^3 \sup_{u\in(a,b)} \|A'(u)\|_p \int_a^b \|B'(u)\|_p du, \tag{3.18}$$

provided that $\sup_{u\in(a,b)} \|A'(u)\|_p < \infty$.

From (1.5) we also have

$$0 \le D\left(\ell, \int_a^{\cdot} \|B'(u)\|_p du\right) \le \frac{1}{\pi^2}(b-a)^{3+1/2}\left(\int_a^b \|B'(u)\|_p^2 du\right)^{1/2},$$

and by (3.13) we obtain

$$\|D(A,B)\|_p \le \sup_{u\in(a,b)} \|A'(u)\|_p D\left(\ell, \int_a^{\cdot} \|B'(u)\|_p du\right) \le \frac{1}{\pi^2}(b-a)^{3+1/2} \sup_{u\in(a,b)} \|A'(u)\|_p \left(\int_a^b \|B'(u)\|_p^2 du\right)^{1/2}. \tag{3.19}$$

Corollary 2. *For $p \ge 1$, let A, $B : [a,b] \to \mathcal{B}_p(H)$ be strongly differentiable on the interval (a,b). Then*

$$\begin{aligned} &\|D(A,B)\|_p \\ &\le \frac{1}{2}(b-a) \sup_{u\in(a,b)} \|A'(u)\|_p \int_a^b (b-t)(t-a)\|B'(u)\|_p dt \\ &\le \frac{1}{2}(b-a)^3 \sup_{u\in(a,b)} \|A'(u)\|_p \\ &\times \begin{cases} \frac{1}{4}\int_a^b \|B'(u)\|_p du, \\ (b-a)^{1/\beta}[B(\beta+1,\beta+1)]^{1/\beta}\left(\int_a^b \|B'(u)\|_p^\alpha du\right)^{1/\alpha}, \\ \frac{1}{6}(b-a)\sup_{u\in(a,b)} \|B'(u)\|_p, \end{cases} \end{aligned} \tag{3.20}$$

where $B(\cdot,\cdot)$ is Beta function and α, $\beta > 1$ with $\frac{1}{\alpha} + \frac{1}{\beta} = 1$.

4. Inequalities for Operator Monotone Functions

A real valued continuous function h on $[0,\infty)$ is said to be operator monotone if $h(A) \ge h(B)$ holds for any $A \ge B \ge 0$.

We have the following representation of operator monotone functions, see for instance [2, p. 144-145]:

Theorem 5. *A function $h : [0,\infty) \to \mathbb{R}$ is operator monotone in $[0,\infty)$ if and only if it has the representation*

$$h(t) = h(0) + bt + \int_0^\infty \frac{t\lambda}{t+\lambda} d\mu(\lambda), \tag{4.1}$$

where $b \geq 0$ *and a positive measure* μ *on* $[0,\infty)$ *such that*

$$\int_0^\infty \frac{\lambda}{1+\lambda} d\mu(\lambda) < \infty. \tag{4.2}$$

We have the following representation result:

Lemma 1. *Let* $h : [0,\infty) \to \mathbb{R}$ *be operator monotone in* $[0,\infty)$. *Assume that* $U \geq 0$, *then for all selfadjoint operators* V *we have*

$$Dh(U)(V) = bV + \int_0^\infty \lambda^2 \left[(\lambda + U)^{-1} V (\lambda + U)^{-1}\right] d\mu(\lambda). \tag{4.3}$$

Proof. From (4.1) we get

$$h(t) = h(0) + bt + \int_0^\infty \left(\lambda - \frac{\lambda^2}{t+\lambda}\right) d\mu(\lambda).$$

Assume that $U \geq 0$, then for all selfadjoint operator V we have, by the representation of h and for t in a small open interval around 0, that

$$\begin{aligned}
&h(U+tV) - h(U) \\
&= btV + \int_0^\infty \left(\lambda - \lambda^2 (U+tV+\lambda)^{-1}\right) d\mu(\lambda) - \int_0^\infty \left(\lambda - \lambda^2 (U+\lambda)^{-1}\right) d\mu(\lambda) \\
&= btV + \int_0^\infty \lambda^2 \left[(\lambda+U)^{-1} - (\lambda+U+tV)^{-1}\right] d\mu(\lambda) \\
&= btV + \int_0^\infty \lambda^2 \left[(\lambda+U)^{-1} (\lambda+U+tV-\lambda-U)(\lambda+U+tV)^{-1}\right] d\mu(\lambda) \\
&= btV + t\int_0^\infty \lambda^2 \left[(\lambda+U)^{-1} V (\lambda+U+tV)^{-1}\right] d\mu(\lambda).
\end{aligned}$$

