p-SCHATTEN NORMS HÖLDER TYPE INEQUALITIES FOR ČEBYŠEV'S FUNCTIONAL

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Abstract. An operator $A\in\mathcal{B}(H)$ is said to belong to the *von Neumann-Schatten class* $\mathcal{B}_p(H)$, $1\leq p<\infty$ if the *p-Schatten norm* is finite

$$\|A\|_p:=[\operatorname{tr}(|A|^p)]^{1/p}<\infty.$$

For two continuous functions $A,\ B:[a,b]\to\mathcal{B}(H)$ we define the *noncommutative Čebyšev functional*

$$D(A,B):=(b-a)\int_a^b A(t)B(t)\,dt-\int_a^b A(t)\,dt\int_a^b B(t)\,dt.$$

In this paper we show among others that, for $p,\ q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, let $A:[a,b]\to\mathcal{B}_p(H)$, $B:[a,b]\to\mathcal{B}_q(H)$ be strongly differentiable functions on the interval (a,b). If $r,\ s>1$ with $\frac{1}{r}+\frac{1}{s}=1$, then

$$\begin{aligned}\|D(A,B)\|_1&\leq\frac{1}{2}(b-a)\left[\int_a^b(b-t)(t-a)\|A'(t)\|_p^r\,dt\right]^{1/r}\\&\quad\times\left[\int_a^b(b-t)(t-a)\|B'(t)\|_q^s\,dt\right]^{1/s}\\&\leq\frac{1}{8}(b-a)^3\left(\int_a^b\|A'(t)\|_p^r\,dt\right)^{1/r}\left(\int_a^b\|B'(t)\|_q^s\,dt\right)^{1/s}.\end{aligned}$$

Some examples of interest for the operator monotone functions are also given.

1. Introduction

For two Lebesgue integrable functions $f,\ g:[a,b]\to\mathbb{C}$, in order to compare the integral mean of the product with the product of the integral means, we consider the *Čebyšev functional* defined by

$$D(f,g):=(b-a)\int_a^b f(t)g(t)\,dt-\int_a^b f(t)\,dt\int_a^b g(t)\,dt.$$

In 1934, G. Grüss [11] showed that

$$|D(f,g)|\leq\frac{1}{4}(b-a)^2(M-m)(N-n),\tag{1.1}$$

provided $m,\ M,\ n,\ N$ are real numbers with the property that

$$-\infty<m\leq f\leq M<\infty,\quad -\infty<n\leq g\leq N<\infty\quad\text{a.e. on}\quad[a,b].\tag{1.2}$$

The constant $\frac{1}{4}$ is best possible in (1.1) in the sense that it cannot be replaced by a smaller one.

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Another lesser known inequality for $D(f,g)$ was derived in 1882 by Čebyšev [8] under the assumption that f', g' exist and are continuous on $[a,b]$, and is given by

$$|D(f,g)| \leq \frac{1}{12}\|f'\|_\infty \|g'\|_\infty (b-a)^4, \tag{1.3}$$

where $\|f'\|_\infty := \sup_{t\in[a,b]} |f'(t)| < \infty$.

The constant $\frac{1}{12}$ cannot be improved in general in (1.3).

Čebyšev's inequality (1.3) also holds if f, $g : [a,b] \to \mathbb{R}$ are assumed to be absolutely continuous and f', $g' \in L_\infty[a,b]$.

In 1970, A. M. Ostrowski [15] proved, amongst others, the following result that is in a sense a combination of the Čebyšev and Grüss results:

$$|D(f,g)| \leq \frac{1}{8}(b-a)^3 (M-m)\|g'\|_\infty, \tag{1.4}$$

provided f is Lebesgue integrable on $[a,b]$ and satisfying (1.2) while $g : [a,b] \to \mathbb{R}$ is absolutely continuous and $g' \in L_\infty[a,b]$. Here the constant $\frac{1}{8}$ is also sharp.

In 1973, A. Lupaş [12] (see also [13, p. 210]) obtained the following result as well:

$$|D(f,g)| \leq \frac{1}{\pi^2}\|f'\|_2 \|g'\|_2 (b-a)^3, \tag{1.5}$$

provided f, g are absolutely continuous and f', $g' \in L_2[a,b]$.

Here the constant $\frac{1}{\pi^2}$ is the best possible as well.

In [4], P. Cerone and S. S. Dragomir proved the following inequalities:

$$\begin{aligned} &|D(f,g)| \\ &\leq (b-a) \times \begin{cases} \inf\limits_{\gamma\in\mathbb{R}} \|g-\gamma\|_\infty \int_a^b \left| f(t) - \frac{1}{b-a}\int_a^b f(s)\,ds \right| dt, \\ \\ \inf\limits_{\gamma\in\mathbb{R}} \|g-\gamma\|_q \left(\int_a^b \left| f(t) - \frac{1}{b-a}\int_a^b f(s)\,ds \right|^p dt \right)^{\frac{1}{p}} \\ \qquad\qquad\qquad \text{where } p > 1,\ 1/p + 1/q = 1. \end{cases} \end{aligned} \tag{1.6}$$

For $\gamma = 0$, we get from the first inequality in (1.6)

$$|D(f,g)| \leq (b-a)\|g\|_\infty \int_a^b \left| f(t) - \frac{1}{b-a}\int_a^b f(s)\,ds \right| dt \tag{1.7}$$

for which the constant 1 cannot be replaced by a smaller constant.

If $m \leq g \leq M$ for a.e. $x \in [a,b]$, then $\left\| g - \frac{m+M}{2} \right\|_\infty \leq \frac{1}{2}(M-m)$ and by the first inequality in (1.6) we can deduce the following result obtained by Cheng and Sun [9]

$$|D(f,g)| \leq \frac{1}{2}(b-a)(M-m)\int_a^b \left| f(t) - \frac{1}{b-a}\int_a^b f(s)\,ds \right| dt. \tag{1.8}$$

The constant $\frac{1}{2}$ is best in (1.8) as shown by Cerone and Dragomir in [5].

The following result holds [10].

Theorem 1. *Let $f : [a,b] \to \mathbb{C}$ be of bounded variation on $[a,b]$ and $g : [a,b] \to \mathbb{C}$ a Lebesgue integrable function on $[a,b]$. Then*

$$|D(f,g)| \leq \frac{1}{2}(b-a)\bigvee_a^b(f)\int_a^b \left| g(t) - \frac{1}{b-a}\int_a^b g(s)\,ds \right| dt, \tag{1.9}$$

where $\bigvee_a^b (f)$ *denotes the total variation of* f *on the interval* $[a,b]$. *The constant* $\frac{1}{2}$ *is best possible in (1.9).*

In order to extend some of the above results for the p-Schatten norm of bounded linear operators in complex Hilbert spaces we need the following preparations.

