p-SCHATTEN NORM INEQUALITIES OF OPIAL-LASOTA'S TYPE

SILVESTRU SEVER DRAGOMIR[1,2]

Abstract. An operator $A \in \mathcal{B}(H)$ is said to belong to the *von Neumann-Schatten class* $\mathcal{B}_p(H)$, $1 \leq p < \infty$ if the *p-Schatten norm* is finite

$$\|A\|_p := [\operatorname{tr}(|A|^p)]^{1/p} < \infty.$$

In this paper we show among others that, for $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, assume that $\{A_i\}_{i=0}^{N} \subset B_p(H) \cap B_q(H)$ are sequences of operators with $A_0 = A_N = 0$, then for $n \in \{2, ..., N-1\}$,

$$\sum_{i=1}^{N-1} \|\Delta A_i A_i\|_1 \leq \left(\sum_{i=0}^{N-1} p_i(n) \|\Delta A_i\|_p^2\right)^{1/2} \left(\sum_{i=0}^{N-1} q_i(n) \|\Delta A_i\|_q^2\right)^{1/2}$$
$$\leq \frac{1}{2} \sum_{i=0}^{N-1} \left(p_i(n) \|\Delta A_i\|_p^2 + q_i(n) \|\Delta A_i\|_q^2\right),$$

where $\Delta A_i := A_{i+1} - A_i$, $i \in \{0, ..., N-1\}$ is the forward difference and

$$p_i(n) := \begin{cases} i, & \text{if } 0 \leq i \leq n-1, \\ N-i, & \text{if } n \leq i \leq N-1 \end{cases}$$

while

$$q_i(n) := \begin{cases} n-i-1, & \text{if } 0 \leq i \leq n-1, \\ i+1-n, & \text{if } n \leq i \leq N-1. \end{cases}$$

1. Introduction

We recall the following Opial type inequalities:

Theorem 1. *Assume that $u : [a, b] \subset \mathbb{R} \to \mathbb{R}$ is an absolutely continuous function on the interval $[a, b]$ and such that $u' \in L_2[a, b]$.*

(i) *If $u(a) = u(b) = 0$, then*

$$\int_a^b |u(t) u'(t)| \, dt \leq \frac{1}{4}(b-a) \int_a^b |u'(t)|^2 \, dt, \tag{1.1}$$

with equality if and only if

$$u(t) = \begin{cases} c(t-a) & \text{if } a \leq t \leq \frac{a+b}{2}, \\ c(b-t) & \text{if } \frac{a+b}{2} < t \leq b \end{cases}$$

where c is an arbitrary constant;

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(ii) *If* $u(a)=0$, *then*

$$\int_a^b |u(t)u'(t)|\,dt \le \frac{1}{2}(b-a)\int_a^b |u'(t)|^2\,dt, \tag{1.2}$$

with equality if and only if $u(t)=c(t-a)$ *for some constant* c;

(iii) *If* $\int_a^b u(t)\,dt=0$, *then the inequality (1.1) holds with equality if and only if*

$$u(t)=c\left(t-\frac{a+b}{2}\right)$$

for any constant c.

The inequality (1.1) was obtained by Olech in [14] in which he gave a simplified proof of an inequality originally due in a slightly less general form to Zdzislaw Opial [15].

Embedded in Olech's proof is the half-interval form of Opial's inequality, also discovered by Beesack [2], which is satisfied by those u vanishing only at a.

The inequality (1.1) in the case (iii), namely in the case that u satisfies the condition $\int_a^b u(t)\,dt=0$ was obtained by Brown and Plum in [6].

As mentioned in [6] the inequality (1.1) also holds if $u(a)+u(b)=0$.

For a sequence $\{x_i\}_{i=0}^n$, we consider the forward operator Δ defined by $\Delta x_i = x_{i+1}-x_i$, $i=0,...,n-1$. The summation by parts formula also holds

$$\sum_{k=m}^{n} a_k\Delta b_k = a_n b_{n+1} - a_m b_m - \sum_{k=m}^{n-1} b_{k+1}\Delta a_k. \tag{1.3}$$

In [12], Lasota provided discrete versions of Opial inequality (1.1) about the forward difference operator as follows:

Theorem 2. *Let* $\{x_i\}_{i=0}^N$ *be a sequence of real numbers with* $x_0=x_N=0$. *Then, the following inequality holds*

$$\sum_{i=1}^{N-1} |x_i\Delta x_i| \le \frac{1}{2}\left\lfloor\frac{N+1}{2}\right\rfloor \sum_{i=0}^{N-1} |\Delta x_i|^2, \tag{1.4}$$

where $\lfloor\cdot\rfloor$ *is the integer part function. If* N *is even, then the inequality (1.4) is sharp.*

Also, we have the following results, see [1]:

Theorem 3. *Let* $\{x_i\}_{i=0}^N$ *be a sequence of real numbers. If* $x_0=0$, *then*

$$\sum_{i=1}^{\tau-1} |x_i\Delta x_i| \le \frac{1}{2}(\tau-1)\sum_{i=0}^{\tau-1} |\Delta x_i|^2,\ \tau\in\{2,...,N\}. \tag{1.5}$$

If $x_N=0$, *then*

$$\sum_{i=\tau}^{N-1} |x_i\Delta x_i| \le \frac{1}{2}(N-\tau+1)\sum_{i=0}^{N-1} |\Delta x_i|^2,\ \tau\in\{1,...,N-1\}. \tag{1.6}$$

For other discrete Opial type inequalities, see [10], [11] and [16]-[20].

