p-SCHATTEN NORM INEQUALITIES FOR A PERTURBED ČEBYŠEV'S FUNCTIONAL

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Abstract. An operator $A \in \mathcal{B}(H)$ is said to belong to the *von Neumann-Schatten class* $\mathcal{B}_p(H)$, $1 \leq p < \infty$ if the *p-Schatten norm* is finite

$$\|A\|_p := [\operatorname{tr}(|A|^p)]^{1/p} < \infty.$$

In this paper we show among others that, if $p,\, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ and $A : [a,b] \to \mathcal{B}_p(H)$ is continuous while $B : [a,b] \to \mathcal{B}_q(H)$ is strongly differentiable on the interval (a,b), then

$$\begin{aligned} &\left\| (b-a) \int_a^b A(t) B(t)\, dt - \int_a^b A(t)\, dt \int_a^b B(t)\, dt \right. \\ &\left. - (b-a) \left(\int_a^b \left(t - \frac{a+b}{2} \right) A(t)\, dt \right) D \right\|_1 \\ &\leq \begin{cases} \int_a^b \|A(t) - C\|_p \max\{t-a, b-t\}\, dt \int_a^b \|B'(s) - D\|_q\, ds, \\ \\ \frac{1}{(\beta+1)^{1/\beta}} \int_a^b \|A(t) - C\|_p \left[(t-a)^{\beta+1} + (b-t)^{\beta+1} \right]^{1/\beta} dt \\ \times \left(\int_a^b \|B'(s) - D\|_q^\alpha\, ds \right)^{1/\alpha}, \ \alpha, \beta > 1, \ \frac{1}{\alpha} + \frac{1}{\beta} = 1, \\ \\ \frac{1}{2} \int_a^b \|A(t) - C\|_p \left[(t-a)^2 + (b-t)^2 \right] dt \sup_{s \in [a,b]} \|B'(s) - D\|_q . \end{cases} \end{aligned}$$

Some examples of interest for the operator monotone functions are also given.

1. Introduction

In [3] the authors obtained the following Grüss' type inequalities for functions with values in Banach spaces.

Theorem 1. *Let F be a Banach space over the real or complex number field $\mathbb{K}$, $\Omega \in \mathbb{R}^n$ a measurable set and $\rho : \Omega \to [0,\infty)$ a Lebesgue integrable function with $\int_\Omega \rho(x)\, dx = 1$. If $\alpha : \Omega \to \mathbb{K}$ is a Lebesgue integrable function such that there exists $\gamma,\, \Gamma \in \mathbb{K}$ with*

$$\left| \alpha(x) - \frac{\gamma + \Gamma}{2} \right| \leq \frac{1}{2} |\Gamma - \gamma| \tag{1.1}$$

or, equivalently,

$$\operatorname{Re}\left[(\Gamma - \alpha(x)) \left(\overline{\alpha(x)} - \overline{\gamma} \right) \right] \geq 0 \tag{1.2}$$

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for a.e. $x \in \Omega$, and $g : \Omega \to F$ is a Bochner measurable function such that $\rho\alpha g$ and ρg are Bochner integrable on Ω, then,

$$\left\| \int_{\Omega} \rho(x)\alpha(x) g(x) dx - \int_{\Omega} \rho(x)\alpha(x) dx \int_{\Omega} \rho(x) g(x) dx \right\| \leq \frac{1}{2} |\Gamma - \gamma| \int_{\Omega} \rho(x) \left\| g(x) - \int_{\Omega} \rho(y) g(y) dy \right\| dx. \tag{1.3}$$

The constant $\frac{1}{2}$ in (1.3) is the best possible.

The following dual result also holds:

Theorem 2. *Let F and Ω, ρ be as above. If $g : \Omega \to X$ is Bochner measurable on Ω and there exist vector $v \in X$ and $r > 0$ such that*

$$\| g(x) - v \| \leq r \text{ for a.e. } x \in \Omega$$

and $\alpha : \Omega \to \mathbb{K}$ is a Lebesgue integrable function with $\rho\alpha g$, ρg Bochner integrable functions on Ω, then we have the sharp inequalities

$$\begin{aligned} &\left\| \int_{\Omega} \rho(x)\alpha(x) g(x) dx - \int_{\Omega} \rho(x)\alpha(x) dx \int_{\Omega} \rho(x) g(x) dx \right\| \\ &\leq r \int_{\Omega} \rho(x) \left| \alpha(x) - \int_{\Omega} \rho(y)\alpha(y) dy \right| dx \\ &\leq r \left[\int_{\Omega} \rho(x) |\alpha(x)|^2 dx - \left| \int_{\Omega} \rho(x)\alpha(x) dx \right|^2 \right]^{\frac{1}{2}}. \end{aligned} \tag{1.4}$$

Now, consider the function f defined on the open and convex subset C of the Banach space E with values in the Banach space F and $\Omega = [0,1]$. Also let $\rho(t) = 1$ and $g(t) = f((1-t)x + ty)$ for $t \in [0,1]$ and x, $y \in C$. Then we can state the following particular case of interest:

Corollary 1. *Assume that $f : C \subset E \to F$ is continuous on C and x, $y \in C$, $x \neq y$. If $p : [0,1] \to \mathbb{K}$ is a Lebesgue integrable function such that there exists γ, $\Gamma \in \mathbb{K}$ with*

$$\left| p(t) - \frac{\gamma + \Gamma}{2} \right| \leq \frac{1}{2} |\Gamma - \gamma| \tag{1.5}$$

or, equivalently,

$$\operatorname{Re}\left[(\Gamma - p(t)) \left(\overline{p(t)} - \overline{\gamma} \right) \right] \geq 0 \tag{1.6}$$

for a.e. $t \in [0,1]$, then,

$$\begin{aligned} &\left\| \int_0^1 p(t) f((1-t)x + ty) dt - \int_0^1 p(t) dt \int_0^1 f((1-t)x + ty) dt \right\| \\ &\leq \frac{1}{2} |\Gamma - \gamma| \int_0^1 \left\| f((1-t)x + ty) - \int_0^1 f((1-s)x + sy) ds \right\| dt. \end{aligned} \tag{1.7}$$

The constant $\frac{1}{2}$ in (1.7) is the best possible.

