NONCOMMUTATIVE GRÜSS TYPE INEQUALITIES FOR FUNCTIONS IN BANACH ALGEBRAS

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Abstract. Let $\mathcal{B}$ be a complex Banach algebra. Assume that x, $y : [a,b] \to \mathcal{B}$ are continuous and y is strongly differentiable on (a,b). In this paper we show among others that

$$\begin{aligned}
&\left\| (b-a)\int_a^b x(t)\,y(t)\,dt - \int_a^b x(t)\,dt \int_a^b y(t)\,dt \right\| \\
&\le \int_a^b \|x(t)-v\| \left[\int_a^t (s-a)\left\|y'(s)\right\| ds + \int_t^b (b-s)\left\|y'(s)\right\| ds \right] dt \\
&\le \begin{cases} \int_a^b \|x(t)-v\| \max\{t-a, b-t\}\,dt \int_a^b \|y'(s)\|\,ds, \\ \\ \frac{1}{(q+1)^{1/q}} \int_a^b \|x(t)-v\| \left[(t-a)^{q+1} + (b-t)^{q+1} \right]^{1/q} dt \\ \times \left(\int_a^b \|y'(s)\|^p\,ds \right)^{1/p}, \\ \\ \frac{1}{2}\int_a^b \|x(t)-v\| \left[(t-a)^2 + (b-t)^2 \right] dt \sup_{s\in[a,b]} \|y'(s)\| \end{cases}
\end{aligned}$$

for all $v \in \mathcal{B}$. Some applications for analytic functions of elements in Banach algebras with examples for exponential function are also given.

1. Introduction

If p, f are integrable on $[a,b]$ and

$$n \le p \le N \text{ and } m \le f \le M \text{ on } [a,b]$$

for some constants n, N, m, M, then

$$\begin{aligned}
&\left| (b-a)\int_a^b p(t)\,f(t)\,dt - \int_a^b p(t)\,dt \int_a^b f(t)\,dt \right| \\
&\le \frac{1}{4}(b-a)^2 (N-n)(M-m),
\end{aligned}$$

which is well known in the literature as *Grüss' inequality.*

The constant $\frac{1}{4}$ is best possible in the sense that it cannot be replaced by a smaller quantity.

In order to obtain similar results for two functions with values in Banach algebras, we need the following preparations.

Let $\mathcal{B}$ be an algebra. An *algebra norm* on $\mathcal{B}$ is a map $\|\cdot\| : \mathcal{B} \to [0,\infty)$ such that $(\mathcal{B}, \|\cdot\|)$ is a normed space, and, further: $\|ab\| \le \|a\|\,\|b\|$ for any $a, b \in \mathcal{B}$. The normed algebra $(\mathcal{B}, \|\cdot\|)$ is a *Banach algebra* if $\|\cdot\|$ is a *complete norm.* We assume

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that the Banach algebra is *unital,* this means that $\mathcal{B}$ has an identity 1 and that $\|1\| = 1$.

Let $\mathcal{B}$ be a unital algebra. An element $a \in \mathcal{B}$ is *invertible* if there exists an element $b \in \mathcal{B}$ with $ab = ba = 1$. The element b is unique; it is called the *inverse* of a and written a^{-1} or $\frac{1}{a}$. The set of invertible elements of $\mathcal{B}$ is denoted by $\operatorname{Inv}(\mathcal{B})$. If $a, b \in \operatorname{Inv}(\mathcal{B})$ then $ab \in \operatorname{Inv}(\mathcal{B})$ and $(ab)^{-1} = b^{-1}a^{-1}$.

For a unital Banach algebra we also have:

(i) If $a \in \mathcal{B}$ and $\lim_{n\to\infty} \|a^n\|^{1/n} < 1$, then $1 - a \in \operatorname{Inv}(\mathcal{B})$;
(ii) $\{b \in \mathcal{B}: \ \|1 - b\| < 1\} \subset \operatorname{Inv}(\mathcal{B})$;
(iii) $\operatorname{Inv}(\mathcal{B})$ is an *open subset* of $\mathcal{B}$;
(iv) The map $\operatorname{Inv}(\mathcal{B}) \ni a \longmapsto a^{-1} \in \operatorname{Inv}(\mathcal{B})$ is continuous.

For simplicity, we denote $z1$, where $z \in \mathbb{C}$ and 1 is the identity of $\mathcal{B}$, by z. The *resolvent set* of $a \in \mathcal{B}$ is defined by

$$\rho(a) := \{z \in \mathbb{C} : \ z - a \in \operatorname{Inv}(\mathcal{B})\};$$

the *spectrum* of a is $\sigma(a)$, the complement of $\rho(a)$ in $\mathbb{C}$, and the *resolvent function* of a is $R_a : \rho(a) \to \operatorname{Inv}(\mathcal{B})$, $R_a(z) := (z - a)^{-1}$. For each $z, w \in \rho(a)$ we have the identity

$$R_a(w) - R_a(z) = (z - w) R_a(z) R_a(w).$$

We also have that

$$\sigma(a) \subset \{z \in \mathbb{C} : \ |z| \le \|a\|\}.$$

The *spectral radius* of a is defined as

$$\nu(a) = \sup\{|z| : z \in \sigma(a)\}.$$

Let $\mathcal{B}$ a unital Banach algebra and $a \in \mathcal{B}$. Then

(i) The resolvent set $\rho(a)$ is open in $\mathbb{C}$;
(ii) For any *bounded linear functionals* $\lambda : \mathcal{B} \to \mathbb{C}$, the function $\lambda \circ R_a$ is analytic on $\rho(a)$;
(iii) The spectrum $\sigma(a)$ is compact and nonempty in $\mathbb{C}$;
(iv) For each $n \in \mathbb{N}$ and $r > \nu(a)$, we have $a^n = \frac{1}{2\pi i} \int_{|\xi|=r} \xi^n (\xi - a)^{-1} d\xi$;
(v) We have $\nu(a) = \lim_{n\to\infty} \|a^n\|^{1/n}$.

Let $\mathcal{B}$ be a unital Banach algebra, $a \in \mathcal{B}$ and G be a domain of $\mathbb{C}$ with $\sigma(a) \subset G$. If $f : G \to \mathbb{C}$ is analytic on G, we define an element $f(a)$ in $\mathcal{B}$ by

$$f(a) := \frac{1}{2\pi i} \int_\delta f(\xi) (\xi - a)^{-1} d\xi, \tag{1.1}$$

where $\delta \subset G$ is taken to be close rectifiable curve in G and such that $\sigma(a) \subset \operatorname{ins}(\delta)$, the inside of δ.

It is well known (see for instance [3, pp. 201-204]) that $f(a)$ does not depend on the choice of δ and the *Spectral Mapping Theorem* (SMT)

$$\sigma(f(a)) = f(\sigma(a)) \tag{1.2}$$

holds.

Concerning other basic definitions and facts in the theory of Banach algebras, the reader can consult the classical books [13] and [15].

For some recent norm inequalities for functions on Banach algebras, see [8], [2] and [5]-[12].

2. Main Results

We have the following equality:

Lemma 1. *Let $y:[a,b]\to\mathcal{B}$ be a strongly differentiable function on the interval (a,b) and $x:[a,b]\to\mathcal{B}$ a continuous function, then for all $v\in\mathcal{B}$*

$$\begin{aligned}(2.1)\qquad &(b-a)\int_a^b x(t)\,y(t)\,dt-\int_a^b x(t)\,dt\int_a^b y(t)\,dt\\ &=\int_a^b (x(t)-v)\left(\int_a^t (s-a)\,y'(s)\,ds+\int_t^b (s-b)\,y'(s)\,ds\right)dt.\end{aligned}$$

Proof. We start to the Montgomery identity for a strongly differentiable function $y:[a,b]\to\mathcal{B}$

$$(2.2)\qquad y(t)(b-a)-\int_a^b y(s)\,ds=\int_a^t (s-a)\,y'(s)\,ds+\int_t^b (s-b)\,y'(s)\,ds$$

that holds for all $t\in[a,b]$.