In order to extend these results for p-Schatten norms we need the following preparations.

Let $(H;\langle .,.\rangle)$ be a complex Hilbert space and $\mathcal{B}(H)$ the Banach algebra of all bounded linear operators on H. If $\{e_i\}_{i\in I}$ an orthonormal basis of H, we say that $A\in\mathcal{B}(H)$ is of *trace class* if

$$\|A\|_1 := \sum_{i\in I}\langle |A|\, e_i, e_i\rangle < \infty. \tag{1.7}$$

The definition of $\|A\|_1$ does not depend on the choice of the orthonormal basis $\{e_i\}_{i\in I}$. We denote by $\mathcal{B}_1(H)$ the set of trace class operators in $\mathcal{B}(H)$.

We define the *trace* of a trace class operator $A\in\mathcal{B}_1(H)$ to be

$$\operatorname{tr}(A) := \sum_{i\in I}\langle Ae_i, e_i\rangle, \tag{1.8}$$

where $\{e_i\}_{i\in I}$ an orthonormal basis of H. Note that this coincides with the usual definition of the trace if H is finite-dimensional. We observe that the series (1.8) converges absolutely and it is independent from the choice of basis.

The following result collects some properties of the trace:

Theorem 4. *We have:*

(i) If $A\in\mathcal{B}_1(H)$ then $A^*\in\mathcal{B}_1(H)$ and

$$\operatorname{tr}(A^*) = \overline{\operatorname{tr}(A)}; \tag{1.9}$$

(ii) If $A\in\mathcal{B}_1(H)$ and $T\in\mathcal{B}(H)$, then AT, $TA\in\mathcal{B}_1(H)$ and

$$\operatorname{tr}(AT) = \operatorname{tr}(TA) \ \ \text{and}\ \ |\operatorname{tr}(AT)| \le \|A\|_1\|T\|; \tag{1.10}$$

(iii) $\operatorname{tr}(\cdot)$ is a bounded linear functional on $\mathcal{B}_1(H)$ with $\|\operatorname{tr}\|=1$;
(iv) If A, $B\in\mathcal{B}_2(H)$ then AB, $BA\in\mathcal{B}_1(H)$ and $\operatorname{tr}(AB)=\operatorname{tr}(BA)$;
(v) $\mathcal{B}_{fin}(H)$ is a dense subspace of $\mathcal{B}_1(H)$.

An operator $A\in\mathcal{B}(H)$ is said to belong to the *von Neumann-Schatten class* $\mathcal{B}_p(H)$, $1\le p<\infty$ if the *p-Schatten norm* is finite [21, p. 60-64]

$$\|A\|_p := \left[\operatorname{tr}(|A|^p)\right]^{1/p} < \infty.$$

For $1<p<q<\infty$ we have that

$$\mathcal{B}_1(H)\subset\mathcal{B}_p(H)\subset\mathcal{B}_q(H)\subset\mathcal{B}(H) \tag{1.11}$$

and

$$\|A\|_1 \ge \|A\|_p \ge \|A\|_q \ge \|A\|. \tag{1.12}$$

For $p\ge 1$ the functional $\|\cdot\|_p$ is a *norm* on the $*$-ideal $\mathcal{B}_p(H)$ and $\left(\mathcal{B}_p(H), \|\cdot\|_p\right)$ is a Banach space.

Also, see for instance [21, p. 60-64],

$$\|A\|_p = \|A^*\|_p,\ A\in\mathcal{B}_p(H) \tag{1.13}$$

$$\|AB\|_p \le \|A\|_p\|B\|_p,\ A,B\in\mathcal{B}_p(H) \tag{1.14}$$

and

$$\|AB\|_p \le \|A\|_p\|B\|,\ \ \|BA\|_p \le \|B\|\,\|A\|_p,\ A\in\mathcal{B}_p(H),\ B\in\mathcal{B}(H). \tag{1.15}$$

This implies that

$$\|CAB\|_p \le \|C\|\,\|A\|_p\|B\|,\ A\in\mathcal{B}_p(H),\ B,\ C\in\mathcal{B}(H). \tag{1.16}$$

In terms of *p-Schatten norm* we have the *Hölder inequality* for $p,\ q>1$ with $\frac{1}{p}+\frac{1}{q}=1$

$$(|\mathrm{tr}\,(AB)|\leq)\ \|AB\|_1\leq\|A\|_p\,\|B\|_q\,,\ A\in\mathcal{B}_p\,(H)\,,\ B\in\mathcal{B}_q\,(H)\,. \tag{1.17}$$

For the theory of trace functionals and their applications the reader is referred to [18] and [21].

For some classical trace inequalities see [7], [8], and [13], which are continuations of the work of Bellman [2].