Dividing by $t \neq 0$, we get

$$\frac{h(U+tV) - h(U)}{t} = bV + \int_0^\infty \lambda^2 \left[(\lambda+U)^{-1} V (\lambda+U+tV)^{-1}\right] d\mu(\lambda)$$

and by taking the limit over $t \to 0$, we get

$$Dh(U)(V) = bV + \int_0^\infty \lambda^2 \left[(\lambda+U)^{-1} V (\lambda+U)^{-1}\right] d\mu(\lambda)$$

for all selfadjoint operator V we have (4.3). □

Theorem 6. *Let* $h : [0,\infty) \to \mathbb{R}$ *be operator monotone in* $[0,\infty)$. *Assume that* $U \geq u > 0$, *then for all selfadjoint operators* $V \in B_p(H)$, $p \geq 1$ *we have*

$$\|Dh(U)(V)\|_p \leq h'(u) \|V\|_p. \tag{4.4}$$

Proof. From (4.3) and using (2.10) we get

$$\begin{aligned}
\|Dh(U)(V) - bV\|_p &\leq \int_0^\infty \lambda^2 \left\|(\lambda+U)^{-1} V (\lambda+U)^{-1}\right\|_p d\mu(\lambda) \\
&\leq \|V\|_p \int_0^\infty \lambda^2 \left\|(\lambda+U)^{-1}\right\|^2 d\mu(\lambda).
\end{aligned} \tag{4.5}$$

Observe that $\lambda + U \geq \lambda + u > 0$ for $\lambda \in [0,\infty)$. Then $0 < (\lambda+U)^{-1} \leq (\lambda+u)^{-1}$, which implies that $\left\|(\lambda+U)^{-1}\right\| \leq (\lambda+u)^{-1}$, namely $\left\|(\lambda+U)^{-1}\right\|^2 \leq (\lambda+u)^{-2}$ for $\lambda \in [0,\infty)$.

Therefore by (4.5) we get

$$\left\| Dh\left(U\right)\left(V\right)-bV\right\|_{p}\leq\left\| V\right\|_{p}\int_{0}^{\infty}\lambda^{2}\left(\lambda+u\right)^{-2}d\mu\left(\lambda\right). \tag{4.6}$$

If we take the derivative over t in (4.1) then we have

$$h'\left(t\right)=b+\int_{0}^{\infty}\frac{\lambda\left(t+\lambda\right)-\lambda t}{\left(t+\lambda\right)^{2}}d\mu\left(\lambda\right)=b+\int_{0}^{\infty}\frac{\lambda^{2}}{\left(t+\lambda\right)^{2}}d\mu\left(\lambda\right) \tag{4.7}$$

for $t>0$.

From (4.7) we get

$$\int_{0}^{\infty}\lambda^{2}\left(\lambda+u\right)^{-2}d\mu\left(\lambda\right)=h'\left(u\right)-b,$$

and by (4.6) we derive

$$\left\| Dh\left(U\right)\left(V\right)-bV\right\|_{p}\leq\left\| V\right\|_{p}h'\left(u\right)-b\left\| V\right\|_{p}.$$

Finally, by the triangle inequality and by the fact that $b\geq 0$, we obtain that

$$\left\| Dh\left(U\right)\left(V\right)\right\|_{p}-b\left\| V\right\|_{p}\leq\left\| Dh\left(U\right)\left(V\right)-bV\right\|_{p},$$

which proves the desired result (4.4). □

For a continuous function h on $(0,\infty)$ and $C,\ E>0$ we consider the auxiliary function $h_{C,E}:[0,1]\to\mathbb{R}$ defined by

$$h_{C,E}\left(t\right):=h\left(\left(1-t\right)C+tE\right),\ t\in\left[0,1\right].$$

We have the following representations of the derivatives:

Lemma 2. *Assume that the operator function generated by h is Fréchet differentiable in each $C\geq 0$, then for $E\geq 0$ we have that $h_{C,E}$ is differentiable on $[0,1]$ and*

$$h'_{C,E}\left(t\right)=D\left(h\right)\left(\left(1-t\right)C+tE\right)\left(E-C\right) \tag{4.8}$$

for $t\in[0,1]$, where in 0 and 1 the derivatives are the right and left derivatives.