If there exists a vector v and $r > 0$ such that

$$\| f((1-t)x + ty) - v \| \leq r \text{ for a.e. } t \in [0,1], \tag{1.8}$$

then for $q:[0,1]\to\mathbb{C}$ *Lebesgue integrable, we have the sharp inequalities*

$$\left\|\int_0^1 q(t)\, f((1-t)x+ty)\,dt-\int_0^1 q(t)\,dt\int_0^1 f((1-t)x+ty)\,dt\right\| \leq r\int_0^1\left|q(t)-\int_0^1 q(s)\,ds\right|dt\leq r\left[\int_0^1|q(t)|^2\,dt-\left|\int_\Omega q(t)\,dt\right|^2\right]^{\frac{1}{2}}. \tag{1.9}$$

We observe that, if there exists two vectors $z,\ w\in F$ such that

$$\left\|f((1-t)x+ty)-\frac{z+w}{2}\right\|\leq\frac{1}{2}\|w-z\|\ \text{ for a.e. } t\in[0,1], \tag{1.10}$$

then for $q:[0,1]\to\mathbb{C}$ Lebesgue integrable, we have the sharp inequalities

$$\left\|\int_0^1 q(t)\, f((1-t)x+ty)\,dt-\int_0^1 q(t)\,dt\int_0^1 f((1-t)x+ty)\,dt\right\| \leq\frac{1}{2}\|w-z\|\int_0^1\left|q(t)-\int_0^1 q(s)\,ds\right|dt \leq\frac{1}{2}\|w-z\|\left[\int_0^1|q(t)|^2\,dt-\left|\int_\Omega q(t)\,dt\right|^2\right]^{\frac{1}{2}}. \tag{1.11}$$

In order to extend some of the above results for the p-Schatten norm of bounded linear operators in complex Hilbert spaces we need the following preparations.

Let $(H;\langle.,.\rangle)$ be a complex Hilbert space and $\mathcal{B}(H)$ the Banach algebra of all bounded linear operators on H. If $\{e_i\}_{i\in I}$ an orthonormal basis of H, we say that $A\in\mathcal{B}(H)$ is of *trace class* if

$$\|A\|_1:=\sum_{i\in I}\langle|A|\,e_i,e_i\rangle<\infty. \tag{1.12}$$

The definition of $\|A\|_1$ does not depend on the choice of the orthonormal basis $\{e_i\}_{i\in I}$. We denote by $\mathcal{B}_1(H)$ the set of trace class operators in $\mathcal{B}(H)$.

We define the *trace* of a trace class operator $A\in\mathcal{B}_1(H)$ to be

$$\operatorname{tr}(A):=\sum_{i\in I}\langle Ae_i,e_i\rangle, \tag{1.13}$$

where $\{e_i\}_{i\in I}$ an orthonormal basis of H. Note that this coincides with the usual definition of the trace if H is finite-dimensional. We observe that the series (1.13) converges absolutely and it is independent from the choice of basis.

The following result collects some properties of the trace:

Theorem 3. *We have:*

(i) If $A\in\mathcal{B}_1(H)$ *then* $A^*\in\mathcal{B}_1(H)$ *and*

$$\operatorname{tr}(A^*)=\overline{\operatorname{tr}(A)}; \tag{1.14}$$

(ii) If $A\in\mathcal{B}_1(H)$ *and* $T\in\mathcal{B}(H)$, *then* $AT,\ TA\in\mathcal{B}_1(H)$ *and*

$$\operatorname{tr}(AT)=\operatorname{tr}(TA)\ \textit{ and }\ |\operatorname{tr}(AT)|\leq\|A\|_1\|T\|; \tag{1.15}$$

(iii) $\operatorname{tr}(\cdot)$ *is a bounded linear functional on* $\mathcal{B}_1(H)$ *with* $\|\operatorname{tr}\|=1$;

(iv) If $A,\ B\in\mathcal{B}_2(H)$ *then* $AB,\ BA\in\mathcal{B}_1(H)$ *and* $\operatorname{tr}(AB)=\operatorname{tr}(BA)$;

(v) $\mathcal{B}_{fin}(H)$ *is a dense subspace of* $\mathcal{B}_1(H)$.

An operator $A\in\mathcal{B}(H)$ is said to belong to the *von Neumann-Schatten class* $\mathcal{B}_p(H)$, $1\leq p<\infty$ if the *p-Schatten norm* is finite [17, p. 60-64]

$$\|A\|_p := \left[\operatorname{tr}(|A|^p)\right]^{1/p} < \infty.$$

For $1<p<q<\infty$ we have that

$$\mathcal{B}_1(H)\subset\mathcal{B}_p(H)\subset\mathcal{B}_q(H)\subset\mathcal{B}(H) \tag{1.16}$$

and

$$\|A\|_1\geq\|A\|_p\geq\|A\|_q\geq\|A\|. \tag{1.17}$$

For $p\geq 1$ the functional $\|\cdot\|_p$ is a *norm* on the $*$-ideal $\mathcal{B}_p(H)$ and $\left(\mathcal{B}_p(H),\|\cdot\|_p\right)$ is a Banach space.

Also, see for instance [17, p. 60-64],

$$\|A\|_p=\|A^*\|_p,\ A\in\mathcal{B}_p(H) \tag{1.18}$$

$$\|AB\|_p\leq\|A\|_p\|B\|_p,\ A,B\in\mathcal{B}_p(H) \tag{1.19}$$

and

$$\|AB\|_p\leq\|A\|_p\|B\|,\ \|BA\|_p\leq\|B\|\|A\|_p,\ A\in\mathcal{B}_p(H),\ B\in\mathcal{B}(H). \tag{1.20}$$

This implies that

$$\|CAB\|_p\leq\|C\|\|A\|_p\|B\|,\ A\in\mathcal{B}_p(H),\ B,\ C\in\mathcal{B}(H). \tag{1.21}$$

In terms of *p-Schatten norm* we have the *Hölder inequality* for $p,\ q>1$ with $\frac{1}{p}+\frac{1}{q}=1$

$$(|\operatorname{tr}(AB)|\leq)\ \|AB\|_1\leq\|A\|_p\|B\|_q,\ A\in\mathcal{B}_p(H),\ B\in\mathcal{B}_q(H). \tag{1.22}$$

For the theory of trace functionals and their applications the reader is referred to [16] and [17].

For some classical trace inequalities see [6], [7], and [14], which are continuations of the work of Bellman [1].