2. 1-Schatten Norm Inequalities

We have the following result for two sequences:

Theorem 5. *For $p,\ q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, assume that $\{A_i\}_{i=0}^N\subset B_p\,(H)$, $\{B_i\}_{i=0}^N\subset B_q\,(H)$ are sequences of operators with $B_0=0$, then for $n\in\{2,...,N\}$,*

$$\begin{aligned}\sum_{i=1}^{n-1}\|\Delta A_iB_i\|_1 &\leq\left(\sum_{i=0}^{n-1}i\,\|\Delta A_i\|_p^2\right)^{1/2}\left(\sum_{i=0}^{n-1}(n-i-1)\,\|\Delta B_i\|_q^2\right)^{1/2}\\&\leq\frac{1}{2}\sum_{i=0}^{n-1}\left[i\,\|\Delta A_i\|_p^2+(n-i-1)\,\|\Delta B_i\|_q^2\right]\\&\leq\frac{1}{2}\,(n-1)\times\begin{cases}n\max_{i\in\{0,...,n-1\}}\left\{\|\Delta A_i\|_p^2\,,\|\Delta B_i\|_q^2\right\},\\[2ex]\sum_{i=0}^{n-1}\left(\|\Delta A_i\|_p^2+\|\Delta B_i\|_q^2\right).\end{cases}\end{aligned} \tag{2.1}$$

Proof. Let $n\in\{2,...,N\}$. Since $B_0=0$, hence $B_i=\sum_{j=0}^{i-1}\Delta B_j$ for $i=1,...,n-1$. Then by Hölder's inequality (1.17)

$$\begin{aligned}\sum_{i=1}^{n-1}\|\Delta A_iB_i\|_1&\leq\sum_{i=1}^{n-1}\|\Delta A_i\|_p\,\|B_i\|_q\\&=\sum_{i=1}^{n-1}\|\Delta A_i\|_p\left\|\sum_{j=0}^{i-1}\Delta B_j\right\|_q\\&=\sum_{i=1}^{n-1}\sqrt{i}\,\|\Delta A_i\|_p\,\frac{1}{\sqrt{i}}\left\|\sum_{j=0}^{i-1}\Delta B_j\right\|_q=:\Gamma.\end{aligned}$$

By the discrete Cauchy-Bunyakowsky-Schwarz (CBS) inequality, we have

$$\Gamma\leq\left(\sum_{i=1}^{n-1}i\,\|\Delta A_i\|_p^2\right)^{1/2}\left[\sum_{i=1}^{n-1}\frac{1}{i}\left\|\sum_{j=0}^{i-1}\Delta B_j\right\|_q^2\right]^2=:\Delta.$$

By (CBS) inequality we also have

$$\frac{1}{i}\left\|\sum_{j=0}^{i-1}\Delta B_j\right\|_q^2\leq\sum_{j=0}^{i-1}\|\Delta B_j\|_q^2\,,$$

which gives

$$\Delta \leq \left(\sum_{i=1}^{n-1} i \left\|\Delta A_i\right\|_q^2\right)^{1/2} \left(\sum_{i=1}^{n-1} \left(\sum_{j=0}^{i-1} \left\|\Delta B_j\right\|_q^2\right)\right)^{1/2}. \tag{2.2}$$

From (1.3), we have for $m = 1$ and n is replaced by $n-1$ that

$$\sum_{i=1}^{n-1} a_i \Delta b_i = a_{n-1} b_n - a_1 b_1 - \sum_{i=1}^{n-2} b_{i+1} \Delta a_i,$$

which by taking $a_i = \sum_{j=0}^{i-1} \left\|\Delta B_j\right\|_q^2$, $b_i = i$, produces that

$$\begin{aligned}
&\sum_{i=1}^{n-1} \left(\sum_{j=0}^{i-1} \left\|\Delta B_j\right\|_q^2\right) \\
&= n \sum_{j=0}^{n-2} \left\|\Delta B_j\right\|_q^2 - \left\|\Delta B_0\right\|_q^2 - \sum_{i=1}^{n-2} (i+1) \left\|\Delta B_i\right\|_q^2 \\
&= n \left\|\Delta B_0\right\|_q^2 - \left\|\Delta B_0\right\|_q^2 + n \sum_{j=1}^{n-2} \left\|\Delta B_j\right\|_q^2 - \sum_{i=1}^{n-2} (i+1) \left\|\Delta B_i\right\|_q^2 \\
&= (n-1) \left\|\Delta B_0\right\|_q^2 + \sum_{i=1}^{n-2} (n-i-1) \left\|\Delta B_i\right\|_q^2 = \sum_{i=0}^{n-2} (n-i-1) \left\|\Delta B_i\right\|_q^2.
\end{aligned}$$

Now, it is obvious that

$$\sum_{i=1}^{n-1} i \left\|\Delta A_i\right\|_p^2 = \sum_{i=0}^{n-1} i \left\|\Delta A_i\right\|_p^2$$

and

$$\sum_{i=0}^{n-2} (n-i-1) \left\|\Delta B_i\right\|_q^2 = \sum_{i=0}^{n-1} (n-i-1) \left\|\Delta B_i\right\|_q^2.$$

By utilising (2.2) we derive the first part of (2.1). The second part follows by the A-G-means inequality,

$$\sqrt{ab} \leq \frac{a+b}{2}, \; a, \; b \geq 0.$$

Now, we have

$$\begin{aligned}
&\sum_{i=0}^{n-1} \left[i \left\|\Delta A_i\right\|_p^2 + (n-i-1) \left\|\Delta B_i\right\|_q^2\right] \\
&\leq \max_{i \in \{0,\dots,n-1\}} \left\{\left\|\Delta A_i\right\|_p^2, \left\|\Delta B_i\right\|_q^2\right\} \sum_{i=0}^{n-1} [i + (n-i-1)] \\
&= n(n-1) \max_{i \in \{0,\dots,n-1\}} \left\{\left\|\Delta A_i\right\|_p^2, \left\|\Delta B_i\right\|_q^2\right\},
\end{aligned}$$

which proves the first branch.