Proof. We prove only for the interior points $t\in(0,1)$. Let h be in a small interval around 0 such that $t+h\in(0,1)$. Then for $h\neq 0$,

$$\begin{aligned}&\frac{h_{C,E}\left(t+h\right)-h\left(t\right)}{h}\\&=\frac{h\left(\left(1-\left(t+h\right)\right)C+\left(t+h\right)E\right)-h\left(\left(1-t\right)C+tE\right)}{h}\\&=\frac{h\left(\left(1-t\right)C+tE+h\left(E-C\right)\right)-h\left(\left(1-t\right)C+tE\right)}{h}\end{aligned}$$

and by taking the limit over $h\to 0$, we get

$$\begin{aligned}h'_{C,E}\left(t\right)&=\lim_{h\to 0}\frac{h_{C,E}\left(t+h\right)-h\left(t\right)}{h}\\&=\lim_{h\to 0}\left[\frac{h\left(\left(1-t\right)C+tE+h\left(E-C\right)\right)-h\left(\left(1-t\right)C+tE\right)}{h}\right]\\&=D\left(h\right)\left(\left(1-t\right)C+tE\right)\left(E-C\right),\end{aligned}$$

which proves (4.8). □

Corollary 3. *Let $h:[0,\infty)\to\mathbb{R}$ be operator monotone in $[0,\infty)$. Then for all $C\geq c>0,\ E\geq d>0$ with $C,E\in B_p(H)$, $p\geq 1$ we have*

$$\begin{aligned}\left\|h'_{C,E}(t)\right\|_p &= \|D(h)((1-t)C+tE)(E-C)\|_p \\ &\leq h'((1-t)c+td)\|E-C\|_p\end{aligned} \tag{4.9}$$

for all $t\in[0,1]$.

The proof follows by Theorem 6 and Lemma 2.

For the operator monotone functions $h,\ k:[0,\infty)\to\mathbb{R}$ and two operators C, $E\geq 0$ we consider the *Čebyšev functional*

$$\begin{aligned}D(h,k,C,E) &:= \int_0^1 h((1-t)C+tE)\,k((1-t)C+tE)\,dt \\ &\quad -\int_0^1 h((1-t)C+tE)\,dt\int_0^1 k((1-t)C+tE)\,dt.\end{aligned}$$

For $p,\ q>1$ with $\frac{1}{p}+\frac{1}{q}=1$ assume that $C\geq c>0,\ E\geq d>0$ with C, $E\in B_p(H)\cap B_q(H)$, then by (3.1) and (4.9) we derive

$$\begin{aligned}&\|D(h,k,C,E)\|_1 \\ &\leq \frac{1}{8}\|E-C\|_p\|E-C\|_q\sup_{t\in(0,1)}h'((1-t)c+td)\int_0^1 k'((1-t)c+td)\,dt.\end{aligned} \tag{4.10}$$

Since

$$\int_0^1 k'((1-t)c+td)\,dt=\begin{cases}\frac{h(d)-h(c)}{d-c} & \text{if } d\neq c\\ \\ h'(c) & \text{if } d=c,\end{cases}$$

hence by (4.10) we get

$$\begin{aligned}\|D(h,k,C,E)\|_1 &\leq \frac{1}{8}\|E-C\|_p\|E-C\|_q \\ &\quad\times\sup_{t\in(0,1)}h'((1-t)c+td)\times\begin{cases}\frac{h(d)-h(c)}{d-c} & \text{if } d\neq c\\ \\ h'(c) & \text{if } d=c.\end{cases}\end{aligned} \tag{4.11}$$

For $p=q=2$ we derive

$$\begin{aligned}\|D(h,k,C,E)\|_1 &\leq \frac{1}{8}\|E-C\|_2^2 \\ &\quad\times\sup_{t\in(0,1)}h'((1-t)c+td)\times\begin{cases}\frac{h(d)-h(c)}{d-c} & \text{if } d\neq c\\ \\ h'(c) & \text{if } d=c.\end{cases}\end{aligned} \tag{4.12}$$

If $C\geq c>0,\ E\geq d>0$ with $C,\ E\in B_p(H)$ for $p\geq 1$, then by (3.13) and (4.9) we derive

$$\begin{aligned}\|D(h,k,C,E)\|_p &\leq \frac{1}{8}\|E-C\|_p^2 \\ &\quad\times\sup_{t\in(0,1)}h'((1-t)c+td)\times\begin{cases}\frac{h(d)-h(c)}{d-c} & \text{if } d\neq c\\ \\ h'(c) & \text{if } d=c.\end{cases}\end{aligned} \tag{4.13}$$

Consider the operator monotone functions $h,\, k : [0,\infty) \to \mathbb{R}$, $h(t) = t^m$, $k(t) = t^n$ with $m,\, n \in (0,1)$, then by (4.11) we derive

$$\begin{aligned}(4.14)\qquad &\left\| \int_0^1 ((1-t)C + tE)^{m+n}\,dt \right. \\ &\left. - \int_0^1 ((1-t)C + tE)^m\,dt \int_0^1 ((1-t)C + tE)^n\,dt \right\|_1 \\ &\le \frac{1}{8}\frac{m\|E-C\|_p \|E-C\|_q}{\min\{c^{1-m}, d^{1-m}\}} \times \begin{cases} \frac{d^n - c^n}{d-c} \text{ if } d \ne c \\ \\ \frac{n}{c^{1-n}} \text{ if } d = c, \end{cases}\end{aligned}$$

for $p,\, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ and $C \ge c > 0$, $E \ge d > 0$ with $C,\, E \in B_p(H) \cap B_q(H)$.