Indeed, integrating by parts in Bochner's integral [14], we have

$$\int_a^t (s-a)\,y'(s)\,ds=(t-a)\,y(t)-\int_a^t y(s)\,ds$$

and

$$\int_t^b (s-b)\,y'(s)\,ds=(b-t)\,y(t)-\int_t^b y(s)\,ds$$

which by addition gives (2.2).

If we multiply this identity at right by $x(t)$ and integrate over t in $[a,b]$, then we get

$$\begin{aligned}(2.3)\qquad &(b-a)\int_a^b x(t)\,y(t)\,dt-\int_a^b x(t)\,dt\int_a^b y(s)\,ds\\ &=\int_a^b x(t)\left(\int_a^t (s-a)\,y'(s)\,ds+\int_t^b (s-b)\,y'(s)\,ds\right)dt.\end{aligned}$$

Now, if we replace $x(t)$ by $x(t)-v$, we get

$$\begin{aligned}&(b-a)\int_a^b [x(t)-v]\,y(t)\,dt-\int_a^b [x(t)-v]\,dt\int_a^b y(t)\,dt\\ &=(b-a)\int_a^b x(t)\,y(t)\,dt-(b-a)\,v\int_a^b y(t)\,dt\\ &-\int_a^b x(t)\,dt\int_a^b y(t)\,dt+(b-a)\,v\int_a^b y(t)\,dt\\ &=(b-a)\int_a^b x(t)\,y(t)\,dt-\int_a^b x(t)\,dt\int_a^b y(s)\,ds\end{aligned}$$

and by (2.3) we derive (2.1). □

Lemma 2. *With the assumptions of Lemma 2 we have*

$$\begin{aligned}(2.4)\quad &(b-a)\int_a^b x(t)\,y(t)\,dt-\int_a^b x(t)\,dt\int_a^b y(t)\,dt\\ &-(b-a)\left(\int_a^b\left(t-\frac{a+b}{2}\right)x(t)\,dt\right)w\\ &=\int_a^b (x(t)-v)\left(\int_a^t (s-a)\left[y'(s)-w\right]ds+\int_t^b (s-b)\left[y'(s)-w\right]ds\right)dt\end{aligned}$$

for all $w\in\mathcal{B}$.

Proof. If we replace $y(t)$ with $y(t)-tw$ in (2.1), then we get

$$\begin{aligned}(2.5)\quad &(b-a)\int_a^b x(t)\left[y(t)-tw\right]dt-\int_a^b x(t)\,dt\int_a^b (y(t)-tw)\,dt\\ &=\int_a^b (x(t)-v)\left(\int_a^t (s-a)\left[y'(s)-w\right]ds+\int_t^b (s-b)\left[y'(s)-w\right]ds\right)dt.\end{aligned}$$

Observe that

$$\begin{aligned}(2.6)\quad &(b-a)\int_a^b x(t)\left[y(t)-tw\right]dt-\int_a^b x(t)\,dt\int_a^b (y(t)-tw)\,dt\\ &=(b-a)\left[\int_a^b x(t)\,y(t)\,dt-\int_a^b tx(t)\,wdt\right]\\ &-\int_a^b x(t)\,dt\left[\int_a^b y(t)\,dt-\frac{1}{2}\left(b^2-a^2\right)w\right]\\ &=(b-a)\int_a^b x(t)\,y(t)\,dt-(b-a)\int_a^b tx(t)\,wdt\\ &-\int_a^b x(t)\,dt\int_a^b y(t)\,dt+\frac{1}{2}\left(b^2-a^2\right)\int_a^b x(t)\,dtw\\ &=(b-a)\int_a^b x(t)\,y(t)\,dt-\int_a^b x(t)\,dt\int_a^b y(t)\,dt\\ &-(b-a)\int_a^b\left(t-\frac{a+b}{2}\right)x(t)\,wdt\end{aligned}$$

and by (2.5), we get (2.4). □

Remark 1. *If* x *is symmetric on* $[a,b]$, *namely* $x(a+b-t)=x(t)$ *for all* $t\in[a,b]$, *then* $h(t):=\left(t-\frac{a+b}{2}\right)x(t)$ *is antisymmetric, which gives that*

$$\int_a^b\left(t-\frac{a+b}{2}\right)x(t)\,dt=0,$$

and by (2.4) we derive

$$(b-a)\int_a^b x(t)\,y(t)\,dt-\int_a^b x(t)\,dt\int_a^b y(t)\,dt \tag{2.7}$$
$$=\int_a^b (x(t)-v)\left(\int_a^t (s-a)\left[y'(s)-w\right]ds+\int_t^b (s-b)\left[y'(s)-w\right]ds\right)dt$$

for all $v,\ w\in\mathcal{B}$.

We have the inequalities:

Theorem 1. *Let* $y:[a,b]\to\mathcal{B}$ *be a strongly differentiable function on the interval* (a,b) *and* $x:[a,b]\to\mathcal{B}$ *a continuous function, then for all* $v\in\mathcal{B}$

$$\left\|(b-a)\int_a^b x(t)\,y(t)\,dt-\int_a^b x(t)\,dt\int_a^b y(t)\,dt\right\| \tag{2.8}$$
$$\le\int_a^b \|x(t)-v\|\left[\int_a^t (s-a)\|y'(s)\|\,ds+\int_t^b (b-s)\|y'(s)\|\,ds\right]dt$$
$$=B(x,y,v).$$

We have the bounds

$$B(x,y,v) \tag{2.9}$$
$$\le\begin{cases}\int_a^b\|x(t)-v\|\max\{t-a,b-t\}\,dt\int_a^b\|y'(s)\|\,ds,\\[2ex] \frac{1}{(q+1)^{1/q}}\int_a^b\|x(t)-v\|\left[(t-a)^{q+1}+(b-t)^{q+1}\right]^{1/q}dt\\ \times\left(\int_a^b\|y'(s)\|^p\,ds\right)^{1/p},\\[2ex] \frac{1}{2}\int_a^b\|x(t)-v\|\left[(t-a)^2+(b-t)^2\right]dt\sup_{s\in[a,b]}\|y'(s)\|.\end{cases}$$

Proof. If we take the norm in (2.1), then we get

$$\left\|(b-a)\int_a^b x(t)\,y(t)\,dt-\int_a^b x(t)\,dt\int_a^b y(t)\,dt\right\| \tag{2.10}$$
$$\le\int_a^b\left\|(x(t)-v)\left(\int_a^t (s-a)\,y'(s)\,ds+\int_t^b (s-b)\,y'(s)\,ds\right)\right\|dt$$
$$\le\int_a^b\|(x(t)-v)\|\left\|\int_a^t (s-a)\,y'(s)\,ds+\int_t^b (s-b)\,y'(s)\,ds\right\|dt$$
$$\le\int_a^b\|(x(t)-v)\|\left[\left\|\int_a^t (s-a)\,y'(s)\,ds\right\|+\left\|\int_t^b (s-b)\,y'(s)\,ds\right\|\right]dt$$
$$\le\int_a^b\|(x(t)-v)\|\left[\int_a^t (s-a)\|y'(s)\|\,ds+\int_t^b (b-s)\|y'(s)\|\,ds\right]dt,$$

which proves the inequality (2.8).