For the second branch, we have

$$\begin{aligned}
&\sum_{i=0}^{n-1}\left[i\left\|\Delta A_i\right\|_p^2+(n-i-1)\left\|\Delta B_i\right\|_q^2\right] \\
&\leq \max_{i\in\{0,\ldots,n\}}\{i,n-i-1\}\sum_{i=0}^{n-1}\left[\left\|\Delta A_i\right\|_p^2+\left\|\Delta B_i\right\|_q^2\right] \\
&=(n-1)\sum_{i=0}^{n-1}\left[\left\|\Delta A_i\right\|_p^2+\left\|\Delta B_i\right\|_q^2\right].
\end{aligned}$$

□

The case of one sequence, is as follows:

Corollary 1. *For* p, $q>1$ *with* $\frac{1}{p}+\frac{1}{q}=1$, *assume that* $\{A_i\}_{i=0}^N\subset B_p(H)\cap B_q(H)$ *with* $A_0=0$, *then for* $n\in\{2,...,N\}$, *we have*

$$\begin{aligned}
\sum_{i=1}^{n-1}\left\|\Delta A_iA_i\right\|_1 &\leq \left(\sum_{i=0}^{n-1}i\left\|\Delta A_i\right\|_p^2\right)^{1/2}\left(\sum_{i=0}^{n-1}(n-i-1)\left\|\Delta A_i\right\|_q^2\right)^{1/2} \\
&\leq \frac{1}{2}\sum_{i=0}^{n-1}\left[i\left\|\Delta A_i\right\|_p^2+(n-i-1)\left\|\Delta A_i\right\|_q^2\right] \\
&\leq \frac{1}{2}(n-1)\times\begin{cases} n\max_{i\in\{0,\ldots,n-1\}}\left\{\left\|\Delta A_i\right\|_p^2,\left\|\Delta A_i\right\|_q^2\right\}, \\ \\ \sum_{i=0}^{n-1}\left(\left\|\Delta A_i\right\|_p^2+\left\|\Delta A_i\right\|_q^2\right).\end{cases}
\end{aligned}\tag{2.3}$$

In particular, if $\{A_i\}_{i=0}^N\subset B_2(H)$ *with* $A_0=0$, *then for* $n\in\{2,...,N\}$, *we have*

$$\begin{aligned}
\sum_{i=1}^{n-1}\left\|\Delta A_iA_i\right\|_1 &\leq \left(\sum_{i=0}^{n-1}i\left\|\Delta A_i\right\|_2^2\right)^{1/2}\left(\sum_{i=0}^{n-1}(n-i-1)\left\|\Delta A_i\right\|_2^2\right)^{1/2} \\
&\leq \frac{1}{2}(n-1)\sum_{i=0}^{n-1}\left\|\Delta A_i\right\|_2^2.
\end{aligned}\tag{2.4}$$

By using the property (1.10) we have by (2.4) and the triangle inequality the following trace inequality

$$\begin{aligned}
&\left|\operatorname{tr}\left(\sum_{i=1}^{n-1}\Delta A_iA_i\right)\right| \\
&\leq \left[\operatorname{tr}\left(\sum_{i=0}^{n-1}i\left|\Delta A_i\right|^2\right)\right]^{1/2}\left[\operatorname{tr}\left(\sum_{i=0}^{n-1}(n-i-1)\left|\Delta A_i\right|^2\right)\right]^{1/2} \\
&\leq \frac{1}{2}(n-1)\operatorname{tr}\left(\sum_{i=0}^{n-1}\left|\Delta A_i\right|^2\right)
\end{aligned}\tag{2.5}$$

provided that $\{A_i\}_{i=0}^N\subset B_2(H)$ with $A_0=0$ and $n\in\{2,...,N\}$.

We also have:

Theorem 6. *For $p,\ q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, assume that $\{A_i\}_{i=0}^{N}\subset B_p(H)$, $\{B_i\}_{i=0}^{N}\subset B_q(H)$ are sequences of operators with $B_N=0$, then for $n\in\{1,...,N-1\}$,*

$$\begin{aligned}
(2.6)\qquad &\sum_{i=n}^{N-1}\left\|\Delta A_i B_i\right\|_1 \\
&\le \left(\sum_{i=n}^{N-1}(N-i)\left\|\Delta A_i\right\|_p^2\right)^{1/2}\left(\sum_{i=n}^{N-1}(i+1-n)\left\|\Delta B_i\right\|_q^2\right)^{1/2} \\
&\le \frac{1}{2}\sum_{i=n}^{N-1}\left[(N-i)\left\|\Delta A_i\right\|_p^2+(i+1-n)\left\|\Delta B_i\right\|_q^2\right] \\
&\le \frac{1}{2}(N-n)\times\begin{cases}(N-n+1)\max_{i\in\{n,...,N-1\}}\left\{\left\|\Delta A_i\right\|_p^2,\left\|\Delta B_i\right\|_q^2\right\}, \\ \sum_{i=n}^{N-1}\left(\left\|\Delta A_i\right\|_p^2+\left\|\Delta B_i\right\|_q^2\right).\end{cases}
\end{aligned}$$

Proof. If $B_N=0$, then $B_i=-\sum_{j=i}^{N-1}\Delta B_j$ for $i=n+1,...,N-1$. Then by (1.17)

$$\begin{aligned}
\sum_{i=n}^{N-1}\left\|\Delta A_i B_i\right\|_1 &\le \sum_{i=n}^{N-1}\left\|\Delta A_i\right\|_p\left\|B_i\right\|_q=\sum_{i=n}^{N-1}\left\|\Delta A_i\right\|_p\left\|\sum_{j=i}^{N-1}\Delta B_j\right\|_q \\
&=\sum_{i=n}^{N-1}\sqrt{N-i}\left\|\Delta A_i\right\|_p\frac{1}{\sqrt{N-i}}\left\|\sum_{j=i}^{N-1}\Delta B_j\right\|_q=:\Theta.
\end{aligned}$$