Let $(H; \langle .,. \rangle)$ be a complex Hilbert space and $\mathcal{B}(H)$ the Banach algebra of all bounded linear operators on H. If $\{e_i\}_{i\in I}$ an orthonormal basis of H, we say that $A \in \mathcal{B}(H)$ is of *trace class* if

$$\|A\|_1 := \sum_{i\in I} \langle |A| e_i, e_i \rangle < \infty. \tag{1.10}$$

The definition of $\|A\|_1$ does not depend on the choice of the orthonormal basis $\{e_i\}_{i\in I}$. We denote by $\mathcal{B}_1(H)$ the set of trace class operators in $\mathcal{B}(H)$.

We define the *trace* of a trace class operator $A \in \mathcal{B}_1(H)$ to be

$$\operatorname{tr}(A) := \sum_{i\in I} \langle Ae_i, e_i \rangle , \tag{1.11}$$

where $\{e_i\}_{i\in I}$ an orthonormal basis of H. Note that this coincides with the usual definition of the trace if H is finite-dimensional. We observe that the series (1.11) converges absolutely and it is independent from the choice of basis.

The following result collects some properties of the trace:

Theorem 2. *We have:*

(i) If $A \in \mathcal{B}_1(H)$ *then* $A^* \in \mathcal{B}_1(H)$ *and*

$$\operatorname{tr}(A^*) = \overline{\operatorname{tr}(A)}; \tag{1.12}$$

(ii) If $A \in \mathcal{B}_1(H)$ *and* $T \in \mathcal{B}(H)$, *then* $AT, TA \in \mathcal{B}_1(H)$ *and*

$$\operatorname{tr}(AT) = \operatorname{tr}(TA) \text{ and } |\operatorname{tr}(AT)| \le \|A\|_1 \|T\| ; \tag{1.13}$$

(iii) $\operatorname{tr}(\cdot)$ *is a bounded linear functional on* $\mathcal{B}_1(H)$ *with* $\|\operatorname{tr}\| = 1$;

(iv) If $A, B \in \mathcal{B}_2(H)$ *then* $AB, BA \in \mathcal{B}_1(H)$ *and* $\operatorname{tr}(AB) = \operatorname{tr}(BA)$;

(v) $\mathcal{B}_{fin}(H)$ *is a dense subspace of* $\mathcal{B}_1(H)$.

An operator $A \in \mathcal{B}(H)$ is said to belong to the *von Neumann-Schatten class* $\mathcal{B}_p(H)$, $1 \le p < \infty$ if the *p-Schatten norm* is finite [17, p. 60-64]

$$\|A\|_p := [\operatorname{tr}(|A|^p)]^{1/p} < \infty.$$

For $1 < p < q < \infty$ we have that

$$\mathcal{B}_1(H) \subset \mathcal{B}_p(H) \subset \mathcal{B}_q(H) \subset \mathcal{B}(H) \tag{1.14}$$

and

$$\|A\|_1 \ge \|A\|_p \ge \|A\|_q \ge \|A\| . \tag{1.15}$$

For $p \ge 1$ the functional $\|\cdot\|_p$ is a *norm* on the $*$-ideal $\mathcal{B}_p(H)$ and $\left(\mathcal{B}_p(H), \|\cdot\|_p\right)$ is a Banach space.

Also, see for instance [17, p. 60-64],

$$\|A\|_p = \|A^*\|_p , \ A \in \mathcal{B}_p(H) \tag{1.16}$$

$$\|AB\|_p \le \|A\|_p \|B\|_p , \ A, B \in \mathcal{B}_p(H) \tag{1.17}$$

and

$$\|AB\|_p \le \|A\|_p \|B\|, \quad \|BA\|_p \le \|B\| \|A\|_p, \; A \in \mathcal{B}_p(H), \; B \in \mathcal{B}(H). \tag{1.18}$$

This implies that

$$\|CAB\|_p \le \|C\| \|A\|_p \|B\|, \; A \in \mathcal{B}_p(H), \; B, \; C \in \mathcal{B}(H). \tag{1.19}$$

In terms of *p-Schatten norm* we have the *Hölder inequality* for $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$

$$(|\operatorname{tr}(AB)| \le) \|AB\|_1 \le \|A\|_p \|B\|_q, \; A \in \mathcal{B}_p(H), \; B \in \mathcal{B}_q(H). \tag{1.20}$$

For the theory of trace functionals and their applications the reader is referred to [16] and [17].

For some classical trace inequalities see [6], [7], and [14], which are continuations of the work of Bellman [1].

2. p-Schatten Norm Inequalities

For two continuous functions $A, B : [a, b] \to \mathcal{B}(H)$ we define the *noncommutative Čebyšev functional*

$$D(A, B) := (b - a) \int_a^b A(t) B(t) dt - \int_a^b A(t) dt \int_a^b B(t) dt.$$

We have the following result of interest:

Theorem 3. *For $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, let $A : [a, b] \to \mathcal{B}_p(H)$, $B : [a, b] \to \mathcal{B}_q(H)$ be strongly differentiable functions on the interval (a, b). If $r, s > 1$ with $\frac{1}{r} + \frac{1}{s} = 1$, then*

$$\begin{aligned} \|D(A, B)\|_1 &\le \left[D\left(\ell, \int_a^{\cdot} \|A'(u)\|_p^r du \right) \right]^{1/r} \left[D\left(\ell, \int_a^{\cdot} \|B'(u)\|_q^s du \right) \right]^{1/s} \\ &= \frac{1}{2}(b - a) \left[\int_a^b (b - t)(t - a) \|A'(t)\|_p^r dt \right]^{1/r} \\ &\quad \times \left[\int_a^b (b - t)(t - a) \|B'(t)\|_q^s dt \right]^{1/s} \\ &\le \frac{1}{8}(b - a)^3 \left(\int_a^b \|A'(t)\|_p^r dt \right)^{1/r} \left(\int_a^b \|B'(t)\|_q^s dt \right)^{1/s}. \end{aligned} \tag{2.1}$$