2. Main Results

We have the following equality:

Lemma 1. *Let $B:[a,b]\to B(H)$ be a strongly differentiable function on the interval (a,b) and $A:[a,b]\to B(H)$ a continuous function, then for all $C\in B(H)$*

$$\begin{aligned}(b-a)\int_a^b A(t)B(t)\,dt-\int_a^b A(t)\,dt\int_a^b B(t)\,dt \\ =\int_a^b (A(t)-C)\left(\int_a^t (s-a)B'(s)\,ds+\int_t^b (s-b)B'(s)\,ds\right)dt.\end{aligned} \tag{2.1}$$

Proof. We start to the Montgomery identity for a strongly differentiable function $B:[a,b]\to B(H)$

$$B(t)(b-a)-\int_a^b B(s)\,ds=\int_a^t (s-a)B'(s)\,ds+\int_t^b (s-b)B'(s)\,ds \tag{2.2}$$

that holds for all $t\in[a,b]$.

Indeed, integrating by parts, we have

$$\int_a^t (s-a) B'(s) \, ds = (t-a) B(t) - \int_a^t B(s) \, ds$$

and

$$\int_t^b (s-b) B'(s) \, ds = (b-t) B(t) - \int_t^b B(s) \, ds$$

which by addition gives (2.2).

If we multiply this identity by $A(t)$ and integrate over t in $[a, b]$, then we get

$$\begin{aligned} (2.3) \qquad & (b-a) \int_a^b A(t) B(t) \, dt - \int_a^b A(t) \, dt \int_a^b B(s) \, ds \\ & = \int_a^b A(t) \left(\int_a^t (s-a) B'(s) \, ds + \int_t^b (s-b) B'(s) \, ds \right) dt. \end{aligned}$$

Now, if we replace $A(t)$ by $A(t) - C$, then we get

$$\begin{aligned} & (b-a) \int_a^b [A(t) - C] B(t) \, dt - \int_a^b [A(t) - C] \, dt \int_a^b B(t) \, dt \\ & = (b-a) \int_a^b A(t) B(t) \, dt - (b-a) C \int_a^b B(t) \, dt \\ & - \int_a^b A(t) \, dt \int_a^b B(t) \, dt + (b-a) C \int_a^b B(t) \, dt \\ & = (b-a) \int_a^b A(t) B(t) \, dt - \int_a^b A(t) \, dt \int_a^b B(s) \, ds \end{aligned}$$

and by (2.3) we derive (2.1). □

Lemma 2. *With the assumptions of Lemma 2 we have*

$$\begin{aligned} (2.4) \qquad & (b-a) \int_a^b A(t) B(t) \, dt - \int_a^b A(t) \, dt \int_a^b B(t) \, dt \\ & - (b-a) \left(\int_a^b \left(t - \frac{a+b}{2} \right) A(t) \, dt \right) D \\ & = \int_a^b (A(t) - C) \\ & \times \left(\int_a^t (s-a) [B'(s) - D] \, ds + \int_t^b (s-b) [B'(s) - D] \, ds \right) dt \end{aligned}$$

for all $D \in B(H)$.

Proof. If we replace $B(t)$ with $B(t) - tD$ in (2.1), then we get

$$
\begin{aligned}
(2.5)\qquad & (b-a)\int_a^b A(t)\,[B(t)-tD]\,dt - \int_a^b A(t)\,dt \int_a^b (B(t)-tD)\,dt \\
&= \int_a^b (A(t)-C) \\
&\times \left(\int_a^t (s-a)\,[B'(s)-D]\,ds + \int_t^b (s-b)\,[B'(s)-D]\,ds\right) dt.
\end{aligned}
$$

Observe that

$$
\begin{aligned}
(2.6)\qquad & (b-a)\int_a^b A(t)\,[B(t)-tD]\,dt - \int_a^b A(t)\,dt \int_a^b (B(t)-tD)\,dt \\
&= (b-a)\left[\int_a^b A(t)\,B(t)\,dt - \int_a^b tA(t)\,dtD\right] \\
&- \int_a^b A(t)\,dt \left[\int_a^b B(t)\,dt - \frac{1}{2}\left(b^2-a^2\right)D\right] \\
&= (b-a)\int_a^b A(t)\,B(t)\,dt - (b-a)\int_a^b tA(t)\,dtD \\
&- \int_a^b A(t)\,dt \int_a^b B(t)\,dt + \frac{1}{2}\left(b^2-a^2\right)\int_a^b A(t)\,dtD \\
&= (b-a)\int_a^b A(t)\,B(t)\,dt - \int_a^b A(t)\,dt \int_a^b B(t)\,dt \\
&- (b-a)\left(\int_a^b \left(t-\frac{a+b}{2}\right) A(t)\,dt\right) D
\end{aligned}
$$

and by (2.5), we get (2.4). □

Remark 1. *If A is symmetric on $[a,b]$, namely $A(a+b-t) = A(t)$ for all $t \in [a,b]$, then $h(t) := \left(t - \frac{a+b}{2}\right) A(t)$ is antisymmetric, which gives that*

$$\int_a^b \left(t - \frac{a+b}{2}\right) A(t)\,dt = 0,$$

and by (2.4) we derive

$$
\begin{aligned}
(2.7)\qquad & (b-a)\int_a^b A(t)\,B(t)\,dt - \int_a^b A(t)\,dt \int_a^b B(t)\,dt \\
&= \int_a^b (A(t)-C) \\
&\times \left(\int_a^t (s-a)\,[B'(s)-D]\,ds + \int_t^b (s-b)\,[B'(s)-D]\,ds\right) dt
\end{aligned}
$$

for all $C \in B(H)$, $D \in B(H)$.

We have the inequalities:

Theorem 4. *For $p,\ q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, let $B:[a,b]\to B_q(H)$ be a strongly differentiable function on the interval (a,b) and $A:[a,b]\to B_p(H)$ a continuous function, then for all $C\in B_p(H)$*

$$\begin{aligned}(2.8)\quad &\left\|(b-a)\int_a^b A(t)\,B(t)\,dt-\int_a^b A(t)\,dt\int_a^b B(t)\,dt\right\|_1\\ &\le \int_a^b \|A(t)-C\|_p\left[\int_a^t (s-a)\,\|B'(s)\|_q\,ds+\int_t^b (b-s)\,\|B'(s)\|_q\,ds\right]dt\\ &=: B(A,B,C).\end{aligned}$$