By the discrete Cauchy-Bunyakowsky-Schwarz (CBS) inequality, we have

$$\begin{aligned}
(2.7)\qquad \Theta &\le \left(\sum_{i=n}^{N-1}(N-i)\left\|\Delta A_i\right\|_p^2\right)^{1/2}\left(\sum_{i=n}^{N-1}\frac{1}{N-i}\left\|\sum_{j=i}^{N-1}\Delta B_j\right\|_q^2\right)^{1/2} \\
&\le \left(\sum_{i=n}^{N-1}(N-i)\left\|\Delta A_i\right\|_p^2\right)^{1/2}\left(\sum_{i=n}^{N-1}\left(\sum_{j=i}^{N-1}\left\|\Delta B_j\right\|_q^2\right)\right)^{1/2}=:\Lambda
\end{aligned}$$

From (1.3) we have

$$\sum_{i=n}^{N-1}a_i\Delta b_i=a_{N-1}b_N-a_nb_n-\sum_{i=n}^{N-2}b_{i+1}\Delta a_i,$$

and by $a_i = \sum_{j=i}^{N-1} |\Delta B_i|^2$ and $b_i = i$, we have

$$\begin{aligned}
&\sum_{i=n}^{N-1} \left(\sum_{j=i}^{N-1} \|\Delta B_i\|_q^2 \right) \\
&= N \|\Delta B_{N-1}\|_q^2 - n \sum_{j=n}^{N-1} \|\Delta B_i\|_q^2 - \sum_{i=n}^{N-2} (i+1) \left(\sum_{j=i+1}^{N-1} \|\Delta B_i\|_q^2 - \sum_{j=i}^{N-1} \|\Delta B_j\|_q^2 \right) \\
&= N \|\Delta B_{N-1}\|_q^2 + \sum_{i=n}^{N-2} (i+1) \|\Delta B_i\|_q^2 - n \sum_{j=n}^{N-1} \|\Delta B_i\|_q^2 \\
&= N \|\Delta B_{N-1}\|_q^2 + \sum_{i=n}^{N-2} (i+1) \|\Delta B_i\|_q^2 - n \sum_{j=n}^{N-2} \|\Delta B_i\|_q^2 - n \|\Delta B_{N-1}\|_q^2 \\
&= \sum_{i=n}^{N-2} (i+1) \|\Delta B_i\|_q^2 - n \sum_{j=n}^{N-2} \|\Delta B_i\|_q^2 \\
&= \sum_{i=n}^{N-2} (i+1-n) \|\Delta B_i\|_q^2 + (N-n) \|\Delta B_{N-1}\|_q^2 = \sum_{i=n}^{N-1} (i+1-n) \|\Delta B_i\|_q^2 .
\end{aligned}$$

Then

$$\Lambda = \left(\sum_{i=n}^{N-1} (N-i) \|\Delta A_i\|_p^2 \right)^{1/2} \left(\sum_{i=n}^{N-1} (i+1-n) \|\Delta B_i\|_q^2 \right)^{1/2},$$

which, by (2.7), proves the first inequality in (2.6).

The second part follows by A-G-means inequality. The last part is obvious. □

Corollary 2. *For* $p, q > 1$ *with* $\frac{1}{p} + \frac{1}{q} = 1$, *assume that* $\{A_i\}_{i=0}^N \subset B_p(H) \cap B_q(H)$ *with* $A_N = 0$, *then for* $n \in \{1, ..., N-1\}$, *we have*

$$\begin{aligned}
&\sum_{i=n}^{N-1} \|\Delta A_i A_i\|_1 \\
&\le \left(\sum_{i=n}^{N-1} (N-i) \|\Delta A_i\|_p^2 \right)^{1/2} \left(\sum_{i=n}^{N-1} (i+1-n) \|\Delta A_i\|_q^2 \right)^{1/2} \\
&\le \frac{1}{2} \sum_{i=n}^{N-1} \left[(N-i) \|\Delta A_i\|_p^2 + (i+1-n) \|\Delta A_i\|_q^2 \right] \\
&\le \frac{1}{2} (N-n) \times \begin{cases} (N-n+1) \max_{i \in \{n,...,N-1\}} \left\{ \|\Delta A_i\|_p^2 , \|\Delta A_i\|_q^2 \right\}, \\[2ex] \sum_{i=n}^{N-1} \left(\|\Delta A_i\|_p^2 + \|\Delta A_i\|_q^2 \right). \end{cases}
\end{aligned} \tag{2.8}$$

In particular, if $\{A_i\}_{i=0}^{N} \subset B_2(H)$ *with* $A_N = 0$, *then for* $n \in \{1, ..., N-1\}$, *we have*

$$\sum_{i=n}^{N-1} \|\Delta A_i A_i\| \leq \left(\sum_{i=n}^{N-1} (N-i) \|\Delta A_i\|_2^2\right)^{1/2} \left(\sum_{i=n}^{N-1} (i+1-n) \|\Delta A_i\|_2^2\right)^{1/2} \leq \frac{1}{2}(N-n+1) \sum_{i=n}^{N-1} \|\Delta A_i\|_2^2. \tag{2.9}$$

From (2.9) we get

$$\left|\operatorname{tr}\left(\sum_{i=n}^{N-1} \Delta A_i A_i\right)\right| \leq \left[\operatorname{tr}\left(\sum_{i=n}^{N-1} (N-i) |\Delta A_i|^2\right)\right]^{1/2} \left[\operatorname{tr}\left(\sum_{i=n}^{N-1} (i+1-n) |\Delta A_i|^2\right)\right]^{1/2} \leq \frac{1}{2}(N-n+1) \operatorname{tr}\left(\sum_{i=n}^{N-1} |\Delta A_i|^2\right), \tag{2.10}$$

provided $\{A_i\}_{i=0}^{N} \subset B_2(H)$ with $A_N = 0$ and for $n \in \{1, ..., N-1\}$.