In particular, we have for $r = s = 2$ that

$$\|D(A,B)\|_1 \le \left[D\left(\ell, \int_a^{\cdot} \|A'(u)\|_p^2 du\right)\right]^{1/2} \left[D\left(\ell, \int_a^{\cdot} \|B'(u)\|_q^2 du\right)\right]^{1/2} \tag{2.2}$$
$$= \frac{1}{2}(b-a)\left[\int_a^b (b-t)(t-a)\|A'(t)\|_p^2 dt\right]^{1/2}$$
$$\times \left[\int_a^b (b-t)(t-a)\|B'(t)\|_q^2 dt\right]^{1/2}$$
$$\le \frac{1}{8}(b-a)^3 \left(\int_a^b \|A'(t)\|_p^2 dt\right)^{1/2} \left(\int_a^b \|B'(t)\|_q^2 dt\right)^{1/2}.$$

Proof. Observe that

$$\int_a^b \int_a^b [A(t) - A(s)][B(t) - B(s)] dtds$$
$$= \int_a^b \int_a^b (A(t)B(t) - A(s)B(t) - A(t)B(s) + A(s)B(s)) dtds$$
$$= (b-a)\int_a^b A(t)B(t)dt - \int_a^b A(s)ds \int_a^b B(t)dt$$
$$- \int_a^b A(t)dt \int_a^b B(s)ds + (b-a)\int_a^b A(s)B(s)ds$$
$$= 2(b-a)\int_a^b A(t)B(t)dt - 2\int_a^b A(t)dt \int_a^b B(t)dt = 2D(A,B),$$

which give the Korkine's noncommutative identity for functions with values in Banach algebra $\mathcal{B}(H)$

$$D(A,B) = \frac{1}{2}\int_a^b \int_a^b [A(t) - A(s)][B(t) - B(s)] dtds. \tag{2.3}$$

For Korkine's classical identity for real-valued functions, see [13, p. 242].

If we take the 1-Schatten norm in (2.3), use the integral's properties and employ Hölder's inequality (1.20) for p, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, then we get

$$\|D(A,B)\|_1 \le \frac{1}{2}\int_a^b \int_a^b \|[A(t) - A(s)][B(t) - B(s)]\|_1 dtds \tag{2.4}$$
$$\le \frac{1}{2}\int_a^b \int_a^b \|A(t) - A(s)\|_p \|B(t) - B(s)\|_q dtds.$$

Observe that for s, $t \in [a,b]$

$$A(t) - A(s) = \int_s^t A'(u)du,\ B(t) - B(s) = \int_s^t B'(u)du,$$

which implies that

$$\begin{aligned}\|A(t)-A(s)\|_p\|B(t)-B(s)\|_q &= \left\|\int_s^t A'(u)\,du\right\|_p\left\|\int_s^t B'(u)\,du\right\|_q\\ &\le \left|\int_s^t \|A'(u)\|_p\,du\right|\left|\int_s^t \|B'(u)\|_q\,du\right|\end{aligned}$$

for all $s,\ t\in[a,b]$.

Using Hölder's inequality for $r,\ s>1$ with $\frac{1}{r}+\frac{1}{s}=1$, then

$$\begin{aligned}&\left|\int_s^t \|A'(u)\|_p\,du\right|\left|\int_s^t \|B'(u)\|_q\,du\right|\\ &\le |t-s|^{1/s}\left|\int_s^t \|A'(u)\|_p^r\,du\right|^{1/r}|t-s|^{1/r}\left|\int_s^t \|B'(u)\|_q^s\,du\right|^{1/s}\\ &= |t-s|\left|\int_s^t \|A'(u)\|_p^r\,du\right|^{1/r}\left|\int_s^t \|B'(u)\|_q^s\,du\right|^{1/s}.\end{aligned}$$

By the weighted Hölder's inequality for double integral, we also have

$$\begin{aligned}&\int_a^b\int_a^b \|A(t)-A(s)\|_p\|B(t)-B(s)\|_q\,dtds \qquad (2.5)\\ &\le \int_a^b\int_a^b |t-s|\left|\int_s^t \|A'(u)\|_p^r\,du\right|^{1/r}\left|\int_s^t \|B'(u)\|_q^s\,du\right|^{1/s}dtds\\ &\le \left(\int_a^b\int_a^b |t-s|\left(\left|\int_s^t \|A'(u)\|_p^r\,du\right|^{1/r}\right)^r dtds\right)^{1/r}\\ &\times\left(\int_a^b\int_a^b |t-s|\left(\left|\int_s^t \|B'(u)\|_q^s\,du\right|^{1/s}\right)^s dtds\right)^{1/s}\\ &= \left(\int_a^b\int_a^b |t-s|\left|\int_s^t \|A'(u)\|_p^r\,du\right|dtds\right)^{1/r}\\ &\times\left(\int_a^b\int_a^b |t-s|\left|\int_s^t \|B'(u)\|_q^s\,du\right|dtds\right)^{1/s}.\end{aligned}$$

Observe that

$$\begin{aligned}&\int_a^b\int_a^b |t-s|\left|\int_s^t \|A'(u)\|_p^r\,du\right|dtds\\ &= \int_a^b\int_a^b (t-s)\left(\int_s^t \|A'(u)\|_p^r\,du\right)dtds\\ &= \int_a^b\int_a^b (t-s)\left(\int_a^t \|A'(u)\|_p^r\,du-\int_a^s \|A'(u)\|_p^r\,du\right)dtds\\ &= 2D\left(\ell,\int_a^{\cdot} \|A'(u)\|_p^r\,du\right)\end{aligned}$$

and

$$\int_a^b \int_a^b |t-s| \left| \int_s^t \|B'(u)\|_q^s du \right| dtds = 2D\left(\ell, \int_a^{\cdot} \|B'(u)\|_q^s du\right).$$

Therefore, by (2.4)

$$\begin{aligned}
\|D(A,B)\|_1 &\le \frac{1}{2}\int_a^b \int_a^b \|A(t)-A(s)\|_p \|B(t)-B(s)\|_q dtds \\
&\le \frac{1}{2}\left[2D\left(\ell, \int_a^{\cdot} \|A'(u)\|_p^r du\right)\right]^{1/r} \left[2D\left(\ell, \int_a^{\cdot} \|B'(u)\|_q^s du\right)\right]^{1/s} \\
&= \left[D\left(\ell, \int_a^{\cdot} \|A'(u)\|_p^r du\right)\right]^{1/r} \left[D\left(\ell, \int_a^{\cdot} \|B'(u)\|_q^s du\right)\right]^{1/s}.
\end{aligned}$$