We have the bounds

$$(2.9)\quad B(A,B,C) \le \begin{cases} \int_a^b \|A(t)-C\|_p \max\{t-a,b-t\}\,dt\int_a^b \|B'(s)\|_q\,ds,\\[2ex] \frac{1}{(\beta+1)^{1/\beta}}\int_a^b \|A(t)-C\|_p\left[(t-a)^{\beta+1}+(b-t)^{\beta+1}\right]^{1/\beta}dt\\ \times\left(\int_a^b \|B'(s)\|_q^{\alpha}\,ds\right)^{1/\alpha},\ \alpha,\ \beta>1,\ \frac{1}{\alpha}+\frac{1}{\beta}=1,\\[2ex] \frac{1}{2}\int_a^b \|A(t)-C\|_p\left[(t-a)^2+(b-t)^2\right]dt\sup_{s\in[a,b]}\|B'(s)\|_q\,. \end{cases}$$

Proof. If we take the norm in (2.1) and using Hölder's inequality (1.22), then we get

$$\begin{aligned}(2.10)\quad &\left\|(b-a)\int_a^b A(t)\,B(t)\,dt-\int_a^b A(t)\,dt\int_a^b B(t)\,dt\right\|_1\\ &\le \int_a^b \left\|(A(t)-C)\left(\int_a^t (s-a)\,B'(s)\,ds+\int_t^b (s-b)\,B'(s)\,ds\right)\right\|_1 dt\\ &\le \int_a^b \|A(t)-C\|_p\left\|\int_a^t (s-a)\,B'(s)\,ds+\int_t^b (s-b)\,B'(s)\,ds\right\|_q dt\\ &\le \int_a^b \|A(t)-C\|_p\left[\left\|\int_a^t (s-a)\,B'(s)\,ds\right\|_q+\left\|\int_t^b (s-b)\,B'(s)\,ds\right\|_q\right]dt\\ &\le \int_a^b \|A(t)-C\|_p\left[\int_a^t (s-a)\,\|B'(s)\|_q\,ds+\int_t^b (b-s)\,\|B'(s)\|_q\,ds\right]dt,\end{aligned}$$

which proves the inequality (2.8).

Using Hölder's inequality, we have for α, $\beta > 1$ with $\frac{1}{\alpha} + \frac{1}{\beta} = 1$,

$$\int_a^t (s-a) \|B'(s)\|_q \, ds \leq \begin{cases} \sup_{s\in[a,t]} (s-a) \int_a^t \|B'(s)\|_q \, ds, \\ \left(\int_a^t (s-a)^\beta \, ds\right)^{1/\beta} \left(\int_a^t \|B'(s)\|_q^\alpha \, ds\right)^{1/\alpha}, \\ \sup_{s\in[a,t]} \|B'(s)\|_q \int_a^t (s-a)\, ds, \end{cases}$$
$$= \begin{cases} (t-a) \int_a^t \|B'(s)\|_q \, ds, \\ \frac{(t-a)^{1+1/\beta}}{(\beta+1)^{1/\beta}} \left(\int_a^t \|B'(s)\|_q^\alpha \, ds\right)^{1/\alpha}, \\ \frac{(t-a)^2}{2} \sup_{s\in[a,t]} \|B'(s)\|_q \end{cases}$$

and

$$\int_t^b (b-s) \|B'(s)\|_q \, ds \leq \begin{cases} \sup_{s\in[t,b]} (b-s) \int_t^b \|B'(s)\|_q \, ds, \\ \left(\int_t^b (b-s)^\beta \, ds\right)^{1/\beta} \left(\int_t^b \|B'(s)\|_q^\alpha \, ds\right)^{1/\alpha}, \\ \sup_{s\in[t,b]} \|B'(s)\|_q \int_t^b (b-s)\, ds, \end{cases}$$
$$= \begin{cases} (b-t) \int_t^b \|B'(s)\|_q \, ds, \\ \frac{(b-t)^{1+1/\beta}}{(\beta+1)^{1/\beta}} \left(\int_t^b \|B'(s)\|_q^\alpha \, ds\right)^{1/\alpha}, \\ \frac{(b-t)^2}{2} \sup_{s\in[t,b]} \|B'(s)\|_q . \end{cases}$$

Therefore

$$\int_a^t (s-a) \|B'(s)\|_q \, ds + \int_t^b (b-s) \|B'(s)\|_q \, ds$$
$$\leq \begin{cases} (t-a) \int_a^t \|B'(s)\|_q \, ds + (b-t) \int_t^b \|B'(s)\|_q \, ds, \\ \frac{1}{(\beta+1)^{1/\beta}} \left[(t-a)^{1+1/\beta} \left(\int_a^t \|B'(s)\|_q^\alpha \, ds\right)^{1/\alpha} \right. \\ \left. + (b-t)^{1+1/\beta} \left(\int_t^b \|B'(s)\|_q^\alpha \, ds\right)^{1/\alpha}\right]. \\ \frac{1}{2} \left[(t-a)^2 \sup_{s\in[a,t]} \|B'(s)\|_q + (b-t)^2 \sup_{s\in[t,b]} \|B'(s)\|_q\right], \end{cases}$$
$$\leq \begin{cases} \max\{t-a, b-t\} \int_a^b \|B'(s)\|_q \, ds, \\ \frac{1}{(\beta+1)^{1/\beta}} \left[(t-a)^{\beta+1} + (b-t)^{\beta+1}\right]^{1/\beta} \left(\int_a^b \|B'(s)\|_q^\alpha \, ds\right)^{1/\alpha}, \\ \frac{1}{2} \left[(t-a)^2 + (b-t)^2\right] \sup_{s\in[a,b]} \|B'(s)\|_q \end{cases}$$

and by (2.10) we get the desired result (2.8). □

Remark 2. *We have the inequalities*

$$\left\| (b-a) \int_a^b A(t) B(t) dt - \int_a^b A(t) dt \int_a^b B(t) dt \right\|_1 \tag{2.11}$$
$$\leq \int_a^b \|A(t)\|_p \left[\int_a^t (s-a) \|B'(s)\|_q ds + \int_t^b (b-s) \|B'(s)\|_q ds \right] dt$$
$$\leq \begin{cases} \int_a^b \|A(t)\|_p \max\{t-a, b-t\} dt \int_a^b \|B'(s)\|_q ds, \\ \\ \frac{1}{(\beta+1)^{1/\beta}} \int_a^b \|A(t)\|_p \left[(t-a)^{\beta+1} + (b-t)^{\beta+1} \right]^{1/\beta} dt \\ \times \left(\int_a^b \|B'(s)\|_q^\alpha ds \right)^{1/\alpha}, \ \alpha, \ \beta > 1, \ \frac{1}{\alpha} + \frac{1}{\beta} = 1, \\ \\ \frac{1}{2} \int_a^b \|A(t)\|_p \left[(t-a)^2 + (b-t)^2 \right] dt \sup_{s \in [a,b]} \|B'(s)\|_q \end{cases}$$

provided that $B : [a,b] \to B(H)$ is a strongly differentiable function on the interval (a,b) and $A : [a,b] \to B(H)$ is a continuous function on $[a,b]$.