Using Hölder's inequality, we have for $p,\, q>1$ with $\frac{1}{p}+\frac{1}{q}=1$,

$$\int_a^t (s-a)\left\|y'(s)\right\| ds \le \begin{cases} \sup_{s\in[a,t]}(s-a)\int_a^t \left\|y'(s)\right\| ds \\ \left(\int_a^t (s-a)^q ds\right)^{1/q}\left(\int_a^t \left\|y'(s)\right\|^p ds\right)^{1/p} \\ \sup_{s\in[a,t]}\left\|y'(s)\right\| \int_a^t (s-a)\,ds \end{cases}$$
$$= \begin{cases} (t-a)\int_a^t \left\|y'(s)\right\| ds \\ \frac{(t-a)^{1+1/q}}{(q+1)^{1/q}}\left(\int_a^t \left\|y'(s)\right\|^p ds\right)^{1/p} \\ \frac{(t-a)^2}{2}\sup_{s\in[a,t]}\left\|y'(s)\right\| \end{cases}$$

and

$$\int_t^b (b-s)\left\|y'(s)\right\| ds \le \begin{cases} \sup_{s\in[t,b]}(b-s)\int_t^b \left\|y'(s)\right\| ds \\ \left(\int_t^b (b-s)^q ds\right)^{1/q}\left(\int_t^b \left\|y'(s)\right\|^p ds\right)^{1/p} \\ \sup_{s\in[t,b]}\left\|y'(s)\right\| \int_t^b (b-s)\,ds \end{cases}$$
$$= \begin{cases} (b-t)\int_t^b \left\|y'(s)\right\| ds \\ \frac{(b-t)^{1+1/q}}{(q+1)^{1/q}}\left(\int_t^b \left\|y'(s)\right\|^p ds\right)^{1/p} \\ \frac{(b-t)^2}{2}\sup_{s\in[t,b]}\left\|y'(s)\right\| . \end{cases}$$

Therefore

$$\int_a^t (s-a)\left\|y'(s)\right\| ds + \int_t^b (b-s)\left\|y'(s)\right\| ds$$
$$\le \begin{cases} (t-a)\int_a^t \left\|y'(s)\right\| ds + (b-t)\int_t^b \left\|y'(s)\right\| ds \\ \frac{1}{(q+1)^{1/q}}\left[(t-a)^{1+1/q}\left(\int_a^t \left\|y'(s)\right\|^p ds\right)^{1/p} \right. \\ \left. +(b-t)^{1+1/q}\left(\int_t^b \left\|y'(s)\right\|^p ds\right)^{1/p}\right] \\ \frac{1}{2}\left[(t-a)^2\sup_{s\in[a,t]}\left\|y'(s)\right\| + (b-t)^2\sup_{s\in[t,b]}\left\|y'(s)\right\|\right] \end{cases}$$
$$\le \begin{cases} \max\{t-a,b-t\}\int_a^b \left\|y'(s)\right\| ds \\ \frac{1}{(q+1)^{1/q}}\left[(t-a)^{q+1}+(b-t)^{q+1}\right]^{1/q}\left(\int_a^b \left\|y'(s)\right\|^p ds\right)^{1/p} \\ \frac{1}{2}\left[(t-a)^2+(b-t)^2\right]\sup_{s\in[a,b]}\left\|y'(s)\right\| \end{cases}$$

and by (2.10) we get the desired result (2.8). □

Remark 2. *We have the inequalities*

$$\left\|(b-a)\int_a^b x(t)\,y(t)\,dt-\int_a^b x(t)\,dt\int_a^b y(t)\,dt\right\| \tag{2.11}$$
$$\leq \int_a^b \|x(t)\|\left[\int_a^t (s-a)\,\|y'(s)\|\,ds+\int_t^b (b-s)\,\|y'(s)\|\,ds\right]dt$$
$$\leq \begin{cases} \int_a^b \|x(t)\| \max\{t-a,b-t\}\,dt\int_a^b \|y'(s)\|\,ds, \\ \\ \frac{1}{(q+1)^{1/q}}\int_a^b \|x(t)\|\left[(t-a)^{q+1}+(b-t)^{q+1}\right]^{1/q}dt \\ \times\left(\int_a^b \|y'(s)\|^p\,ds\right)^{1/p}, \\ \\ \frac{1}{2}\int_a^b \|x(t)\|\left[(t-a)^2+(b-t)^2\right]dt\sup_{s\in[a,b]}\|y'(s)\| \end{cases}$$

provided that $y:[a,b]\to\mathcal{B}$ is a strongly differentiable function on the interval (a,b) and $x:[a,b]\to\mathcal{B}$ is a continuous function on $[a,b]$.

Corollary 1. *With the assumptions of Theorem 1 and if there exists $v\in\mathcal{B}$ and $M>0$ such that*

$$\|x(t)-v\|\leq M \text{ for all } t\in[a,b], \tag{2.12}$$

then

$$\left\|(b-a)\int_a^b x(t)\,y(t)\,dt-\int_a^b x(t)\,dt\int_a^b y(t)\,dt\right\| \tag{2.13}$$
$$\leq 2M\int_a^b (b-t)(t-a)\,\|y'(t)\|\,dt$$
$$\leq M\times\begin{cases} \frac{1}{2}(b-a)^2\int_a^b \|y'(t)\|\,dt, \\ \\ 2(b-a)^{2+1/q}\left[B(q+1,q+1)\right]^{1/q}\left(\int_a^b \|y'(t)\|^p\,dt\right)^{1/p}, \\ \\ \frac{1}{3}(b-a)^3\sup_{t\in[a,b]}\|y'(t)\| \end{cases}$$

for $p,\,q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, where $B(\cdot,\cdot)$ is the Beta function.

Proof. By (2.12) we get

$$\int_a^b \|x(t)-v\|\left[\int_a^t (s-a)\,\|y'(s)\|\,ds+\int_t^b (b-s)\,\|y'(s)\|\,ds\right]dt \tag{2.14}$$
$$\leq M\left[\int_a^b\left(\int_a^t (s-a)\,\|y'(s)\|\,ds\right)dt+\int_a^b\left(\int_t^b (b-s)\,\|y'(s)\|\,ds\right)dt\right].$$

Using integration by parts, we can state that

$$\begin{aligned}
&\int_a^b \left(\int_a^t (s-a)\left\|y'(s)\right\| ds\right) dt \\
&= \left(\int_a^t (s-a)\left\|y'(s)\right\| ds\right) t\Big|_a^b - \int_a^b (t-a)\, t\left\|y'(t)\right\| dt \\
&= b\int_a^b (s-a)\left\|y'(s)\right\| ds - \int_a^b (t-a)\, t\left\|y'(t)\right\| dt \\
&= \int_a^b (b-t)(t-a)\left\|y'(t)\right\| dt
\end{aligned}$$

and

$$\begin{aligned}
&\int_a^b \left(\int_t^b (b-s)\left\|y'(s)\right\| ds\right) dt \\
&= \left(\int_t^b (b-s)\left\|y'(s)\right\| ds\right) t\Big|_a^b + \int_a^b (b-t)\, t\left\|y'(t)\right\| dt \\
&= -a\int_a^b (b-s)\left\|y'(s)\right\| ds + \int_a^b (b-t)\, t\left\|y'(t)\right\| dt \\
&= \int_a^b (b-t)(t-a)\left\|y'(t)\right\| dt.
\end{aligned}$$

Therefore

$$\begin{aligned}
&\int_a^b \left(\int_a^t (s-a)\left\|y'(s)\right\| ds\right) dt + \int_a^b \left(\int_t^b (b-s)\left\|y'(s)\right\| ds\right) dt \\
&= 2\int_a^b (b-t)(t-a)\left\|y'(t)\right\| dt
\end{aligned}$$

and by (2.14) and (2.8) we get the first inequality in (2.13).