Since

$$(t-s)\left(\int_s^t \|A'(u)\|_p^r du\right) = (t-s)\left(\int_a^t \|A'(u)\|_p^r du - \int_a^s \|A'(u)\|_p^r du\right),$$

hence by Korkine's identity for real valued functions $f(t) = \ell(t)$ and $g(t) = \int_a^t \|A'(u)\|_p^r du$, we have

$$\begin{aligned}
\frac{1}{2}\int_a^b \int_a^b (t-s)\left(\int_s^t \|A'(u)\|_p^r du\right) &= (b-a)\int_a^b \ell(t)\left(\int_a^t \|A'(u)\|_p^r du\right) dt \\
&\quad - \int_a^b \ell(t)\, dt \int_a^b \left(\int_a^t \|A'(u)\|_p^r du\right) dt \\
&= D\left(\ell, \int_a^{\cdot} \|A'(u)\|_p^r du\right).
\end{aligned} \tag{2.6}$$

From (2.6) we have

$$D\left(\ell, \int_a^{\cdot} \|A'(u)\|_p^r du\right) = \frac{1}{2}(b-a)\int_a^b (b-t)(t-a)\|A'(t)\|_p^r dt$$

and

$$D\left(\ell, \int_a^{\cdot} \|B'(u)\|_q^s du\right) = \frac{1}{2}(b-a)\int_a^b (b-t)(t-a)\|B'(t)\|_q^s dt.$$

Therefore

$$\begin{aligned}
&\left[D\left(\ell, \int_a^{\cdot} \|A'(u)\|_p^r du\right)\right]^{1/r} \left[D\left(\ell, \int_a^{\cdot} \|B'(u)\|_q^s du\right)\right]^{1/s} \\
&= \left[\frac{1}{2}(b-a)\int_a^b (b-t)(t-a)\|A'(t)\|_p^r dt\right]^{1/r} \\
&\times \left[\frac{1}{2}(b-a)\int_a^b (b-t)(t-a)\|B'(t)\|_q^s dt\right]^{1/s} \\
&= \frac{1}{2}(b-a)\left[\int_a^b (b-t)(t-a)\|A'(t)\|_p^r dt\right]^{1/r} \\
&\times \left[\int_a^b (b-t)(t-a)\|B'(t)\|_q^s dt\right]^{1/s}
\end{aligned}$$

and the first part of the theorem is proved.

Now, observe, by the *A-G-inequality*

$$\alpha\beta \le \frac{1}{4}(\alpha+\beta)^2,$$

we have

$$\int_a^b (b-t)(t-a)\|A'(t)\|_p^r\,dt \le \frac{1}{4}(b-a)^2\int_a^b \|A'(t)\|_p^r\,dt$$

and

$$\int_a^b (b-t)(t-a)\|B'(t)\|_q^s\,dt \le \frac{1}{4}(b-a)^2\int_a^b \|B'(t)\|_q^s\,dt,$$

which gives the last part of (2.1). □

Corollary 1. *Let A, $B:[a,b]\to\mathcal{B}_p(H)$ be strongly differentiable functions on the interval (a,b). If r, $s>1$ with $\frac{1}{r}+\frac{1}{s}=1$, then*

$$\|D(A,B)\|_1 \le \frac{1}{8}(b-a)^3\left(\int_a^b \|A'(t)\|_2^r\,dt\right)^{1/r}\left(\int_a^b \|B'(t)\|_2^s\,dt\right)^{1/s}. \tag{2.7}$$

In particular, we have for $r=s=2$ that

$$\|D(A,B)\|_1 \le \frac{1}{8}(b-a)^3\left(\int_a^b \|A'(t)\|_2^2\,dt\right)^{1/2}\left(\int_a^b \|B'(t)\|_2^2\,dt\right)^{1/2}. \tag{2.8}$$

Remark 1. *Assume that α, $\beta>1$ with $\frac{1}{\alpha}+\frac{1}{\beta}=1$ and γ, $\delta>1$ with $\frac{1}{\gamma}+\frac{1}{\delta}=1$. Then by Hölder's inequality we get*

$$\begin{aligned}&\int_a^b (b-t)(t-a)\|A'(t)\|_p^r\,dt\\&\le (b-a)^{2+1/\beta}\left[B(\beta+1,\beta+1)\right]^{1/\beta}\left(\int_a^b \|A'(t)\|_p^{\alpha r}\,dt\right)^{1/\alpha}\end{aligned}$$

and

$$\begin{aligned}&\int_a^b (b-t)(t-a)\|B'(t)\|_q^s\,dt\\&\le (b-a)^{2+1/\delta}\left[B(\delta+1,\delta+1)\right]^{1/\delta}\left(\int_a^b \|B'(t)\|_q^{\gamma s}\,dt\right)^{1/\gamma}.\end{aligned}$$

Then

$$\begin{aligned}&\left[\int_a^b (b-t)(t-a)\|A'(t)\|_p^r\,dt\right]^{1/r}\\&\le (b-a)^{(2\beta+1)/(\beta r)}\left[B(\beta+1,\beta+1)\right]^{1/(\beta r)}\left(\int_a^b \|A'(t)\|_p^{\alpha r}\,dt\right)^{1/(\alpha r)}\end{aligned}$$

and

$$\left[\int_a^b (b-t)(t-a)\|B'(t)\|_q^s\,dt\right]^{1/s}$$
$$\leq (b-a)^{(2\delta+1)/(\delta s)}\left[B(\delta+1,\delta+1)\right]^{1/(\delta s)}\left(\int_a^b \|B'(t)\|_q^{\gamma s}\,dt\right)^{1/(\gamma s)}.$$

Therefore

$$\frac{1}{2}(b-a)\left[\int_a^b (b-t)(t-a)\|A'(t)\|_p^r\,dt\right]^{1/r}$$
$$\times\left[\int_a^b (b-t)(t-a)\|B'(t)\|_q^s\,dt\right]^{1/s}$$
$$\leq \frac{1}{2}(b-a)(b-a)^{(2\beta+1)/(\beta r)}\left[B(\beta+1,\beta+1)\right]^{1/(\beta r)}\left(\int_a^b \|A'(t)\|_p^{\alpha r}\,dt\right)^{1/(\alpha r)}$$
$$\times (b-a)^{(2\delta+1)/(\delta s)}\left[B(\delta+1,\delta+1)\right]^{1/(\delta s)}\left(\int_a^b \|B'(t)\|_q^{\gamma s}\,dt\right)^{1/(\gamma s)}$$
$$= \frac{1}{2}\left[B(\beta+1,\beta+1)\right]^{1/(\beta r)}\left[B(\delta+1,\delta+1)\right]^{1/(\delta s)}$$
$$\times (b-a)^{1+(2\beta+1)/(\beta r)+(2\delta+1)/(\delta s)}$$
$$\times\left(\int_a^b \|A'(t)\|_p^{\alpha r}\,dt\right)^{1/(\alpha r)}\left(\int_a^b \|B'(t)\|_q^{\gamma s}\,dt\right)^{1/(\gamma s)}$$