We also have the following result that incorporates both cases:

Theorem 7. *For* $p,\ q > 1$ *with* $\frac{1}{p} + \frac{1}{q} = 1$, *assume that* $\{A_i\}_{i=0}^{N} \subset B_p(H)$, $\{B_i\}_{i=0}^{N} \subset B_q(H)$ *are sequences of operators with* $B_0 = B_N = 0$, *then for* $n \in \{2, ..., N-1\}$,

$$\sum_{i=1}^{N-1} \|\Delta A_i B_i\|_1 \leq \left(\sum_{i=0}^{N-1} p_i(n) \|\Delta A_i\|_p^2\right)^{1/2} \left(\sum_{i=0}^{N-1} q_i(n) \|\Delta B_i\|_q^2\right)^{1/2} \leq \frac{1}{2} \sum_{i=0}^{N-1} \left(p_i(n) \|\Delta A_i\|_p^2 + q_i(n) \|\Delta B_i\|_q^2\right), \tag{2.11}$$

where

$$p_i(n) := \begin{cases} i, & \text{if } 0 \leq i \leq n-1, \\ N-i, & \text{if } n \leq i \leq N-1 \end{cases}$$

and

$$q_i(n) := \begin{cases} n-i-1, & \text{if } 0 \leq i \leq n-1, \\ i+1-n, & \text{if } n \leq i \leq N-1. \end{cases}$$

Proof. We have for $n \in \{2, ..., N-1\}$ that

$$\sum_{i=1}^{n-1} \|\Delta A_i B_i\|_1 \leq \sum_{i=1}^{n-1} \|\Delta A_i\|_p \|B_i\|_q \leq \left(\sum_{i=0}^{n-1} i |\Delta A_i|^2\right)^{1/2} \left(\sum_{i=0}^{n-1} (n-i-1) |\Delta B_i|^2\right)^{1/2}$$

and

$$\begin{aligned}\sum_{i=n}^{N-1}\left\|\Delta A_i B_i\right\|_1 &\le \sum_{i=n}^{N-1}\left\|\Delta A_i\right\|_p\left\|B_i\right\|_q\\ &\le \left(\sum_{i=n}^{N-1}(N-i)\left|\Delta A_i\right|^2\right)^{1/2}\left(\sum_{i=n}^{N-1}(i+1-n)\left|\Delta B_i\right|^2\right)^{1/2}.\end{aligned}$$

If we add these inequalities, then we get, by the elementary inequality

$$ab+cd\le\left(a^2+c^2\right)^{1/2}\left(b^2+d^2\right)^{1/2},\ a,\ b,\ c,\ d\ge 0$$

that

$$\begin{aligned}\sum_{i=1}^{N-1}\left\|\Delta A_i B_i\right\|_1 &\le \left(\sum_{i=0}^{n-1} i\left\|\Delta A_i\right\|_p^2\right)^{1/2}\left(\sum_{i=0}^{n-1}(n-i-1)\left\|\Delta B_i\right\|_q^2\right)^{1/2}\\ &+\left(\sum_{i=n}^{N-1}(N-i)\left\|\Delta A_i\right\|_p^2\right)^{1/2}\left(\sum_{i=n}^{N-1}(i+1-n)\left\|\Delta B_i\right\|_q^2\right)^{1/2}\\ &\le\left(\sum_{i=0}^{n-1} i\left\|\Delta A_i\right\|_p^2+\sum_{i=n}^{N-1}(N-i)\left\|\Delta A_i\right\|_p^2\right)^{1/2}\\ &\times\left(\sum_{i=0}^{n-1}(n-i-1)\left\|\Delta B_i\right\|_q^2+\sum_{i=n}^{N-1}(i+1-n)\left\|\Delta B_i\right\|_q^2\right)^{1/2}\\ &=\left(\sum_{i=0}^{N-1}p_i\left\|\Delta A_i\right\|_p^2\right)^{1/2}\left(\sum_{i=0}^{N-1}q_i\left\|\Delta B_i\right\|_q^2\right)^{1/2},\end{aligned}$$

which proves the first inequality in (2.11).