Using Hölder's inequality for $p,\ q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, then we get

$$\begin{aligned}
&\int_a^b (b-t)(t-a)\left\|y'(t)\right\| dt \\
&\leq \begin{cases}
\sup_{t\in[a,b]}\left[(b-t)(t-a)\right]\int_a^b \left\|y'(t)\right\| dt, \\
\left(\int_a^b \left[(b-t)(t-a)\right]^q dt\right)^{1/q}\left(\int_a^b \left\|y'(t)\right\|^p dt\right)^{1/p}, \\
\int_a^b (b-t)(t-a)\, dt \sup_{t\in[a,b]}\left\|y'(t)\right\|,
\end{cases} \\
&= \begin{cases}
\frac{1}{4}(b-a)^2\int_a^b \left\|y'(t)\right\| dt, \\
(b-a)^{2+1/q}\left[B(q+1,q+1)\right]^{1/q}\left(\int_a^b \left\|y'(t)\right\|^p dt\right)^{1/p}, \\
\frac{1}{6}(b-a)^3\sup_{t\in[a,b]}\left\|y'(t)\right\|
\end{cases}
\end{aligned}$$

and the last part of (2.13). □

The perturbed Grüss' type inequality also holds:

Theorem 2. *Let $y:[a,b]\to\mathcal{B}$ be a strongly differentiable function on the interval (a,b) and $x:[a,b]\to\mathcal{B}$ a continuous function, then for all $v,\ w\in\mathcal{B}$*

$$\begin{aligned}(2.15)\qquad &\left\|(b-a)\int_a^b x(t)\,y(t)\,dt-\int_a^b x(t)\,dt\int_a^b y(t)\,dt\right.\\ &\left.-(b-a)\left(\int_a^b\left(t-\frac{a+b}{2}\right)x(t)\,dt\right)w\right\|\\ &\le\int_a^b\|x(t)-v\|\\ &\times\left[\int_a^t(s-a)\left\|y'(s)-w\right\|ds+\int_t^b(b-s)\left\|y'(s)-w\right\|ds\right]dt\\ &=B(x,y,v,w).\end{aligned}$$

We have the bounds for $p,\ q>1$ with $\frac{1}{p}+\frac{1}{q}=1$,

$$(2.16)\qquad B(x,y,v,w)\le\begin{cases}\int_a^b\|x(t)-v\|\max\{t-a,b-t\}\,dt\int_a^b\left\|y'(s)-w\right\|ds\\[2ex] \frac{1}{(q+1)^{1/q}}\int_a^b\|x(t)-v\|\left[(t-a)^{q+1}+(b-t)^{q+1}\right]^{1/q}dt\\ \times\left(\int_a^b\left\|y'(s)-w\right\|^p ds\right)^{1/p}\\[2ex] \frac{1}{2}\int_a^b\|x(t)-v\|\left[(t-a)^2+(b-t)^2\right]dt\sup_{s\in[a,b]}\left\|y'(s)-w\right\|.\end{cases}$$

Moreover, if x is symmetric on $[a,b]$, then

$$(2.17)\qquad \left\|(b-a)\int_a^b x(t)\,y(t)\,dt-\int_a^b x(t)\,dt\int_a^b y(t)\,dt\right\|\le B(x,y,v,w)$$

for all $v,\ w\in\mathcal{B}$.

The proof is similar to the one of Theorem 1 and we omit the details.

Corollary 2. *With the assumptions of Theorem 2 and if there exists $w\in\mathcal{B}$ such that*

$$(2.18)\qquad \left\|y'(s)-w\right\|\le L\text{ for all }t\in(a,b),$$

then for $p,\ q>1$ *with* $\frac{1}{p}+\frac{1}{q}=1$,

$$\begin{aligned}
(2.19)\qquad & \left\| (b-a)\int_a^b x(t)\,y(t)\,dt-\int_a^b x(t)\,dt\int_a^b y(t)\,dt \right. \\
& \left. -(b-a)\left(\int_a^b\left(t-\frac{a+b}{2}\right)x(t)\,dt\right)w\right\| \\
& \leq \frac{1}{2}L\int_a^b \|x(t)-v\|\left[(t-a)^2+(b-t)^2\right]dt \\
& \leq \frac{1}{2}L\times\begin{cases}
(b-a)^2\int_a^b\|x(t)-v\|\,dt \\[2ex]
(b-a)^{2+1/q}\left(\int_a^b\|x(t)-v\|^p\,dt\right)^{1/p} \\
\times\left(\int_0^1\left[t^2+(1-t)^2\right]^q dt\right)^{1/q} \\[2ex]
\frac{2}{3}(b-a)^3\sup_{t\in[a,b]}\|x(t)-v\|
\end{cases}
\end{aligned}$$

for all $v\in\mathcal{B}$.

Moreover, if x *is symmetric on* $[a,b]$, *then for* $p,\ q>1$ *with* $\frac{1}{p}+\frac{1}{q}=1$,

$$\begin{aligned}
(2.20)\qquad & \left\| (b-a)\int_a^b x(t)\,y(t)\,dt-\int_a^b x(t)\,dt\int_a^b y(t)\,dt \right\| \\
& \leq \frac{1}{2}L\int_a^b \|x(t)-v\|\left[(t-a)^2+(b-t)^2\right]dt \\
& \leq \frac{1}{2}L\times\begin{cases}
(b-a)^2\int_a^b\|x(t)-v\|\,dt, \\[2ex]
(b-a)^{2+1/q}\left(\int_a^b\|x(t)-v\|^p\,dt\right)^{1/p} \\
\times\left(\int_0^1\left[t^2+(1-t)^2\right]^q dt\right)^{1/q}, \\[2ex]
\frac{2}{3}(b-a)^3\sup_{t\in[a,b]}\|x(t)-v\|
\end{cases}
\end{aligned}$$

for all $v\in\mathcal{B}$.

Remark 3. *If there exists* $v,\ w\in\mathcal{B}$ *such that the conditions (2.12) and (2.18) are satisfied, then*

$$\begin{aligned}
(2.21)\qquad & \left\| (b-a)\int_a^b x(t)\,y(t)\,dt-\int_a^b x(t)\,dt\int_a^b y(t)\,dt \right. \\
& \left. -(b-a)\left(\int_a^b\left(t-\frac{a+b}{2}\right)x(t)\,dt\right)w\right\| \\
& \leq \frac{1}{3}LM(b-a)^3.
\end{aligned}$$

Moreover, if x *is symmetric on* $[a,b]$, *then*

$$(2.22)\qquad \left\| (b-a)\int_a^b x(t)\,y(t)\,dt-\int_a^b x(t)\,dt\int_a^b y(t)\,dt \right\| \leq \frac{1}{3}LM(b-a)^3.$$

3. Applications for Analytic Functions

Assume that $f : G \to \mathbb{C}$ is analytic on the convex domain G and $x, y \in \mathcal{B}$, $x \neq y$ with $\sigma(x), \sigma(y) \subset G$. By the convexity of G we have that $\sigma((1-t)x + ty) \subset G$ for all $t \in [0,1]$ and we can define the auxiliary function $f_{x,y} : [0,1] \to \mathcal{B}$ by

$$f_{x,y}(t) := f((1-t)x + ty), \; t \in [0,1]. \tag{3.1}$$

Lemma 3. *Assume that $f : G \to \mathbb{C}$ is analytic on the convex domain G and x, $y \in \mathcal{B}$, $x \neq y$ with $\sigma(x), \sigma(y) \subset G$. The function $f_{x,y} : [0,1] \to \mathcal{B}$ is differentiable on $(0,1)$ as a function of t and we have*

$$f'_{x,y}(t) = D(f)((1-t)x + ty)(y - x) \tag{3.2}$$

for all $t \in (0,1)$, where $D(f)(\cdot)(\cdot)$ is the Fréchet derivative of function f as a function defined on the Banach algebra $\mathcal{B}$ by equation (1.1).