and by (2.1) we get

$$\|D(A,B)\|_1 \leq \frac{1}{2}\left[B(\beta+1,\beta+1)\right]^{1/(\beta r)}\left[B(\delta+1,\delta+1)\right]^{1/(\delta s)} \times (b-a)^{1+(2\beta+1)/(\beta r)+(2\delta+1)/(\delta s)} \times\left(\int_a^b \|A'(t)\|_p^{\alpha r}\,dt\right)^{1/(\alpha r)}\left(\int_a^b \|B'(t)\|_q^{\gamma s}\,dt\right)^{1/(\gamma s)}, \tag{2.9}$$

where $r,\ s>1$ *with* $\frac{1}{r}+\frac{1}{s}=1,\ \alpha,\ \beta>1$ *with* $\frac{1}{\alpha}+\frac{1}{\beta}=1$ *and* $\gamma,\ \delta>1$ *with* $\frac{1}{\gamma}+\frac{1}{\delta}=1.$

We also have:

Corollary 2. *For* $p,\ q>1$ *with* $\frac{1}{p}+\frac{1}{q}=1,$ *let* $A:[a,b]\to\mathcal{B}_p(H),\ B:[a,b]\to\mathcal{B}_q(H)$ *be strongly differentiable functions on the interval* (a,b). *If* $r,\ s>1$ *with* $\frac{1}{r}+\frac{1}{s}=1,$ *then*

$$\|D(A,B)\|_1 \leq \frac{1}{12}(b-a)^4\sup_{u\in(a,b)}\|A'(u)\|_p\sup_{u\in(a,b)}\|B'(u)\|_q, \tag{2.10}$$

provided $\sup_{u\in(a,b)}\|A'(u)\|_p,\ \sup_{u\in(a,b)}\|B'(u)\|_q<\infty.$

In particular, we have for $p=q=2$ *that*

$$\|D(A,B)\|_1 \leq \frac{1}{12}(b-a)^4\sup_{u\in(a,b)}\|A'(u)\|_2\sup_{u\in(a,b)}\|B'(u)\|_2, \tag{2.11}$$

provided $\sup_{u\in(a,b)}\|A'(u)\|_2$, $\sup_{u\in(a,b)}\|B'(u)\|_2<\infty$.

Proof. From (1.3) we have

$$D\left(\ell,\int_a^{\cdot}\|A'(u)\|_p^r\,du\right)\leq\frac{1}{12}(b-a)^4\sup_{u\in(a,b)}\|A'(u)\|_p^r$$

and

$$D\left(\ell,\int_a^{\cdot}\|B'(u)\|_q^s\,du\right)\leq\frac{1}{12}(b-a)^4\sup_{u\in(a,b)}\|B'(u)\|_q^s.$$

By (2.1) we then get

$$\begin{aligned}\|D(A,B)\|_1&\leq\left[D\left(\ell,\int_a^{\cdot}\|A'(u)\|_p^r\,du\right)\right]^{1/r}\left[D\left(\ell,\int_a^{\cdot}\|B'(u)\|_q^s\,du\right)\right]^{1/s}\\&\leq\left[\frac{1}{12}(b-a)^4\sup_{u\in(a,b)}\|A'(u)\|_p^r\right]^{1/r}\\&\quad\times\left[\frac{1}{12}(b-a)^4\sup_{u\in(a,b)}\|B'(u)\|_q^s\right]^{1/s}\\&=\frac{1}{12}(b-a)^4\sup_{u\in(a,b)}\|A'(u)\|_p\sup_{u\in(a,b)}\|B'(u)\|_q,\end{aligned}$$

which proves the desired result. □

Corollary 3. *For* $p,\ q>1$ *with* $\frac{1}{p}+\frac{1}{q}=1$, *let* $A:[a,b]\to\mathcal{B}_p(H)$, $B:[a,b]\to\mathcal{B}_q(H)$ *be strongly differentiable functions on the interval* (a,b). *If* $r,\ s>1$ *with* $\frac{1}{r}+\frac{1}{s}=1$, *then*

$$\begin{aligned}&\|D(A,B)\|_1\\&\leq\frac{1}{\pi^2}(b-a)^{3+1/2}\left(\int_a^b\|A'(u)\|_p^{2r}\,du\right)^{1/(2r)}\left(\int_a^b\|B'(u)\|_q^{2s}\,du\right)^{1/(2s)},\end{aligned}\tag{2.12}$$

provided that $\int_a^b\|A'(u)\|_p^{2r}\,du$, $\int_a^b\|B'(u)\|_q^{2s}\,du<\infty$.

In particular, we have for $p=q=2$ *that*

$$\begin{aligned}&\|D(A,B)\|_1\\&\leq\frac{1}{\pi^2}(b-a)^{3+1/2}\left(\int_a^b\|A'(u)\|_2^{2r}\,du\right)^{1/(2r)}\left(\int_a^b\|B'(u)\|_2^{2s}\,du\right)^{1/(2s)}.\end{aligned}\tag{2.13}$$

Proof. If we use the inequality (1.5) we get

$$D\left(\ell,\int_a^{\cdot}\|A'(u)\|_p^r\,du\right)\leq\frac{1}{\pi^2}\left(\int_a^b\|A'(u)\|_p^{2r}\,du\right)^{1/2}(b-a)^{3+1/2}$$

and

$$D\left(\ell,\int_a^{\cdot}\|B'(u)\|_q^s\,du\right)\leq\frac{1}{\pi^2}\left(\int_a^b\|B'(u)\|_q^{2s}\,du\right)^{1/2}(b-a)^{3+1/2}.$$