The second inequality follows by A-G-means inequality. □

Corollary 3. *For $p,\ q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, assume that $\{A_i\}_{i=0}^N\subset B_p(H)\cap B_q(H)$ are sequences of operators with $A_0=A_N=0$, then for $n\in\{2,...,N-1\}$,*

$$\begin{aligned}\sum_{i=1}^{N-1}\left\|\Delta A_i A_i\right\|_1 &\le\left(\sum_{i=0}^{N-1}p_i(n)\left\|\Delta A_i\right\|_p^2\right)^{1/2}\left(\sum_{i=0}^{N-1}q_i(n)\left\|\Delta A_i\right\|_q^2\right)^{1/2}\\ &\le\frac{1}{2}\sum_{i=0}^{N-1}\left(p_i(n)\left\|\Delta A_i\right\|_p^2+q_i(n)\left\|\Delta A_i\right\|_q^2\right),\end{aligned}\tag{2.12}$$

In particular, if $\{A_i\}_{i=0}^N\subset B_2(H)$ with $A_0=A_N=0$, then

$$\begin{aligned}\sum_{i=1}^{N-1}\left\|\Delta A_i A_i\right\|_1 &\le\left(\sum_{i=0}^{N-1}p_i(n)\left\|\Delta A_i\right\|_2^2\right)^{1/2}\left(\sum_{i=0}^{N-1}q(n)_i\left\|\Delta A_i\right\|_2^2\right)^{1/2}\\ &\le\frac{1}{2}\sum_{i=0}^{N-1}s_i(n)\left\|\Delta A_i\right\|_2^2\end{aligned}\tag{2.13}$$

where

$$s_i(n) := \begin{cases} n-1, & \text{if } 0 \le i \le n-1, \\ N-n+1, & \text{if } n \le i \le N-1. \end{cases}$$

Remark 1. *If we take in (2.13)* $n = \left\lfloor \frac{N+1}{2} \right\rfloor + 1$, *then by (2.13) we get*

$$\begin{aligned} \sum_{i=1}^{N-1} \|\Delta A_i A_i\|_1 &\le \left(\sum_{i=0}^{N-1} p_i \left(\left\lfloor \frac{N+1}{2} \right\rfloor + 1 \right) \|\Delta A_i\|_2^2 \right)^{1/2} \\ &\times \left(\sum_{i=0}^{N-1} q \left(\left\lfloor \frac{N+1}{2} \right\rfloor + 1 \right) \|\Delta A_i\|_2^2 \right)^{1/2} \\ &\le \frac{1}{2} \sum_{i=0}^{N-1} s_i \left(\left\lfloor \frac{N+1}{2} \right\rfloor + 1 \right) \|\Delta A_i\|_2^2 \\ &\le \frac{1}{2} \left\lfloor \frac{N+1}{2} \right\rfloor \sum_{i=0}^{N-1} \|\Delta A_i\|_2^2, \end{aligned} \tag{2.14}$$

where

$$s_i(n) := \begin{cases} \left\lfloor \frac{N+1}{2} \right\rfloor, & \text{if } 0 \le i \le \left\lfloor \frac{N+1}{2} \right\rfloor, \\ N - \left\lfloor \frac{N+1}{2} \right\rfloor, & \text{if } \left\lfloor \frac{N+1}{2} \right\rfloor + 1 \le i \le N-1 \end{cases} \le \left\lfloor \frac{N+1}{2} \right\rfloor.$$

From (2.14) we get the following trace inequalities

$$\begin{aligned} \left| \operatorname{tr} \left(\sum_{i=1}^{N-1} \Delta A_i A_i \right) \right| &\le \left[\operatorname{tr} \left(\sum_{i=0}^{N-1} p_i \left(\left\lfloor \frac{N+1}{2} \right\rfloor + 1 \right) |\Delta A_i|^2 \right) \right]^{1/2} \\ &\times \left[\operatorname{tr} \left(\sum_{i=0}^{N-1} q \left(\left\lfloor \frac{N+1}{2} \right\rfloor + 1 \right) |\Delta A_i|^2 \right) \right]^{1/2} \\ &\le \frac{1}{2} \operatorname{tr} \left(\sum_{i=0}^{N-1} s_i \left(\left\lfloor \frac{N+1}{2} \right\rfloor + 1 \right) |\Delta A_i|^2 \right) \\ &\le \frac{1}{2} \left\lfloor \frac{N+1}{2} \right\rfloor \operatorname{tr} \left(\sum_{i=0}^{N-1} |\Delta A_i|^2 \right). \end{aligned} \tag{2.15}$$

3. p-Schatten Norm Inequalities

We also have:

Theorem 8. *For $p \geq 1$, assume that $\{A_i\}_{i=0}^N, \{B_i\}_{i=0}^N \subset B_p(H)$ are sequences of operators with $B_0 = 0$, then for $n \in \{2, ..., N\}$,*

$$\sum_{i=1}^{n-1} \|\Delta A_i B_i\|_p \leq \left(\sum_{i=0}^{n-1} i \|\Delta A_i\|_p^2\right)^{1/2} \left(\sum_{i=0}^{n-1} (n-i-1) \|\Delta B_i\|_p^2\right)^{1/2} \tag{3.1}$$
$$\leq \frac{1}{2} \sum_{i=0}^{n-1} \left[i \|\Delta A_i\|_p^2 + (n-i-1) \|\Delta B_i\|_p^2\right]$$
$$\leq \frac{1}{2}(n-1) \times \begin{cases} n \max_{i\in\{0,\ldots,n-1\}} \left\{\|\Delta A_i\|_p^2, \|\Delta B_i\|_p^2\right\}, \\ \\ \sum_{i=0}^{n-1} \left(\|\Delta A_i\|_p^2 + \|\Delta B_i\|_p^2\right). \end{cases}$$

Proof. Let $n \in \{2, ..., N\}$. Since $B_0 = 0$, hence $B_i = \sum_{j=0}^{i-1} \Delta B_j$ for $i = 1, ..., n-1$. Then by the inequality (1.14) we have

$$\sum_{i=1}^{n-1} \|\Delta A_i B_i\|_p \leq \sum_{i=1}^{n-1} \|\Delta A_i\|_p \|B_i\|_p$$
$$= \sum_{i=1}^{n-1} \|\Delta A_i\|_p \left\|\sum_{j=0}^{i-1} \Delta B_j\right\|_p$$
$$= \sum_{i=1}^{n-1} \sqrt{i} \|\Delta A_i\|_p \frac{1}{\sqrt{i}} \left\|\sum_{j=0}^{i-1} \Delta B_j\right\|_p.$$