Corollary 2. *With the assumptions of Theorem 4 and if there exists $C \in B_p(H)$ and $M > 0$ such that*

$$\|A(t) - C\|_p \leq M \text{ for all } t \in [a,b], \tag{2.12}$$

then

$$\left\| (b-a) \int_a^b A(t) B(t) dt - \int_a^b A(t) dt \int_a^b B(t) dt \right\|_1 \tag{2.13}$$
$$\leq 2M \int_a^b (b-t)(t-a) \|B'(t)\|_q dt$$
$$\leq M \times \begin{cases} \frac{1}{2} (b-a)^2 \int_a^b \|B'(t)\|_q dt, \\ \\ 2(b-a)^{2+1/\beta} [B(\beta+1, \beta+1)]^{1/\beta} \left(\int_a^b \|B'(t)\|_q^\alpha dt \right)^{1/\alpha}, \\ \\ \frac{1}{3} (b-a)^3 \sup_{t \in [a,b]} \|B'(t)\|_q \end{cases}$$

for α, $\beta > 1$ with $\frac{1}{\alpha} + \frac{1}{\beta} = 1$, where $B(\cdot, \cdot)$ is the Beta function.

Proof. By (2.12) we get

$$\int_a^b \|A(t) - C\|_p \left[\int_a^t (s-a) \|B'(s)\|_q ds + \int_t^b (b-s) \|B'(s)\|_q ds \right] dt \tag{2.14}$$
$$\leq M \left[\int_a^b \left(\int_a^t (s-a) \|B'(s)\|_q ds \right) dt + \int_a^b \left(\int_t^b (b-s) \|B'(s)\|_q ds \right) dt \right].$$

Using integration by parts, we can state that

$$\begin{aligned}
&\int_a^b \left(\int_a^t (s-a)\left\|B'(s)\right\|_q ds\right) dt \\
&= \left(\int_a^t (s-a)\left\|B'(s)\right\|_q ds\right) t\Big|_a^b - \int_a^b (t-a)\, t \left\|B'(t)\right\|_q dt \\
&= b\int_a^b (s-a)\left\|B'(s)\right\|_q ds - \int_a^b (t-a)\, t\left\|B'(t)\right\|_q dt \\
&= \int_a^b (b-t)(t-a)\left\|B'(t)\right\|_q dt
\end{aligned}$$

and

$$\begin{aligned}
&\int_a^b \left(\int_t^b (b-s)\left\|B'(s)\right\|_q ds\right) dt \\
&= \left(\int_t^b (b-s)\left\|B'(s)\right\|_q ds\right) t\Big|_a^b + \int_a^b (b-t)\, t\left\|B'(t)\right\|_q dt \\
&= -a\int_a^b (b-s)\left\|B'(s)\right\|_q ds + \int_a^b (b-t)\, t\left\|B'(t)\right\|_q dt \\
&= \int_a^b (b-t)(t-a)\left\|B'(t)\right\|_q dt.
\end{aligned}$$

Therefore

$$\begin{aligned}
&\int_a^b \left(\int_a^t (s-a)\left\|B'(s)\right\|_q ds\right) dt + \int_a^b \left(\int_t^b (b-s)\left\|B'(s)\right\|_q ds\right) dt \\
&= 2\int_a^b (b-t)(t-a)\left\|B'(t)\right\|_q dt
\end{aligned}$$

and by (2.14) and (2.8) we get the first inequality in (2.13).

Using Hölder's inequality for $\alpha,\ \beta > 1$ with $\frac{1}{\alpha}+\frac{1}{\beta}=1$, then we get

$$\begin{aligned}
&\int_a^b (b-t)(t-a)\left\|B'(t)\right\|_q dt \\
&\le \begin{cases}
\sup_{t\in[a,b]}\left[(b-t)(t-a)\right]\int_a^b \left\|B'(t)\right\|_q dt, \\[2ex]
\left(\int_a^b \left[(b-t)(t-a)\right]^\beta dt\right)^{1/\beta}\left(\int_a^b \left\|B'(t)\right\|_q^\alpha dt\right)^{1/\alpha}, \\[2ex]
\int_a^b (b-t)(t-a)\, dt \sup_{t\in[a,b]}\left\|B'(t)\right\|_q,
\end{cases} \\
&= \begin{cases}
\frac{1}{4}(b-a)^2\int_a^b \left\|B'(t)\right\|_q dt, \\[2ex]
(b-a)^{2+1/\beta}\left[B(\beta+1,\beta+1)\right]^{1/\beta}\left(\int_a^b \left\|B'(t)\right\|_q^\alpha dt\right)^{1/\alpha}, \\[2ex]
\frac{1}{6}(b-a)^3\sup_{t\in[a,b]}\left\|B'(t)\right\|_q
\end{cases}
\end{aligned}$$

and the last part of (2.13). □

The perturbed Grüss' type inequality also holds:

Theorem 5. *Let $B : [a,b] \to B_q(H)$ be a strongly differentiable function on the interval (a,b) and $A : [a,b] \to B_p(H)$ a continuous function, then for all $C \in B_p(H)$, $D \in B_q(H)$ for $p,q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$,*

$$\begin{aligned}(2.15)\quad &\left\| (b-a)\int_a^b A(t)\,B(t)\,dt - \int_a^b A(t)\,dt \int_a^b B(t)\,dt \right. \\ &\left. - (b-a)\left(\int_a^b \left(t - \frac{a+b}{2}\right) A(t)\,dt\right) D \right\|_1 \\ &\le \int_a^b \|A(t) - C\|_p \\ &\times \left[\int_a^t (s-a)\,\|B'(s) - D\|_q\,ds + \int_t^b (b-s)\,\|B'(s) - D\|_q\,ds\right] dt \\ &= B(A,B,C,D).\end{aligned}$$

We have the bounds for α, $\beta > 1$ with $\frac{1}{\alpha} + \frac{1}{\beta} = 1$,

$$(2.16)\quad B(A,B,C,D) \le \begin{cases} \int_a^b \|A(t) - C\|_p \max\{t-a, b-t\}\,dt \int_a^b \|B'(s) - D\|_q\,ds, \\ \\ \frac{1}{(\beta+1)^{1/\beta}} \int_a^b \|A(t) - C\|_p \left[(t-a)^{\beta+1} + (b-t)^{\beta+1}\right]^{1/\beta} dt \\ \times \left(\int_a^b \|B'(s) - D\|_q^\alpha\,ds\right)^{1/\alpha}, \\ \\ \frac{1}{2}\int_a^b \|A(t) - C\|_p \left[(t-a)^2 + (b-t)^2\right] dt \sup_{s\in[a,b]} \|B'(s) - D\|_q. \end{cases}$$

Moreover, if A is symmetric on $[a,b]$, then

$$(2.17)\quad \left\| (b-a)\int_a^b A(t)\,B(t)\,dt - \int_a^b A(t)\,dt \int_a^b B(t)\,dt \right\|_1 \le B(A,B,C,D)$$

for all $C \in B(H)$, $D \in B(H)$.