By (2.1) we then get

$$\begin{aligned}\|D(A,B)\|_1 &\leq \left[D\left(\ell, \int_a^{\cdot} \|A'(u)\|_p^r du\right)\right]^{1/r} \left[D\left(\ell, \int_a^{\cdot} \|B'(u)\|_q^s du\right)\right]^{1/s} \\ &\leq \left[\frac{1}{\pi^2}\left(\int_a^b \|A'(u)\|_p^{2r} du\right)^{1/2} (b-a)^{3+1/2}\right]^{1/r} \\ &\times \left[\frac{1}{\pi^2}\left(\int_a^b \|B'(u)\|_q^{2s} du\right)^{1/2} (b-a)^{3+1/2}\right]^{1/s} \\ &= \frac{1}{\pi^2}(b-a)^{3+1/2}\left(\int_a^b \|A'(u)\|_p^{2r} du\right)^{1/(2r)} \left(\int_a^b \|B'(u)\|_q^{2s} du\right)^{1/(2s)}\end{aligned}$$

which proves the desired result. □

We also have:

Theorem 4. *For $p \geq 1$, let A, $B : [a,b] \to \mathcal{B}_p(H)$ be strongly differentiable functions on the interval (a,b). If r, $s > 1$ with $\frac{1}{r} + \frac{1}{s} = 1$, then*

$$\begin{aligned}\|D(A,B)\|_p &\leq \left[D\left(\ell, \int_a^{\cdot} \|A'(u)\|_p^r du\right)\right]^{1/r} \left[D\left(\ell, \int_a^{\cdot} \|B'(u)\|_p^s du\right)\right]^{1/s} \\ &= \frac{1}{2}(b-a)\left[\int_a^b (b-t)(t-a)\|A'(t)\|_p^r dt\right]^{1/r} \\ &\times \left[\int_a^b (b-t)(t-a)\|B'(t)\|_p^s dt\right]^{1/s} \\ &\leq \frac{1}{8}(b-a)^3 \left(\int_a^b \|A'(t)\|_p^r dt\right)^{1/r} \left(\int_a^b \|B'(t)\|_q^s dt\right)^{1/s}.\end{aligned} \tag{2.14}$$

In particular, we have for $r = s = 2$ that

$$\begin{aligned}\|D(A,B)\|_p &\leq \left[D\left(\ell, \int_a^{\cdot} \|A'(u)\|_p^2 du\right)\right]^{1/2} \left[D\left(\ell, \int_a^{\cdot} \|B'(u)\|_p^2 du\right)\right]^{1/2} \\ &= \frac{1}{2}(b-a)\left[\int_a^b (b-t)(t-a)\|A'(t)\|_p^2 dt\right]^{1/2} \\ &\times \left[\int_a^b (b-t)(t-a)\|B'(t)\|_p^2 dt\right]^{1/2} \\ &\leq \frac{1}{8}(b-a)^3 \left(\int_a^b \|A'(t)\|_p^2 dt\right)^{1/2} \left(\int_a^b \|B'(t)\|_p^2 dt\right)^{1/2}.\end{aligned} \tag{2.15}$$

Proof. If we take the p-Schatten norm in (2.3), use the integral's properties and employ inequality (1.17) for $p \geq 1$, then we get

$$\begin{aligned}\|D(A,B)\|_p &\leq \frac{1}{2}\int_a^b\int_a^b \|[A(t)-A(s)][B(t)-B(s)]\|_p\,dtds\\ &\leq \frac{1}{2}\int_a^b\int_a^b \|A(t)-A(s)\|_p\,\|B(t)-B(s)\|_p\,dtds.\end{aligned}$$

By utilising now a similar argument to the one in the proof of Theorem 3 we derive the desired result (2.14). □

Corollary 4. *Let A, $B:[a,b]\to\mathcal{B}_p(H)$ be strongly differentiable functions on the interval (a,b). If r, $s>1$ with $\frac{1}{r}+\frac{1}{s}=1$, then*

$$\|D(A,B)\|_p \leq \frac{1}{12}(b-a)^4 \sup_{u\in(a,b)}\|A'(u)\|_p \sup_{u\in(a,b)}\|B'(u)\|_p, \tag{2.16}$$

provided that $\sup_{u\in(a,b)}\|A'(u)\|_p$, $\sup_{u\in(a,b)}\|B'(u)\|_p<\infty$.

Also,

$$\begin{aligned}&\|D(A,B)\|_p \\ &\leq \frac{1}{\pi^2}(b-a)^{3+1/2}\left(\int_a^b\|A'(u)\|_p^{2r}du\right)^{1/(2r)}\left(\int_a^b\|B'(u)\|_p^{2s}du\right)^{1/(2s)},\end{aligned} \tag{2.17}$$

provided that $\int_a^b\|A'(u)\|_p^{2r}du$, $\int_a^b\|B'(u)\|_p^{2s}du<\infty$.

3. Inequalities for Operator Monotone Functions

A real valued continuous function h on $[0,\infty)$ is said to be operator monotone if $h(A)\geq h(B)$ holds for any $A\geq B\geq 0$.

We have the following representation of operator monotone functions, see for instance [2, p. 144-145]:

Theorem 5. *A function $h:[0,\infty)\to\mathbb{R}$ is operator monotone in $[0,\infty)$ if and only if it has the representation*

$$h(t)=h(0)+bt+\int_0^\infty\frac{t\lambda}{t+\lambda}d\mu(\lambda), \tag{3.1}$$

where $b\geq 0$ and a positive measure μ on $[0,\infty)$ such that

$$\int_0^\infty\frac{\lambda}{1+\lambda}d\mu(\lambda)<\infty. \tag{3.2}$$

We have the following representation result:

Lemma 1. *Let $h:[0,\infty)\to\mathbb{R}$ be operator monotone in $[0,\infty)$. Assume that $U\geq 0$, then for all selfadjoint operators V we have*

$$Dh(U)(V)=bV+\int_0^\infty\lambda^2\left[(\lambda+U)^{-1}V(\lambda+U)^{-1}\right]d\mu(\lambda). \tag{3.3}$$

Proof. From (3.1) we get

$$h(t)=h(0)+bt+\int_0^\infty\left(\lambda-\frac{\lambda^2}{t+\lambda}\right)d\mu(\lambda).$$

Assume that $U \geq 0$, then for all selfadjoint operator V we have, by the representation of h and for t in a small open interval around 0, that

$$\begin{aligned}
&h(U+tV)-h(U)\\
&=btV+\int_0^\infty\left(\lambda-\lambda^2(U+tV+\lambda)^{-1}\right)d\mu(\lambda)-\int_0^\infty\left(\lambda-\lambda^2(U+\lambda)^{-1}\right)d\mu(\lambda)\\
&=btV+\int_0^\infty\lambda^2\left[(\lambda+U)^{-1}-(\lambda+U+tV)^{-1}\right]d\mu(\lambda)\\
&=btV+\int_0^\infty\lambda^2\left[(\lambda+U)^{-1}(\lambda+U+tV-\lambda-U)(\lambda+U+tV)^{-1}\right]d\mu(\lambda)\\
&=btV+t\int_0^\infty\lambda^2\left[(\lambda+U)^{-1}V(\lambda+U+tV)^{-1}\right]d\mu(\lambda).
\end{aligned}$$