We also have the lateral derivatives

$$f'_{x,y}(0+) = D(f)(x)(y-x) \text{ and } f'_{x,y}(1-) = D(f)(y)(y-x). \tag{3.3}$$

Proof. We prove only for the interior points $t \in (0,1)$. Let h be in a small interval around 0 such that $t + h \in (0,1)$. Then for $h \neq 0$,

$$\begin{aligned} &\frac{f_{x,y}(t+h) - f(t)}{h} \\ &= \frac{f((1-(t+h))x + (t+h)y) - f((1-t)x + ty)}{h} \\ &= \frac{f((1-t)x + ty + h(y-x)) - f((1-t)x + ty)}{h} \end{aligned}$$

and by taking the limit over $h \to 0$, we get

$$\begin{aligned} f'_{x,y}(t) &= \frac{df_{x,y}(t)}{dt} = \lim_{h \to 0} \frac{f_{x,y}(t+h) - f(t)}{h} \\ &= \lim_{h \to 0} \left[\frac{f((1-t)x + ty + h(y-x)) - f((1-t)x + ty)}{h} \right] \\ &= D(f)((1-t)x + ty)(y-x), \end{aligned}$$

which proves (3.2).

The proof is similar for the lateral derivatives. □

Lemma 4. *Assume that $f : G \to \mathbb{C}$ is analytic on the domain G and $x \in \mathcal{B}$, with $\sigma(x) \subset G$, then for $v \in \mathcal{B}$ we have*

$$D(f)(x)(v) = \frac{1}{2\pi i} \int_\gamma f(\xi)(\xi - x)^{-1} v (\xi - x)^{-1} d\xi, \tag{3.4}$$

where $\gamma \subset G$ is taken to be close rectifiable curve in G and such that $\sigma(x) \subset \operatorname{ins}(\gamma)$, the inside of γ.

Proof. Let $v \in \mathcal{B}$. Then there exists a small interval around 0 such that for h in this interval $\sigma(x+hv) \subset \operatorname{ins}(\delta) \subset G$. Then

$$\begin{aligned}
& f(x+hv) - f(x) \\
&= \frac{1}{2\pi i}\left(\int_\gamma f(\xi)(\xi - x - hv)^{-1} d\xi - \int_\gamma f(\xi)(\xi - x)^{-1} d\xi\right) \\
&= \frac{1}{2\pi i}\int_\gamma f(\xi)\left[(\xi - x - hv)^{-1} - (\xi - x)^{-1}\right] d\xi \\
&= \frac{1}{2\pi i} h \int_\gamma f(\xi)\left[(\xi - x - hv)^{-1} v (\xi - x)^{-1}\right] d\xi,
\end{aligned}$$

which gives for $h \neq 0$ that

$$\frac{f(x+hv) - f(x)}{h} = \frac{1}{2\pi i}\int_\gamma f(\xi)(\xi - x - hv)^{-1} v(\xi - x)^{-1} d\xi.$$

By taking the limit over $h \to 0$ and using the properties of the integral, we get (3.4). □

Lemma 5. *Assume that $f : G \to \mathbb{C}$ is analytic on the convex domain G and x, $y \in \mathcal{B}$, $x \neq y$ with $\sigma(x), \sigma(y) \subset G$ while $\gamma \subset G$ is taken to be close rectifiable curve in G and such that $\sigma(x), \sigma(y) \subset \operatorname{ins}(\gamma)$. Then*

$$f'_{x,y}(t) = \frac{1}{2\pi i}\int_\gamma f(\xi)(\xi - (1-t)x - ty)^{-1}(y - x)(\xi - (1-t)x - ty)^{-1} d\xi \tag{3.5}$$

for all $t \in (0,1)$.

We also have the lateral derivatives

$$f'_{x,y}(0+) = \frac{1}{2\pi i}\int_\gamma f(\xi)(\xi - x)^{-1}(y - x)(\xi - x)^{-1} d\xi, \tag{3.6}$$

and

$$f'_{x,y}(1-) = \frac{1}{2\pi i}\int_\gamma f(\xi)(\xi - y)^{-1}(y - x)(\xi - y)^{-1} d\xi. \tag{3.7}$$

The proof is obvious by Lemmas 3 and 4.

Lemma 6. *With the assumptions of Lemma 5 we have the bounds*

$$\begin{aligned}
& \left\|f'_{x,y}(t)\right\| \\
&\leq \frac{1}{2\pi}\|y - x\|\int_\gamma |f(\xi)|\left\|(\xi - (1-t)x - ty)^{-1}\right\|^2 |d\xi| \\
&\leq \frac{1}{2\pi}\|y - x\|\int_\gamma |f(\xi)|(|\xi| - \|(1-t)x + ty\|)^{-2} |d\xi| \\
&\leq \frac{1}{2\pi}\|y - x\|\int_\gamma |f(\xi)|[(1-t)(|\xi| - \|x\|) + t(|\xi| - \|y\|)]^{-2} |d\xi| \\
&\leq \frac{1}{2\pi}\|y - x\|\int_\gamma |f(\xi)|\left[(1-t)(|\xi| - \|x\|)^{-2} + t(|\xi| - \|y\|)^{-2}\right] |d\xi| \\
&\leq \frac{1}{2\pi}\|y - x\|\int_\gamma \frac{|f(\xi)|}{\min\left\{(|\xi| - \|x\|)^2, (|\xi| - \|y\|)^2\right\}} |d\xi|
\end{aligned} \tag{3.8}$$

for all $t \in [0,1]$.

Proof. By taking the norm in (3.5) we get

$$\begin{aligned}
(3.9)\quad & \left\| f'_{x,y}(t) \right\| \\
& \leq \frac{1}{2\pi} \int_{\gamma} |f(\xi)| \left\| (\xi - (1-t)x - ty)^{-1} (y-x) (\xi - (1-t)x - ty)^{-1} \right\| |d\xi| \\
& \leq \frac{1}{2\pi} \|y-x\| \int_{\gamma} |f(\xi)| \left\| (\xi - (1-t)x - ty)^{-1} \right\|^2 |d\xi| \\
& = \frac{1}{2\pi} \|y-x\| \int_{\gamma} |f(\xi)| |\xi|^{-2} \left\| \left(1 - (1-t)\frac{x}{\xi} - t\frac{y}{\xi} \right)^{-1} \right\|^2 |d\xi|
\end{aligned}$$

for all $t \in [0,1]$, which proves the first inequality in (3.8).

Since

$$\left\| (1-t)\frac{x}{\xi} + t\frac{y}{\xi} \right\| \leq (1-t)\left\| \frac{x}{\xi} \right\| + t\left\| \frac{y}{\xi} \right\| < 1 - t + t = 1$$

for $\xi \in \gamma$, hence

$$\left(1 - (1-t)\frac{x}{\xi} - t\frac{y}{\xi} \right)^{-1} = \sum_{k=0}^{\infty} \left[(1-t)\frac{x}{\xi} + t\frac{y}{\xi} \right]^k .$$

Therefore

$$\begin{aligned}
\left\| \left(1 - (1-t)\frac{x}{\xi} - t\frac{y}{\xi} \right)^{-1} \right\| & \leq \sum_{k=0}^{\infty} \left\| (1-t)\frac{x}{\xi} + t\frac{y}{\xi} \right\|^k \\
& = \left(1 - \left\| (1-t)\frac{x}{\xi} + t\frac{y}{\xi} \right\| \right)^{-1} \\
& = \left(\frac{|\xi|}{|\xi|} - \left\| (1-t)\frac{x}{\xi} + t\frac{y}{\xi} \right\| \right)^{-1} \\
& = |\xi| \left(|\xi| - \|(1-t)x + ty\| \right)^{-1}
\end{aligned}$$

for $\xi \in \gamma$, which implies that

$$\left\| \left(1 - (1-t)\frac{x}{\xi} - t\frac{y}{\xi} \right)^{-1} \right\|^2 \leq |\xi|^2 \left(|\xi| - \|(1-t)x + ty\| \right)^{-2}$$

for $\xi \in \gamma$.