The proof is similar to the one of Theorem 4 and we omit the details.

Corollary 3. *With the assumptions of Theorem 5 and if there exists $D \in B_q(H)$ and $L > 0$ such that*

$$(2.18)\quad \|B'(s) - D\|_q \le L \text{ for all } t \in (a,b),$$

then for $\alpha,\ \beta>1$ *with* $\frac{1}{\alpha}+\frac{1}{\beta}=1$,

$$\begin{aligned}(2.19)\quad &\left\|(b-a)\int_a^b A(t)\,B(t)\,dt-\int_a^b A(t)\,dt\int_a^b B(t)\,dt\right.\\ &\left.-(b-a)\left(\int_a^b\left(t-\frac{a+b}{2}\right)A(t)\,dt\right)D\right\|_1\\ &\le\frac{1}{2}L\int_a^b\|A(t)-C\|_p\left[(t-a)^2+(b-t)^2\right]dt\\ &\le\frac{1}{2}L\times\begin{cases}(b-a)^2\int_a^b\|A(t)-C\|_p\,dt,\\ \\ (b-a)^{2+1/\beta}\left(\int_a^b\|A(t)-C\|_p^\alpha\,dt\right)^{1/\alpha}\\ \times\left(\int_0^1\left[t^2+(1-t)^2\right]^\beta dt\right)^{1/\beta},\\ \\ \frac{2}{3}(b-a)^3\sup_{t\in[a,b]}\|A(t)-C\|_p\end{cases}\end{aligned}$$

for all $C\in B_p(H)$.

Moreover, if A *is symmetric on* $[a,b]$, *then for* $\alpha,\ \beta>1$ *with* $\frac{1}{\alpha}+\frac{1}{\beta}=1$,

$$\begin{aligned}(2.20)\quad &\left\|(b-a)\int_a^b A(t)\,B(t)\,dt-\int_a^b A(t)\,dt\int_a^b B(t)\,dt\right\|_1\\ &\le\frac{1}{2}L\int_a^b\|A(t)-C\|_p\left[(t-a)^2+(b-t)^2\right]dt\\ &\le\frac{1}{2}L\times\begin{cases}(b-a)^2\int_a^b\|A(t)-C\|_p\,dt,\\ \\ (b-a)^{2+1/\beta}\left(\int_a^b\|A(t)-C\|_p^\alpha\,dt\right)^{1/\alpha}\\ \times\left(\int_0^1\left[t^2+(1-t)^2\right]^\beta dt\right)^{1/\beta},\\ \\ \frac{2}{3}(b-a)^3\sup_{t\in[a,b]}\|A(t)-C\|_p\end{cases}\end{aligned}$$

for all $C\in B_p(H)$.

Remark 3. *For* $p,\ q>1$ *with* $\frac{1}{p}+\frac{1}{q}=1$, *if there exists* $V,\ Z\in B_q(H)$ *such that*

$$(2.21)\quad \left\|B(t)-\frac{V+Z}{2}\right\|_q\le\frac{1}{2}\|Z-V\|_q\ \text{ for all } t\in[a,b],$$

then by (2.19) we get

$$
\begin{aligned}
(2.22) \quad & \left\| (b-a)\int_a^b A(t)\,B(t)\,dt - \int_a^b A(t)\,dt \int_a^b B(t)\,dt \right. \\
& \left. - (b-a)\left(\int_a^b \left(t-\frac{a+b}{2}\right) A(t)\,dt\right)\frac{V+Z}{2} \right\| \\
& \le \frac{1}{4}\|Z-V\| \int_a^b \|A(t)-C\|_p \left[(t-a)^2+(b-t)^2\right] dt \\
& \le \frac{1}{4}\|Z-V\| \times \begin{cases} (b-a)^2 \int_a^b \|A(t)-C\|_p\,dt, \\ \\ (b-a)^{2+1/\beta}\left(\int_a^b \|A(t)-C\|_p^\alpha\,dt\right)^{1/\alpha} \\ \times\left(\int_0^1 \left[t^2+(1-t)^2\right]^\beta dt\right)^{1/\beta}, \\ \alpha,\ \beta>1,\ \frac{1}{\alpha}+\frac{1}{\beta}=1, \\ \\ \frac{2}{3}(b-a)^3 \sup_{t\in[a,b]} \|A(t)-C\|_p \end{cases}
\end{aligned}
$$

for all $C \in B_p(H)$.

Moreover, if A *is symmetric on* $[a,b]$, *then*

$$
\begin{aligned}
(2.23) \quad & \left\| (b-a)\int_a^b A(t)\,B(t)\,dt - \int_a^b A(t)\,dt \int_a^b B(t)\,dt \right\| \\
& \le \frac{1}{4}\|Z-V\| \int_a^b \|A(t)-C\|_p \left[(t-a)^2+(b-t)^2\right] dt \\
& \le \frac{1}{4}\|Z-V\| \times \begin{cases} (b-a)^2 \int_a^b \|A(t)-C\|_p\,dt, \\ \\ (b-a)^{2+1/\beta}\left(\int_a^b \|A(t)-C\|_p^\alpha\,dt\right)^{1/\alpha} \\ \times\left(\int_0^1 \left[t^2+(1-t)^2\right]^\beta dt\right)^{1/\beta}, \\ \\ \frac{2}{3}(b-a)^3 \sup_{t\in[a,b]} \|A(t)-C\|_p . \end{cases}
\end{aligned}
$$

Remark 4. *If there exists* $C \in B_p(H)$, $D \in B_q(H)$, $M>0$ *and* $L>0$ *such that the conditions (2.12) and (2.18) are satisfied, then*

$$
\begin{aligned}
(2.24) \quad & \left\| (b-a)\int_a^b A(t)\,B(t)\,dt - \int_a^b A(t)\,dt \int_a^b B(t)\,dt \right. \\
& \left. - (b-a)\left(\int_a^b \left(t-\frac{a+b}{2}\right) A(t)\,dt\right) D \right\| \\
& \le \frac{1}{3} LM (b-a)^3 .
\end{aligned}
$$

Moreover, if A is symmetric on $[a,b]$, then

$$\left\| (b-a) \int_a^b A(t) B(t) \, dt - \int_a^b A(t) \, dt \int_a^b B(t) \, dt \right\| \leq \frac{1}{3} LM (b-a)^3 . \tag{2.25}$$

3. Inequalities for Operator Monotone Functions

A real valued continuous function h on $[0,\infty)$ is said to be operator monotone if $h(A) \geq h(B)$ holds for any $A \geq B \geq 0$.