Dividing by $t \neq 0$, we get

$$\frac{h(U+tV)-h(U)}{t}=bV+\int_0^\infty\lambda^2\left[(\lambda+U)^{-1}V(\lambda+U+tV)^{-1}\right]d\mu(\lambda)$$

and by taking the limit over $t \to 0$, we get

$$Dh(U)(V)=bV+\int_0^\infty\lambda^2\left[(\lambda+U)^{-1}V(\lambda+U)^{-1}\right]d\mu(\lambda)$$

for all selfadjoint operator V we have (3.3). □

Lemma 2. *Let $h:[0,\infty)\to\mathbb{R}$ be operator monotone in $[0,\infty)$. Assume that $U \geq u > 0$, then for all selfadjoint operators $V \in B_p(H)$, $p \geq 1$ we have*

$$\|Dh(U)(V)\|_p \leq h'(u)\|V\|_p . \tag{3.4}$$

Proof. From (3.3) and using (1.19) we get

$$\begin{aligned}
\|Dh(U)(V)-bV\|_p &\leq \int_0^\infty\lambda^2\left\|(\lambda+U)^{-1}V(\lambda+U)^{-1}\right\|_p d\mu(\lambda) \qquad (3.5)\\
&\leq \|V\|_p\int_0^\infty\lambda^2\left\|(\lambda+U)^{-1}\right\|^2 d\mu(\lambda).
\end{aligned}$$

Observe that $\lambda+U \geq \lambda+u > 0$ for $\lambda \in [0,\infty)$. Then $0<(\lambda+U)^{-1}\leq(\lambda+u)^{-1}$, which implies that $\left\|(\lambda+U)^{-1}\right\|\leq(\lambda+u)^{-1}$, namely $\left\|(\lambda+U)^{-1}\right\|^2\leq(\lambda+u)^{-2}$ for $\lambda\in[0,\infty)$.

Therefore by (3.5) we get

$$\|Dh(U)(V)-bV\|_p \leq \|V\|_p\int_0^\infty\lambda^2(\lambda+u)^{-2}d\mu(\lambda). \tag{3.6}$$

If we take the derivative over t in (3.1) then we have

$$h'(t)=b+\int_0^\infty\frac{\lambda(t+\lambda)-\lambda t}{(t+\lambda)^2}d\mu(\lambda)=b+\int_0^\infty\frac{\lambda^2}{(t+\lambda)^2}d\mu(\lambda) \tag{3.7}$$

for $t>0$.

From (3.7) we get

$$\int_0^\infty\lambda^2(\lambda+u)^{-2}d\mu(\lambda)=h'(u)-b,$$

and by (3.6) we derive

$$\|Dh(U)(V) - bV\|_p \le \|V\|_p h'(u) - b\|V\|_p .$$

Finally, by the triangle inequality and by the fact that $b \ge 0$, we obtain that

$$\|Dh(U)(V)\|_p - b\|V\|_p \le \|Dh(U)(V) - bV\|_p ,$$

which proves the desired result (3.4). □

For a continuous function h on $(0, \infty)$ and C, $E > 0$ we consider the auxiliary function $h_{C,E} : [0,1] \to \mathbb{R}$ defined by

$$h_{C,E}(t) := h((1-t)C + tE), \ t \in [0,1].$$

We have the following representations of the derivatives:

Lemma 3. *Assume that the operator function generated by h is Fréchet differentiable in each $C \ge 0$, then for $E \ge 0$ we have that $h_{C,E}$ is differentiable on $[0,1]$ and*

$$h'_{C,E}(t) = D(h)((1-t)C + tE)(E - C) \tag{3.8}$$

for $t \in [0,1]$, where in 0 and 1 the derivatives are the right and left derivatives.

Proof. We prove only for the interior points $t \in (0,1)$. Let h be in a small interval around 0 such that $t + h \in (0,1)$. Then for $h \ne 0$,

$$\begin{aligned}
&\frac{h_{C,E}(t+h) - h(t)}{h} \\
&= \frac{h((1-(t+h))C + (t+h)E) - h((1-t)C + tE)}{h} \\
&= \frac{h((1-t)C + tE + h(E-C)) - h((1-t)C + tE)}{h}
\end{aligned}$$

and by taking the limit over $h \to 0$, we get

$$\begin{aligned}
h'_{C,E}(t) &= \lim_{h\to 0} \frac{h_{C,E}(t+h) - h(t)}{h} \\
&= \lim_{h\to 0} \left[\frac{h((1-t)C + tE + h(E-C)) - h((1-t)C + tE)}{h}\right] \\
&= D(h)((1-t)C + tE)(E - C),
\end{aligned}$$

which proves (3.8). □

Corollary 5. *Let $h : [0,\infty) \to \mathbb{R}$ be operator monotone in $[0,\infty)$. Then for all $C \ge c > 0$, $E \ge d > 0$ with $C, E \in B_p(H)$, $p \ge 1$ we have*

$$\begin{aligned}
\left\|h'_{C,E}(t)\right\|_p &= \|D(h)((1-t)C + tE)(E - C)\|_p \\
&\le h'((1-t)c + td)\|E - C\|_p
\end{aligned} \tag{3.9}$$

for all $t \in [0,1]$.

The proof follows by Theorem **??** and Lemma 3.