Therefore

$$\begin{aligned}
& \int_{\gamma} |f(\xi)| |\xi|^{-2} \left\| \left(1 - (1-t)\frac{x}{\xi} - t\frac{y}{\xi} \right)^{-1} \right\|^2 |d\xi| \\
& \leq \int_{\gamma} |f(\xi)| \left(|\xi| - \|(1-t)x + ty\| \right)^{-2} |d\xi|
\end{aligned}$$

and we derive the second inequality in (3.8).

By the triangle inequality we have

$$\begin{aligned}
|\xi| - \|(1-t)x + ty\| & \geq |\xi| - (1-t)\|x\| - t\|y\| \\
& = (1-t)(|\xi| - \|x\|) + t(|\xi| - \|y\|) > 0
\end{aligned}$$

for $\xi \in \gamma$.

This implies that

$$\left(|\xi| - \|(1-t)x + ty\|\right)^{-1} \le \left[(1-t)\left(|\xi| - \|x\|\right) + t\left(|\xi| - \|y\|\right)\right]^{-1}$$

and thus

$$\left(|\xi| - \|(1-t)x + ty\|\right)^{-2} \le \left[(1-t)\left(|\xi| - \|x\|\right) + t\left(|\xi| - \|y\|\right)\right]^{-2}$$

for $\xi \in \gamma$ and $t \in [0,1]$. This proves the third inequality in (3.8).

By the convexity of the power function $(\cdot)^{-2}$ we also have

$$\begin{aligned}&\left[(1-t)\left(|\xi| - \|x\|\right) + t\left(|\xi| - \|y\|\right)\right]^{-2}\\ &\le (1-t)\left(|\xi| - \|x\|\right)^{-2} + t\left(|\xi| - \|y\|\right)^{-2}\end{aligned}$$

for $t \in [0,1]$, which proves the fourth inequality in (3.8).

Finally, observe that

$$(1-t)\left(|\xi| - \|x\|\right)^{-2} + t\left(|\xi| - \|y\|\right)^{-2} \le \max\left\{\left(|\xi| - \|x\|\right)^{-2}, \left(|\xi| - \|y\|\right)^{-2}\right\}$$

and the last part of (3.8) is thus proved. □

We have the following bounds for the p-norm of $f_{x,y}^{'}$.

Proposition 1. *Assume that $f : G \to \mathbb{C}$ is analytic on the convex domain G and x, $y \in \mathcal{B}$, $x \neq y$ with $\sigma(x)$, $\sigma(y) \subset G$ while $\gamma \subset G$ is taken to be close rectifiable curve in G and such that $\sigma(x)$, $\sigma(y) \subset \operatorname{ins}(\gamma)$. Then*

$$\sup_{t\in[0,1]} \left\|f_{x,y}^{'}(t)\right\| \le \frac{1}{2\pi}\|y-x\| \int_\gamma \frac{|f(\xi)|}{\min\left\{\left(|\xi| - \|x\|\right)^2, \left(|\xi| - \|y\|\right)^2\right\}} |d\xi|, \tag{3.10}$$

$$\int_0^1 \left\|f_{x,y}^{'}(t)\right\| dt \le \frac{1}{2\pi}\|y-x\| \int_\gamma \frac{|f(\xi)|}{\left(|\xi| - \|y\|\right)\left(|\xi| - \|x\|\right)} |d\xi| \tag{3.11}$$

and

$$\begin{aligned}&\left(\int_0^1 \left\|f_{x,y}^{'}(t)\right\|^p dt\right)^{1/p} \\ &\le \frac{1}{2\pi}\|y-x\| \left(\int_\gamma |f(\xi)|^q |d\xi|\right)^{1/q} \\ &\times \left(\frac{1}{(2p-1)\left(\|y\| - \|x\|\right)} \int_\gamma \frac{\left(|\xi| - \|x\|\right)^{2p-1} - \left(|\xi| - \|y\|\right)^{2p-1}}{\left(|\xi| - \|x\|\right)^{2p-1}\left(|\xi| - \|y\|\right)^{2p-1}} |d\xi|\right)^{1/p},\end{aligned} \tag{3.12}$$

for p, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. The inequality (3.10) is obvious by (3.8).

From (3.8) we get, by taking the integral and by using Fubini's theorem, that

$$\begin{aligned}&\int_0^1 \left\|f_{x,y}^{'}(t)\right\| dt \\ &\le \frac{1}{2\pi}\|y-x\| \int_\gamma |f(\xi)| \left(\int_0^1 \left[(1-t)\left(|\xi| - \|x\|\right) + t\left(|\xi| - \|y\|\right)\right]^{-2} dt\right) |d\xi|.\end{aligned} \tag{3.13}$$

Observe that

$$\begin{aligned}
&\int_0^1 \left[(1-t)\left(|\xi|-\|x\|\right)+t\left(|\xi|-\|y\|\right)\right]^{-2} dt \\
&= -\frac{1}{\|x\|-\|y\|}\int_0^1 \frac{d}{dt}\left[(1-t)\left(|\xi|-\|x\|\right)+t\left(|\xi|-\|y\|\right)\right]^{-1} dt \\
&= -\frac{1}{\|x\|-\|y\|}\left[(1-t)\left(|\xi|-\|x\|\right)+t\left(|\xi|-\|y\|\right)\right]^{-1}\Big|_0^1 \\
&= \frac{1}{\|y\|-\|x\|}\left[\left(|\xi|-\|y\|\right)^{-1}-\left(|\xi|-\|x\|\right)^{-1}\right] \\
&= \frac{1}{\|y\|-\|x\|}\frac{\|y\|-\|x\|}{\left(|\xi|-\|y\|\right)\left(|\xi|-\|x\|\right)} = \frac{1}{\left(|\xi|-\|y\|\right)\left(|\xi|-\|x\|\right)},
\end{aligned}$$

for $\|y\| \neq \|x\|$, which, by (3.13), proves (3.11).

If $\|y\| = \|x\|$, then (3.11) also holds.

From (3.8) we also have for p, $q > 1$ with $\frac{1}{p}+\frac{1}{q}=1$,

$$\begin{aligned}
&\left\|f'_{x,y}(t)\right\| \\
&\leq \frac{1}{2\pi}\|y-x\|\int_\gamma |f(\xi)|\left[(1-t)\left(|\xi|-\|x\|\right)+t\left(|\xi|-\|y\|\right)\right]^{-2}|d\xi| \\
&\leq \frac{1}{2\pi}\|y-x\|\left(\int_\gamma |f(\xi)|^q |d\xi|\right)^{1/q} \\
&\times\left(\int_\gamma \left[(1-t)\left(|\xi|-\|x\|\right)+t\left(|\xi|-\|y\|\right)\right]^{-2p}\right)^{1/p}
\end{aligned}$$

and by taking the power p we get

$$\begin{aligned}
\left\|f'_{x,y}(t)\right\|^p &\leq \left[\frac{1}{2\pi}\|y-x\|\left(\int_\gamma |f(\xi)|^q |d\xi|\right)^{1/q}\right]^p \\
&\times\left(\int_\gamma \left[(1-t)\left(|\xi|-\|x\|\right)+t\left(|\xi|-\|y\|\right)\right]^{-2p}\right)
\end{aligned}$$

for $t \in [0,1]$.