Now, by a similar argument to the one in the proof of Theorem 5 we deduce the desired result (3.1). □

Corollary 4. *For $p \geq 1$, assume that $\{A_i\}_{i=0}^N \subset B_p(H)$ with $A_0 = 0$, then for $n \in \{2, ..., N\}$,*

$$\sum_{i=1}^{n-1} \|\Delta A_i A_i\|_p \leq \left(\sum_{i=0}^{n-1} i \|\Delta A_i\|_p^2\right)^{1/2} \left(\sum_{i=0}^{n-1} (n-i-1) \|\Delta A_i\|_p^2\right)^{1/2} \tag{3.2}$$
$$\leq \frac{1}{2}(n-1) \sum_{i=0}^{n-1} \|\Delta A_i\|_p^2.$$

We also have:

Theorem 9. *For $p \geq 1$, assume that $\{A_i\}_{i=0}^N, \{B_i\}_{i=0}^N \subset B_p(H)$ are sequences of operators with $B_N = 0$, then for $n \in \{1, ..., N-1\}$,*

$$\begin{aligned} &\sum_{i=n}^{N-1} \|\Delta A_i B_i\|_p \qquad (3.3)\\ &\leq \left(\sum_{i=n}^{N-1} (N-i) \|\Delta A_i\|_p^2\right)^{1/2} \left(\sum_{i=n}^{N-1} (i+1-n) \|\Delta B_i\|_p^2\right)^{1/2} \\ &\leq \frac{1}{2} \sum_{i=n}^{N-1} \left[(N-i) \|\Delta A_i\|_p^2 + (i+1-n) \|\Delta B_i\|_p^2\right] \\ &\leq \frac{1}{2}(N-n) \times \begin{cases} (N-n+1) \max_{i \in \{n,...,N-1\}} \left\{\|\Delta A_i\|_p^2, \|\Delta B_i\|_p^2\right\}, \\ \sum_{i=n}^{N-1} \left(\|\Delta A_i\|_p^2 + \|\Delta B_i\|_p^2\right). \end{cases} \end{aligned}$$

Corollary 5. *For $p \geq 1$, assume that $\{A_i\}_{i=0}^N \subset B_p(H)$ with $A_N = 0$, then for $n \in \{1, ..., N-1\}$,*

$$\begin{aligned} \sum_{i=n}^{N-1} \|\Delta A_i A_i\|_p &\leq \left(\sum_{i=n}^{N-1} (N-i) \|\Delta A_i\|_p^2\right)^{1/2} \left(\sum_{i=n}^{N-1} (i+1-n) \|\Delta A_i\|_p^2\right)^{1/2} \qquad (3.4)\\ &\leq \frac{1}{2}(N-n+1) \sum_{i=n}^{N-1} \|\Delta A_i\|_p^2. \end{aligned}$$

Finally, we can state

Theorem 10. *For $p \geq 1$, assume that $\{A_i\}_{i=0}^N, \{B_i\}_{i=0}^N \subset B_p(H)$ are sequences of operators with $B_0 = B_N = 0$, then for $n \in \{2, ..., N-1\}$,*

$$\begin{aligned} \sum_{i=1}^{N-1} \|\Delta A_i B_i\|_p &\leq \left(\sum_{i=0}^{N-1} p_i(n) \|\Delta A_i\|_p^2\right)^{1/2} \left(\sum_{i=0}^{N-1} q_i(n) \|\Delta B_i\|_p^2\right)^{1/2} \qquad (3.5)\\ &\leq \frac{1}{2} \sum_{i=0}^{N-1} \left(p_i(n) \|\Delta A_i\|_p^2 + q_i(n) \|\Delta B_i\|_p^2\right), \end{aligned}$$

where $p_i(n)$ and $q_i(n)$ are defined in Theorem 7.

Corollary 6. *For $p \geq 1$, assume that $\{A_i\}_{i=0}^N \subset B_p(H)$ with $A_0 = A_N = 0$, then*

$$\begin{aligned} \sum_{i=1}^{N-1} \|\Delta A_i A_i\|_p &\leq \left(\sum_{i=0}^{N-1} p_i(n) \|\Delta A_i\|_p^2\right)^{1/2} \left(\sum_{i=0}^{N-1} q(n)_i \|\Delta A_i\|_p^2\right)^{1/2} \qquad (3.6)\\ &\leq \frac{1}{2} \sum_{i=0}^{N-1} s_i(n) \|\Delta A_i\|_p^2 \leq \frac{1}{2} \left\lfloor \frac{N+1}{2} \right\rfloor \sum_{i=0}^{N-1} \|\Delta A_i\|_p^2, \end{aligned}$$

where $s_i(n)$ are defined in Corollary 3.

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[1]Mathematics, College of Engineering & Science, Victoria University, PO Box 14428, Melbourne City, MC 8001, Australia.

E-mail address: `sever.dragomir@vu.edu.au`

URL: `http://rgmia.org/dragomir`

[2]DST-NRF Centre of Excellence in the Mathematical, and Statistical Sciences, School of Computer Science, & Applied Mathematics, University of the Witwatersrand,, Johannesburg, South Africa.