We have the following representation of operator monotone functions, see for instance [2, p. 144-145]:

Theorem 6. *A function $h : [0,\infty) \to \mathbb{R}$ is operator monotone in $[0,\infty)$ if and only if it has the representation*

$$h(t) = h(0) + bt + \int_0^\infty \frac{t\lambda}{t+\lambda} d\mu(\lambda), \tag{3.1}$$

where $b \geq 0$ and a positive measure μ on $[0,\infty)$ such that

$$\int_0^\infty \frac{\lambda}{1+\lambda} d\mu(\lambda) < \infty. \tag{3.2}$$

We have the following representation result:

Lemma 3. *Let $h : [0,\infty) \to \mathbb{R}$ be operator monotone in $[0,\infty)$. Assume that $U \geq 0$, then for all selfadjoint operators V we have*

$$Dh(U)(V) = bV + \int_0^\infty \lambda^2 \left[(\lambda+U)^{-1} V (\lambda+U)^{-1} \right] d\mu(\lambda). \tag{3.3}$$

Proof. From (3.1) we get

$$h(t) = h(0) + bt + \int_0^\infty \left(\lambda - \frac{\lambda^2}{t+\lambda} \right) d\mu(\lambda).$$

Assume that $U \geq 0$, then for all selfadjoint operator V we have, by the representation of h and for t in a small open interval around 0, that

$$\begin{aligned}
& h(U+tV) - h(U) \\
& = btV + \int_0^\infty \left(\lambda - \lambda^2 (U+tV+\lambda)^{-1} \right) d\mu(\lambda) - \int_0^\infty \left(\lambda - \lambda^2 (U+\lambda)^{-1} \right) d\mu(\lambda) \\
& = btV + \int_0^\infty \lambda^2 \left[(\lambda+U)^{-1} - (\lambda+U+tV)^{-1} \right] d\mu(\lambda) \\
& = btV + \int_0^\infty \lambda^2 \left[(\lambda+U)^{-1} (\lambda+U+tV-\lambda-U)(\lambda+U+tV)^{-1} \right] d\mu(\lambda) \\
& = btV + t \int_0^\infty \lambda^2 \left[(\lambda+U)^{-1} V (\lambda+U+tV)^{-1} \right] d\mu(\lambda).
\end{aligned}$$

Dividing by $t \neq 0$, we get

$$\frac{h(U+tV) - h(U)}{t} = bV + \int_0^\infty \lambda^2 \left[(\lambda+U)^{-1} V (\lambda+U+tV)^{-1} \right] d\mu(\lambda)$$

and by taking the limit over $t \to 0$, we get

$$Dh(U)(V) = bV + \int_0^\infty \lambda^2 \left[(\lambda+U)^{-1} V (\lambda+U)^{-1} \right] d\mu(\lambda)$$

for all selfadjoint operator V we have (3.3). □

Lemma 4. *Let $h : [0,\infty) \to \mathbb{R}$ be operator monotone in $[0,\infty)$. Assume that $U \geq u > 0$, then for all selfadjoint operators $V \in B_p(H)$, $p \geq 1$ we have*

$$\|Dh(U)(V)\|_p \leq h'(u)\|V\|_p . \tag{3.4}$$

Proof. From (3.3) and using (1.21) we get

$$\begin{aligned} \|Dh(U)(V) - bV\|_p &\leq \int_0^\infty \lambda^2 \left\|(\lambda+U)^{-1} V (\lambda+U)^{-1}\right\|_p d\mu(\lambda) \\ &\leq \|V\|_p \int_0^\infty \lambda^2 \left\|(\lambda+U)^{-1}\right\|^2 d\mu(\lambda). \end{aligned} \tag{3.5}$$

Observe that $\lambda + U \geq \lambda + u > 0$ for $\lambda \in [0,\infty)$. Then $0 < (\lambda+U)^{-1} \leq (\lambda+u)^{-1}$, which implies that $\left\|(\lambda+U)^{-1}\right\| \leq (\lambda+u)^{-1}$, namely $\left\|(\lambda+U)^{-1}\right\|^2 \leq (\lambda+u)^{-2}$ for $\lambda \in [0,\infty)$.

Therefore by (3.5) we get

$$\|Dh(U)(V) - bV\|_p \leq \|V\|_p \int_0^\infty \lambda^2 (\lambda+u)^{-2} d\mu(\lambda). \tag{3.6}$$

If we take the derivative over t in (3.1) then we have

$$h'(t) = b + \int_0^\infty \frac{\lambda(t+\lambda) - \lambda t}{(t+\lambda)^2} d\mu(\lambda) = b + \int_0^\infty \frac{\lambda^2}{(t+\lambda)^2} d\mu(\lambda) \tag{3.7}$$

for $t > 0$.

From (3.7) we get

$$\int_0^\infty \lambda^2 (\lambda+u)^{-2} d\mu(\lambda) = h'(u) - b,$$

and by (3.6) we derive

$$\|Dh(U)(V) - bV\|_p \leq \|V\|_p h'(u) - b\|V\|_p .$$

Finally, by the triangle inequality and by the fact that $b \geq 0$, we obtain that

$$\|Dh(U)(V)\|_p - b\|V\|_p \leq \|Dh(U)(V) - bV\|_p ,$$

which proves the desired result (3.4). □

For a continuous function h on $(0,\infty)$ and C, $E > 0$ we consider the auxiliary function $h_{C,E} : [0,1] \to \mathbb{R}$ defined by

$$h_{C,E}(t) := h((1-t)C + tE), \ t \in [0,1].$$

We have the following representations of the derivatives:

Lemma 5. *Assume that the operator function generated by h is Fréchet differentiable in each $C \geq 0$, then for $E \geq 0$ we have that $h_{C,E}$ is differentiable on $[0,1]$ and*

$$h'_{C,E}(t) = D(h)((1-t)C + tE)(E - C) \tag{3.8}$$

for $t \in [0,1]$, where in 0 and 1 the derivatives are the right and left derivatives.