Integrating this inequality on $[0,1]$, we get by Fubini's theorem that

$$\begin{aligned}
(3.14)\quad \int_0^1 \left\| f_{x,y}^{'}(t)\right\|^p dt &\le \left[\frac{1}{2\pi}\|y-x\|\left(\int_\gamma |f(\xi)|^q |d\xi|\right)^{1/q}\right]^p \\
&\times \int_0^1 \left(\int_\gamma [(1-t)(|\xi|-\|x\|)+t(|\xi|-\|y\|)]^{-2p} |d\xi|\right) dt \\
&= \left[\frac{1}{2\pi}\|y-x\|\left(\int_\gamma |f(\xi)|^q |d\xi|\right)^{1/q}\right]^p \\
&\times \int_\gamma \left(\int_0^1 [(1-t)(|\xi|-\|x\|)+t(|\xi|-\|y\|)]^{-2p} dt\right) |d\xi| \\
&= \left[\frac{1}{2\pi}\|y-x\|\left(\int_\gamma |f(\xi)|^q |d\xi|\right)^{1/q}\right]^p \\
&\int_\gamma \left(\int_0^1 [(1-t)(|\xi|-\|x\|)+t(|\xi|-\|y\|)]^{-2p} dt\right) |d\xi| .
\end{aligned}$$

Observe that

$$\begin{aligned}
&\int_\gamma \left(\int_0^1 [(1-t)(|\xi|-\|x\|)+t(|\xi|-\|y\|)]^{-2p} dt\right) |d\xi| \\
&= \int_\gamma \left(\left.\frac{[(1-t)(|\xi|-\|x\|)+t(|\xi|-\|y\|)]^{-2p+1}}{(1-2p)(\|x\|-\|y\|)}\right|_0^1 \right) |d\xi| \\
&= \int_\gamma \frac{(|\xi|-\|y\|)^{-2p+1}-(|\xi|-\|x\|)^{-2p+1}}{(1-2p)(\|x\|-\|y\|)} |d\xi| \\
&= \int_\gamma \frac{\frac{1}{(|\xi|-\|y\|)^{2p-1}}-\frac{1}{(|\xi|-\|x\|)^{2p-1}}}{(1-2p)(\|x\|-\|y\|)} |d\xi| \\
&= \frac{1}{(2p-1)(\|y\|-\|x\|)}\int_\gamma \frac{(|\xi|-\|x\|)^{2p-1}-(|\xi|-\|y\|)^{2p-1}}{(|\xi|-\|x\|)^{2p-1}(|\xi|-\|y\|)^{2p-1}} |d\xi| ,
\end{aligned}$$

then by (3.14) we get

$$\begin{aligned}
&\int_0^1 \left\| f_{x,y}^{'}(t)\right\|^p dt \\
&\le \left[\frac{1}{2\pi}\|y-x\|\left(\int_\gamma |f(\xi)|^q |d\xi|\right)^{1/q}\right]^p \\
&\times \frac{1}{(2p-1)(\|y\|-\|x\|)}\int_\gamma \frac{(|\xi|-\|x\|)^{2p-1}-(|\xi|-\|y\|)^{2p-1}}{(|\xi|-\|x\|)^{2p-1}(|\xi|-\|y\|)^{2p-1}} |d\xi| ,
\end{aligned}$$

which proves (3.12). □

We can state now the main result of this section:

Theorem 3. *Assume that $f:G\to\mathbb{C}$ is analytic on the convex domain G and x, $y\in\mathcal{B}$, $x\neq y$ with $\sigma(x)$, $\sigma(y)\subset G$. If $g:[0,1]\to\mathcal{B}$ is continuous, then for p, $q>1$*

with $\frac{1}{p}+\frac{1}{q}=1,$

$$\left\|\int_0^1 g(t)\,f((1-t)x+ty)\,dt-\int_0^1 g(t)\,dt\int_0^1 f((1-t)x+ty)\,dt\right\| \tag{3.15}$$

$$\leq \frac{1}{2\pi}\|y-x\|\begin{cases}\int_0^1\|g(t)-v\|\max\{t,1-t\}\,dt\int_\gamma \frac{|f(\xi)|}{(|\xi|-\|y\|)(|\xi|-\|x\|)}|d\xi|,\\ \\ \frac{1}{(q+1)^{1/q}}\int_0^1\|g(t)-v\|\left[(t-a)^{q+1}+(b-t)^{q+1}\right]^{1/q}dt\\ \times\left(\int_\gamma|f(\xi)|^q|d\xi|\right)^{1/q}\\ \times\left(\frac{1}{(2p-1)(\|y\|-\|x\|)}\int_\gamma\frac{(|\xi|-\|x\|)^{2p-1}-(|\xi|-\|y\|)^{2p-1}}{(|\xi|-\|x\|)^{2p-1}(|\xi|-\|y\|)^{2p-1}}|d\xi|\right)^{1/p}\\ \\ \frac{1}{2}\int_0^1\|g(t)-v\|\left[t^2+(1-t)^2\right]dt\\ \times\int_\gamma\frac{|f(\xi)|}{\min\left\{(|\xi|-\|x\|)^2,(|\xi|-\|y\|)^2\right\}}|d\xi|,\end{cases}$$

for all $v\in\mathcal{B}$.

The proof follows by Theorem 1 and Proposition 1 for $x(t)=g(t)$, $y(t)=f_{x,y}(t)$, $t\in[0,1]$.

Corollary 3. *With the assumptions of Theorem 3 and if the condition (2.12) is satisfied for* $x(t)=g(t)$, $t\in[0,1]$, *then*

$$\left\|\int_0^1 g(t)\,f((1-t)x+ty)\,dt-\int_0^1 g(t)\,dt\int_0^1 f((1-t)x+ty)\,dt\right\| \tag{3.16}$$

$$\leq\frac{1}{2\pi}\|y-x\|\,M$$

$$\times\begin{cases}\frac{3}{4}\int_\gamma\frac{|f(\xi)|}{(|\xi|-\|y\|)(|\xi|-\|x\|)}|d\xi|,\\ \\ \frac{1}{(q+1)^{1/q}}\int_0^1\left[(t-a)^{q+1}+(b-t)^{q+1}\right]^{1/q}dt\\ \times\left(\int_\gamma|f(\xi)|^q|d\xi|\right)^{1/q}\\ \times\left(\frac{1}{(2p-1)(\|y\|-\|x\|)}\int_\gamma\frac{(|\xi|-\|x\|)^{2p-1}-(|\xi|-\|y\|)^{2p-1}}{(|\xi|-\|x\|)^{2p-1}(|\xi|-\|y\|)^{2p-1}}|d\xi|\right)^{1/p}\\ \\ \frac{1}{3}\int_\gamma\frac{|f(\xi)|}{\min\left\{(|\xi|-\|x\|)^2,(|\xi|-\|y\|)^2\right\}}|d\xi|,\end{cases}$$

where $f:G\to\mathbb{C}$ *is analytic on the convex domain* G *and* $x,y\in\mathcal{B}$, $x\neq y$ *with* $\sigma(x),\sigma(y)\subset G$.

4. The Case of Circular Paths

We consider the circular path $\xi(s)=Re^{2\pi is}$ where $s\in[0,1]$, then $d\xi(s)=2\pi iRe^{2\pi is}ds$, $|d\xi(s)|=2\pi Rds$ and $|\xi|=R$.

Assume that $f:G\to\mathbb{C}$ is analytic on the convex domain G and $x,y\in\mathcal{B}$, $x\neq y$ with $\sigma(x),\sigma(y)\subset D(0,R)\subset G$. Then by Proposition 1 we derive the simpler inequalities

$$\sup_{t\in[0,1]}\left\|f'_{x,y}(t)\right\|\leq\frac{R\|y-x\|}{\min\left\{(R-\|x\|)^2,(R-\|y\|)^2\right\}}\int_0^1\left|f\left(Re^{2\pi is}\right)\right|ds, \tag{4.1}$$

$$\int_0^1 \left\| f'_{x,y}(t) \right\| dt \leq \frac{R\|y-x\|}{(R-\|y\|)(R-\|x\|)} \int_0^1 \left| f\left(Re^{2\pi i s}\right) \right| ds, \tag{4.2}$$

and

$$\begin{aligned} &\left(\int_0^1 \left\| f'_{x,y}(t) \right\|^p dt \right)^{1/p} \\ &\leq R\|y-x\| \left(\int_0^1 \left| f\left(Re^{2\pi i s}\right) \right|^q ds \right)^{1/q} \\ &\times \left(\frac{1}{(2p-1)(\|y\|-\|x\|)} \frac{(R-\|x\|)^{2p-1} - (R-\|y\|)^{2p-1}}{(R-\|x\|)^{2p-1}(R-\|y\|)^{2p-1}} \right)^{1/p}, \end{aligned} \tag{4.3}$$

for $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$.