For the operator monotone functions h, $k:[0,\infty)\to\mathbb{R}$ and two operators C, $E\geq 0$ we consider the *Čebyšev functional*

$$D(h,k,C,E):=\int_0^1 h((1-t)C+tE)\,k((1-t)C+tE)\,dt - \int_0^1 h((1-t)C+tE)\,dt\int_0^1 k((1-t)C+tE)\,dt.$$

For p, $q>1$ with $\frac{1}{p}+\frac{1}{q}=1$ assume that $C\geq c>0$, $E\geq d>0$ with C, $E\in B_p(H)\cap B_q(H)$, then by (2.1) for r, $s>1$ with $\frac{1}{r}+\frac{1}{s}=1$, we derive that

$$\begin{aligned}\|D(h,k,C,E)\|_1 &\leq \frac{1}{2}\|E-C\|_p\|E-C\|_q \\ &\times\left[\int_0^1 t(1-t)\left[h'((1-t)c+td)\right]^r dt\right]^{1/r} \\ &\times\left[\int_0^1 t(1-t)\left[k'((1-t)c+td)\right]^s dt\right]^{1/s} \\ &\leq \frac{1}{8}\|E-C\|_p\|E-C\|_q\left(\int_0^1\left[h'((1-t)c+td)\right]^r dtdt\right)^{1/r} \\ &\times\left(\int_0^1\left[k'((1-t)c+td)\right]^s dt\right)^{1/s}.\end{aligned}\tag{3.10}$$

For $r=s=2$ we get by (3.10) that

$$\begin{aligned}\|D(h,k,C,E)\|_1 &\leq \frac{1}{2}\|E-C\|_p\|E-C\|_q \\ &\times\left[\int_0^1 t(1-t)\left[h'((1-t)c+td)\right]^r dt\right]^{1/2} \\ &\times\left[\int_0^1 t(1-t)\left[k'((1-t)c+td)\right]^s dt\right]^{1/2} \\ &\leq \frac{1}{8}\|E-C\|_p\|E-C\|_q \\ &\times\left(\int_0^1\left[h'((1-t)c+td)\right]^2 dtdt\right)^{1/2} \\ &\times\left(\int_0^1\left[k'((1-t)c+td)\right]^2 dt\right)^{1/2}.\end{aligned}\tag{3.11}$$

For $p=q=2$ in (3.10) we derive

$$\begin{aligned}
\|D(h,k,C,E)\|_1 &\le \frac{1}{2}\|E-C\|_2^2\left[\int_0^1 t(1-t)\left[h'((1-t)c+td)\right]^r dt\right]^{1/r} \\
&\times\left[\int_0^1 t(1-t)\left[k'((1-t)c+td)\right]^s dt\right]^{1/s} \\
&\le \frac{1}{8}\|E-C\|_p^2\left(\int_0^1 \left[h'((1-t)c+td)\right]^r dtdt\right)^{1/r} \\
&\times\left(\int_0^1 \left[k'((1-t)c+td)\right]^s dt\right)^{1/s}.
\end{aligned} \tag{3.12}$$

If in this inequality we take $r=s=2$, then we get

$$\begin{aligned}
\|D(h,k,C,E)\|_1 &\le \frac{1}{2}\|E-C\|_2^2\left[\int_0^1 t(1-t)\left[h'((1-t)c+td)\right]^2 dt\right]^{1/2} \\
&\times\left[\int_0^1 t(1-t)\left[k'((1-t)c+td)\right]^2 dt\right]^{1/2} \\
&\le \frac{1}{8}\|E-C\|_2^2\left(\int_0^1 \left[h'((1-t)c+td)\right]^2 dt\right)^{1/2} \\
&\times\left(\int_0^1 \left[k'((1-t)c+td)\right]^2 dt\right)^{1/2}.
\end{aligned} \tag{3.13}$$

If $r,\ s>1$ with $\frac{1}{r}+\frac{1}{s}=1$, then for the operator monotone functions $h,\ k:[0,\infty)\to\mathbb{R}$ and two operators $C\ge c>0$, $E\ge d>0$ with $C,\ E\in B_p(H),\ p\ge 1$

$$\begin{aligned}
\|D(h,k,C,E)\|_p &\le \frac{1}{2}\|E-C\|_p^2\left[\int_0^1 t(1-t)\left[h'((1-t)c+td)\right]^r dt\right]^{1/r} \\
&\times\left[\int_0^1 t(1-t)\left[k'((1-t)c+td)\right]^s dt\right]^{1/s} \\
&\le \frac{1}{8}\|E-C\|_p^2\left(\int_0^1 \left[h'((1-t)c+td)\right]^r dt\right)^{1/r} \\
&\times\left(\int_0^1 \left[k'((1-t)c+td)\right]^s dt\right)^{1/s}.
\end{aligned} \tag{3.14}$$

Consider the operator monotone functions $h,\, k : [0,\infty) \to \mathbb{R}$, $h(t) = t^m$, $k(t) = t^n$ with $m,\, n \in (0,1)$, then by (3.10) we derive

$$\begin{aligned}(3.15)\quad & \left\| \int_0^1 ((1-t)C + tE)^{m+n} dt \right. \\ & \left. - \int_0^1 ((1-t)C + tE)^m dt \int_0^1 ((1-t)C + tE)^n dt \right\|_1 \\ & \le \frac{1}{8} \|E - C\|_p \|E - C\|_q \times \begin{cases} \left(\frac{d^{(m-1)r+1} - c^{(m-1)r+1}}{[(m-1)r+1](d-c)} \right)^{1/r} & \text{if } d \ne c \\ c^{m-1} \text{ if } d = c \end{cases} \\ & \times \begin{cases} \left(\frac{d^{(n-1)r+1} - c^{(n-1)r+1}}{[(n-1)r+1](d-c)} \right)^{1/r} & \text{if } d \ne c \\ c^{n-1} \text{ if } d = c \end{cases},\end{aligned}$$

for $p,\, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, $(m-1)r + 1$, $(n-1)r + 1 \ne 0$ and $C \ge c > 0$, $E \ge d > 0$ with $C,\, E \in B_p(H) \cap B_q(H)$.

If $C \ge c > 0$, $E \ge d > 0$ with $C,\, E \in B_p(H)$, $p \ge 1$, then from (3.14) we get for $m,\, n \in (0,1)$ that

$$\begin{aligned}(3.16)\quad & \left\| \int_0^1 ((1-t)C + tE)^{m+n} dt \right. \\ & \left. - \int_0^1 ((1-t)C + tE)^m dt \int_0^1 ((1-t)C + tE)^n dt \right\|_p \\ & \le \frac{1}{8} \|E - C\|_p^2 \times \begin{cases} \left(\frac{d^{(m-1)r+1} - c^{(m-1)r+1}}{[(m-1)r+1](d-c)} \right)^{1/r} & \text{if } d \ne c \\ c^{m-1} \text{ if } d = c \end{cases} \\ & \times \begin{cases} \left(\frac{d^{(n-1)s+1} - c^{(n-1)s+1}}{[(n-1)s+1](d-c)} \right)^{1/s} & \text{if } d \ne c \\ c^{n-1} \text{ if } d = c, \end{cases}\end{aligned}$$

provided that $(m-1)r + 1$, $(n-1)r + 1 \ne 0$.

Similar inequalities may be stated by utilising the other results from the previous section, however the details are not presented here.

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