From (4.13) we derive for $m,\, n \in (0,1)$ that

$$\begin{aligned}(4.15)\qquad &\left\| \int_0^1 ((1-t)C + tE)^{m+n}\,dt \right. \\ &\left. - \int_0^1 ((1-t)C + tE)^m\,dt \int_0^1 ((1-t)C + tE)^n\,dt \right\|_p \\ &\le \frac{1}{8}\frac{m\|E-C\|_p^2}{\min\{c^{1-m}, d^{1-m}\}} \times \begin{cases} \frac{d^n - c^n}{d-c} \text{ if } d \ne c \\ \\ \frac{n}{c^{1-n}} \text{ if } d = c, \end{cases}\end{aligned}$$

for $p \ge 1$ and $C \ge c > 0$, $E \ge d > 0$ with $C,\, E \in B_p(H)$.

Similar inequalities may be stated by utilising the other results from the previous section, however the details are not presented here.

References

[1] R. Bellman, Some inequalities for positive definite matrices, in: E.F. Beckenbach (Ed.), *General Inequalities 2*, Proceedings of the 2nd International Conference on General Inequalities, Birkhäuser, Basel, 1980, pp. 89–90.

[2] R. Bhatia, *Matrix Analysis*. Graduate Texts in Mathematics, 169. Springer-Verlag, New York, 1997. xii+347 pp. ISB

[3] N. S. Barnett, C. Buşe, P. Cerone and S. S. Dragomir, Ostrowski's inequality for vector-valued functions and applications, *Computers and Mathematics with Applications* **44** (2002), 559-572.

[4] P. Cerone and S. S. Dragomir, New bounds for the Čebyšev functional, *Appl. Math. Lett.*, **18** (2005), 603-611.

[5] P. Cerone and S. S. Dragomir, A refinement of the Grüss inequality and applications, *Tamkang J. Math.* **38**(2007), No. 1, 37-49. Preprint available at *RGMIA Res. Rep. Coll.*, **5**(2) (2002), Art. 14. [ONLINE `http://rgmia.vu.edu.au/v5n2.htm`

[6] D. Chang, A matrix trace inequality for products of Hermitian matrices, *J. Math. Anal. Appl.* **237** (1999) 721–725.

[7] I. D. Coop, On matrix trace inequalities and related topics for products of Hermitian matrix, *J. Math. Anal. Appl.* **188** (1994) 999–1001.

[8] P. L. Chebyshev, Sur les expressions approximatives des intègrals dèfinis par les outres prises entre les même limites, *Proc. Math. Soc. Charkov,* **2** (1882), 93-98.

[9] X.-L. Cheng and J. Sun, Note on the perturbed trapezoid inequality, *J. Inequal. Pure Appl. Math.,* **3**(2) (2002), Art. 29. [ONLINE`http://jipam.vu.edu.au/article.php?sid=181`

[10] S. S. Dragomir, New Grüss' type inequalities for functions of bounded variation and applications. *Appl. Math. Lett.* **25** (2012), no. 10, 1475–1479.

[11] G. Grüss, Über das maximum des absoluten Betrages von $\frac{1}{b-a}\int_a^b f(x)\,g(x)\,dx - \frac{1}{(b-a)^2}\int_a^b f(x)\,dx \cdot \int_a^b g(x)\,dx$, *Math. Z.,* **39** (1934), 215-226.

[12] A. Lupaş, The best constant in an integral inequality, *Mathematica* (Cluj), **15** (**38**) (1973), No. 2, 219-222.
[13] D. S. Mitrinović, J. E. Pečarić and A. M. Fink, *Classical and New Inequalities in Analysis,* Kluwer Academic Publishers, Dordrecht, 1993.
[14] H. Neudecker, A matrix trace inequality, *J. Math. Anal. Appl.* **166** (1992) 302–303.
[15] A. M. Ostrowski, On an integral inequality, *Aequationes Math.,* **4** (1970), 358-373.
[16] B. Simon, *Trace Ideals and Their Applications,* Cambridge University Press, Cambridge, 1979.
[17] V. A. Zagrebvov, *Gibbs Semigroups,* Operator Theory: Advances and Applications, Volume 273, Birkhäuser, 2019

[1]Mathematics, College of Engineering & Science, Victoria University, PO Box 14428, Melbourne City, MC 8001, Australia.
E-mail address: sever.dragomir@vu.edu.au
URL: http://rgmia.org/dragomir

[2]DST-NRF Centre of Excellence in the Mathematical, and Statistical Sciences, School of Computer Science, & Applied Mathematics, University of the Witwatersrand,, Johannesburg, South Africa.