Assume that $f : G \to \mathbb{C}$ is analytic on the convex domain G and $x, y \in \mathcal{B}$, $x \neq y$ with $\sigma(x), \sigma(y) \subset D(0, R) \subset G$. If $g : [0,1] \to \mathcal{B}$ is continuous, then by (3.15),

$$\begin{aligned} &\left\| \int_0^1 g(t) f((1-t)x+ty)\, dt - \int_0^1 g(t)\, dt \int_0^1 f((1-t)x+ty)\, dt \right\| \\ &\leq R\|y-x\| \\ &\times \begin{cases} \frac{1}{(R-\|y\|)(R-\|x\|)} \int_0^1 \|g(t)-v\| \max\{t, 1-t\}\, dt \int_0^1 \left| f\left(Re^{2\pi i s}\right) \right| ds, \\ \\ \frac{1}{(q+1)^{1/q}} \int_0^1 \|g(t)-v\| \left[(t-a)^{q+1} + (b-t)^{q+1} \right]^{1/q} dt \\ \times \left(\int_0^1 \left| f\left(Re^{2\pi i s}\right) \right|^q ds \right)^{1/q} \\ \times \left(\frac{1}{(2p-1)(\|y\|-\|x\|)} \frac{(R-\|x\|)^{2p-1} - (R-\|y\|)^{2p-1}}{(R-\|x\|)^{2p-1}(R-\|y\|)^{2p-1}} \right)^{1/p} \\ \\ \frac{1}{2} \frac{1}{\min\left\{(R-\|x\|)^2, (R-\|y\|)^2\right\}} \int_0^1 \|g(t)-v\| \left[t^2 + (1-t)^2 \right] dt \\ \times \int_0^1 \left| f\left(Re^{2\pi i s}\right) \right| ds, \end{cases} \end{aligned} \tag{4.4}$$

and by (3.16), we get

$$\begin{aligned} &\left\| \int_0^1 g(t) f((1-t)x+ty)\, dt - \int_0^1 g(t)\, dt \int_0^1 f((1-t)x+ty)\, dt \right\| \\ &\leq R\|y-x\| M \\ &\times \begin{cases} \frac{3}{4} \frac{1}{(R-\|y\|)(R-\|x\|)} \int_0^1 \left| f\left(Re^{2\pi i s}\right) \right| ds, \\ \\ \frac{1}{(q+1)^{1/q}} \int_0^1 \left[(t-a)^{q+1} + (b-t)^{q+1} \right]^{1/q} dt \\ \times \left(\int_0^1 \left| f\left(Re^{2\pi i s}\right) \right|^q ds \right)^{1/q} \\ \times \left(\frac{1}{(2p-1)(\|y\|-\|x\|)} \frac{(R-\|x\|)^{2p-1} - (R-\|y\|)^{2p-1}}{(R-\|x\|)^{2p-1}(R-\|y\|)^{2p-1}} \right)^{1/p}, \\ \\ \frac{1}{3} \frac{1}{\min\left\{(R-\|x\|)^2, (R-\|y\|)^2\right\}} \int_0^1 \left| f\left(Re^{2\pi i s}\right) \right| ds, \end{cases} \end{aligned} \tag{4.5}$$

provided that there is $v \in \mathcal{B}$ such that $\|g(t) - v\| \leq M$ for all $t \in [a, b]$.

The *modified Bessel function of the first kind* $I_\nu(z)$ for real number ν can be defined by the power series as [1, p. 376]

$$I_\nu(z) = \left(\frac{1}{2}z\right)^\nu \sum_{k=0}^{\infty} \frac{\left(\frac{1}{4}z^2\right)^k}{k!\Gamma(\nu+k+1)},$$

where Γ is the *gamma function.* For $n=0$ we have $I_0(z)$ given by

$$I_0(z) = \sum_{k=0}^{\infty} \frac{\left(\frac{1}{4}z^2\right)^k}{(k!)^2}.$$

An integral formula for real number ν is

$$I_\nu(z) = \frac{1}{\pi}\int_0^\pi e^{z\cos\theta}\cos(\nu\theta)\,d\theta - \frac{\sin(\nu\pi)}{\pi}\int_0^\infty e^{-z\cosh t-\nu t}dt,$$

which simplifies for ν an integer n to

$$I_n(z) = \frac{1}{\pi}\int_0^\pi e^{z\cos\theta}\cos(n\theta)\,d\theta.$$

For $n=0$ we have

$$I_0(z) = \frac{1}{\pi}\int_0^\pi e^{z\cos\theta}d\theta.$$

Consider the exponential function $f(a) = \exp a$, $a \in \mathcal{B}$. Assume that $x, y \in \mathcal{B}$ and $\|x\|, \|y\| < R$ for some $R > 0$. Observe that

$$\left|\exp\left(Re^{2\pi it}\right)\right| = \left|\exp\left[R\left(\cos(2\pi t) + i\sin(2\pi t)\right)\right]\right| = \exp\left[R\cos(2\pi t)\right]$$

and, by changing the variable $\theta = 2\pi t$, we get $dt = \frac{1}{2\pi}d\theta$ and

$$\begin{aligned} &\int_0^1 \exp\left[R\cos(2\pi t)\right]dt \\ &= \frac{1}{2\pi}\int_0^{2\pi}\exp\left[R\cos\theta\right]d\theta \\ &= \frac{1}{2}\left(\frac{1}{\pi}\int_0^\pi \exp\left[R\cos\theta\right]d\theta + \frac{1}{\pi}\int_\pi^{2\pi}\exp\left[R\cos\theta\right]d\theta\right) \\ &= \frac{1}{2}\left(I_0(R) + I_0(-R)\right) = I_0(R). \end{aligned} \tag{4.6}$$

If $g : [0,1] \to \mathcal{B}$ is continuous, then for all $u \in [0,1]$ we have by (4.4) for the exponential function, that

$$\begin{aligned} &\left\|\int_0^1 g(t)\exp\left((1-t)x+ty\right)dt - \int_0^1 g(t)\,dt\int_0^1 \exp\left((1-t)x+ty\right)dt\right\| \\ &\le RI_0(R)\left\|y-x\right\| \\ &\times \begin{cases} \frac{1}{(R-\|y\|)(R-\|x\|)}\int_0^1 \left\|g(t)-v\right\|\max\{t,1-t\}\,dt \\ \\ \frac{1}{2}\frac{1}{\min\left\{(R-\|x\|)^2,(R-\|y\|)^2\right\}}\int_0^1 \left\|g(t)-v\right\|\left[t^2+(1-t)^2\right]dt \end{cases} \end{aligned} \tag{4.7}$$

for all $v \in \mathcal{B}$.

If $\|g(t)-v\| \leq M$ for all $t \in [a,b]$, then by (4.5) we get

$$\left\|\int_0^1 g(t)\exp((1-t)x+ty)\,dt-\int_0^1 g(t)\,dt\int_0^1 \exp((1-t)x+ty)\,dt\right\| \tag{4.8}$$
$$\leq RI_0(R)M\|y-x\|\begin{cases}\frac{3}{4}\frac{1}{(R-\|y\|)(R-\|x\|)},\\ \\ \frac{1}{3}\frac{1}{\min\left\{(R-\|x\|)^2,(R-\|y\|)^2\right\}}\end{cases}$$

provided that $x, y \in \mathcal{B}$ and $\|x\|, \|y\| < R$.

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