Proof. We prove only for the interior points $t \in (0,1)$. Let h be in a small interval around 0 such that $t + h \in (0,1)$. Then for $h \neq 0$,

$$\begin{aligned}
&\frac{h_{C,E}(t+h) - h(t)}{h} \\
&= \frac{h((1-(t+h))C + (t+h)E) - h((1-t)C + tE)}{h} \\
&= \frac{h((1-t)C + tE + h(E-C)) - h((1-t)C + tE)}{h}
\end{aligned}$$

and by taking the limit over $h \to 0$, we get

$$\begin{aligned}
h'_{C,E}(t) &= \lim_{h \to 0} \frac{h_{C,E}(t+h) - h(t)}{h} \\
&= \lim_{h \to 0} \left[\frac{h((1-t)C + tE + h(E-C)) - h((1-t)C + tE)}{h} \right] \\
&= D(h)((1-t)C + tE)(E-C),
\end{aligned}$$

which proves (3.8). □

Corollary 4. *Let $h : [0,\infty) \to \mathbb{R}$ be operator monotone in $[0,\infty)$. Then for all $C \geq c > 0$, $E \geq d > 0$ with $C, E \in B_p(H)$, $p \geq 1$ we have*

$$\begin{aligned}
\left\| h'_{C,E}(t) \right\|_p &= \| D(h)((1-t)C + tE)(E-C) \|_p \\
&\leq h'((1-t)c + td) \| E - C \|_p
\end{aligned} \tag{3.9}$$

for all $t \in [0,1]$.

The proof follows by Lemma 4 and Lemma 5.

For a continuos function $A : [0,1] \to B(H)$ and the operator monotone function $k : [0,\infty) \to \mathbb{R}$ and two operators C, $E \geq 0$ we consider the *Čebyšev functional*

$$\begin{aligned}
&D(A,k,C,E) \\
&:= \int_0^1 A(t)\, k((1-t)C + tE)\, dt - \int_0^1 A(t)\, dt \int_0^1 k((1-t)C + tE)\, dt.
\end{aligned}$$

If we use Theorem 4 and Corollary 4 we then get for $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ and the operator monotone function $k : [0,\infty) \to \mathbb{R}$, that

$$\begin{aligned}
&\| D(A,k,C,E) \|_1 \\
&\leq \| E - C \|_q \\
&\times \begin{cases}
\int_0^1 \| A(t) - Q \|_p \max\{t, 1-t\}\, dt \int_0^1 k'((1-t)c + td)\, dt, \\[2ex]
\frac{1}{(\beta+1)^{1/\beta}} \int_0^1 \| A(t) - Q \|_p \left[t^{\beta+1} + (1-t)^{\beta+1} \right]^{1/\beta} dt \\
\times \left(\int_0^1 \left[k'((1-t)c + td) \right]^\alpha dt \right)^{1/\alpha}, \ \alpha,\ \beta > 1,\ \frac{1}{\alpha} + \frac{1}{\beta} = 1, \\[2ex]
\frac{1}{2} \int_0^1 \| A(t) - Q \|_p \left[t^2 + (1-t)^2 \right] dt \sup_{t \in (0,1)} k'((1-t)c + td),
\end{cases}
\end{aligned} \tag{3.10}$$

where $A : [0,1] \to B_p(H)$ is a continuous function, $Q \in B_p(H)$ and $C \geq c > 0$, $E \geq d > 0$ with $C, E \in B_q(H)$.

Since

$$\int_0^1 k'((1-t)c+td)\,dt = \begin{cases} \frac{k(d)-k(c)}{d-c} & \text{if } c \neq d, \\ \\ k'(c) & \text{if } c = d, \end{cases}$$

hence by (3.10) we derive the simpler inequality

$$\left\| \int_0^1 A(t)\,k((1-t)C+tE)\,dt - \int_0^1 A(t)\,dt \int_0^1 k((1-t)C+tE)\,dt \right\|_1 \tag{3.11}$$
$$\leq \|E-C\|_q \int_0^1 \|A(t)-Q\|_p \max\{t,1-t\}\,dt$$
$$\times \begin{cases} \frac{k(d)-k(c)}{d-c} & \text{if } c \neq d, \\ \\ k'(c) & \text{if } c = d. \end{cases}$$

Moreover, if $\|A(t)-Q\|_p \leq M$ for $t \in [0,1]$, then by (3.11) we derive

$$\left\| \int_0^1 A(t)\,k((1-t)C+tE)\,dt - \int_0^1 A(t)\,dt \int_0^1 k((1-t)C+tE)\,dt \right\|_1 \tag{3.12}$$
$$\leq \frac{3}{4}\|E-C\|_q \times \begin{cases} \frac{k(d)-k(c)}{d-c} & \text{if } c \neq d, \\ \\ k'(c) & \text{if } c = d. \end{cases}$$

From (3.11) we derive for the operator monotone function $k(x) = x^r$, $r \in (0,1)$ that

$$\left\| \int_0^1 A(t)\,((1-t)C+tE)^r\,dt - \int_0^1 A(t)\,dt \int_0^1 ((1-t)C+tE)^r\,dt \right\|_1 \tag{3.13}$$
$$\leq \|E-C\|_q \int_0^1 \|A(t)-Q\|_p \max\{t,1-t\}\,dt$$
$$\times \begin{cases} \frac{d^r-c^r}{d-c} & \text{if } c \neq d, \\ \\ \frac{r}{c^{1-r}} & \text{if } c = d. \end{cases}$$

From (3.11) we derive for the operator monotone function $k(x) = \ln x$ that

$$\left\| \int_0^1 A(t)\ln((1-t)C+tE)\,dt - \int_0^1 A(t)\,dt \int_0^1 \ln((1-t)C+tE)\,dt \right\|_1 \tag{3.14}$$
$$\leq \|E-C\|_q \int_0^1 \|A(t)-Q\|_p \max\{t,1-t\}\,dt$$
$$\times \begin{cases} \frac{\ln d-\ln c}{d-c} & \text{if } c \neq d, \\ \\ \frac{1}{c} & \text{if } c = d. \end{cases}$$

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