p-SCHATTEN NORM INEQUALITIES OF OPIAL-LASOTA'S TYPE FOR TWO SEQUENCES

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Abstract. An operator $A \in \mathcal{B}(H)$ is said to belong to the *von Neumann-Schatten class* $\mathcal{B}_p(H)$, $1 \le p < \infty$ if the *p-Schatten norm* is finite

$$\|A\|_p := [\operatorname{tr}(|A|^p)]^{1/p} < \infty.$$

In this paper we show among others that, for $p,\ q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, and $\{A_i\}_{i=0}^N \subset B_p(H)$, $\{B_i\}_{i=0}^N \subset B_q(H)$ that are sequences of operators with $A_0 = B_N = 0$, then

$$\begin{aligned}\sum_{i=1}^{N-1} \|A_i B_i\|_1 &\le \frac{1}{2}\left(\sum_{i=0}^{N-1}(N-i)(N+i+1)\|\Delta A_i\|_p^2\right)^{1/2} \\ &\times \left(\sum_{i=0}^{N-1}(i+1)(2N-i)\|\Delta B_i\|_q^2\right)^{1/2} \\ &\le \frac{1}{2}N(N+1) \\ &\times \begin{cases} \frac{1}{3}(2N+1)\max_{i\in\{0,\dots,N-1\}}\|\Delta A_i\|_p \max_{i\in\{0,\dots,N-1\}}\|\Delta B_i\|_q, \\ \left(\sum_{i=0}^{N-1}\|\Delta A_i\|_p^2\right)^{1/2}\left(\sum_{i=0}^{N-1}\|\Delta B_i\|_q^2\right)^{1/2}, \end{cases}\end{aligned}$$

where $\Delta A_i := A_{i+1} - A_i$, $i \in \{0, \dots, N-1\}$ is the forward difference.

1. Introduction

For a sequence $\{x_i\}_{i=0}^N$, we consider the forward operator Δ defined by $\Delta x_i = x_{i+1} - x_i$, $i = 0, \dots, N-1$. Recall the summation by parts formula stated as

$$\sum_{k=m}^{n} a_k \Delta b_k = a_n b_{n+1} - a_m b_m - \sum_{k=m}^{n-1} b_{k+1}\Delta a_k, \tag{1.1}$$

where a_k and b_k are some sequences for which the products above exist.

In [12], Lasota provided discrete versions of Opial inequality [15] about the forward difference operator as follows:

Theorem 1. *Let $\{x_i\}_{i=0}^N$ be a sequence of real numbers with $x_0 = x_N = 0$. Then, the following inequality holds*

$$\sum_{i=1}^{N-1} |x_i \Delta x_i| \le \frac{1}{2}\left\lfloor \frac{N+1}{2} \right\rfloor \sum_{i=0}^{N-1} |\Delta x_i|^2, \tag{1.2}$$

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where $\lfloor\cdot\rfloor$ is the integer part function. If N is even, then the inequality (1.2) is sharp.

For various Opial type inequalities, see [2]-[6] and [14].
Also, we have the following results, see [1]:

Theorem 2. *Let $\{x_i\}_{i=0}^N$ be a sequence of real numbers. If $x_0=0$, then*

$$\sum_{i=1}^{\tau-1}|x_i\Delta x_i|\le\frac{1}{2}(\tau-1)\sum_{i=0}^{\tau-1}|\Delta x_i|^2,\ \tau\in\{2,...,N\}. \tag{1.3}$$

If $x_N=0$, then

$$\sum_{i=\tau}^{N-1}|x_i\Delta x_i|\le\frac{1}{2}(N-\tau+1)\sum_{i=0}^{N-1}|\Delta x_i|^2,\ \tau\in\{1,...,N-1\}. \tag{1.4}$$

For other discrete Opial type inequalities, see [10], [11] and [16]-[20].
In the recent paper [9] we obtained the following extension for two sequences:

Theorem 3. *Assume that $\{x_i\}_{i=0}^N$, $\{y_i\}_{i=0}^N$ are sequences of complex numbers. If $y_0=y_N=0$, then for $n\in\{2,...,N-1\}$,*

$$\begin{aligned}\sum_{i=1}^{N-1}|\Delta x_i||y_i| &\le\left(\sum_{i=0}^{N-1}p_i(n)|\Delta x_i|^2\right)^{1/2}\left(\sum_{i=0}^{N-1}q_i(n)|\Delta y_i|^2\right)^{1/2}\\ &\le\frac{1}{2}\sum_{i=0}^{N-1}\left(p_i(n)|\Delta x_i|^2+q_i(n)|\Delta y_i|^2\right),\end{aligned} \tag{1.5}$$

where

$$p_i(n):=\begin{cases} i, & \text{if } 0\le i\le n-1,\\ N-i, & \text{if } n\le i\le N-1\end{cases}$$

and

$$q_i(n):=\begin{cases} n-i-1, & \text{if } 0\le i\le n-1,\\ i+1-n, & \text{if } n\le i\le N-1.\end{cases}$$

Corollary 1. *Assume that $\{x_i\}_{i=0}^N$ is a sequence of complex numbers with $x_0=x_N=0$, then*

$$\begin{aligned}\sum_{i=1}^{N-1}|\Delta x_i||x_i| &\le\left(\sum_{i=0}^{N-1}p_i(n)|\Delta x_i|^2\right)^{1/2}\left(\sum_{i=0}^{N-1}q(n)_i|\Delta x_i|^2\right)^{1/2}\\ &\le\frac{1}{2}\sum_{i=0}^{N-1}s_i(n)|\Delta x_i|^2,\end{aligned} \tag{1.6}$$

where

$$s_i(n):=\begin{cases} n-1, & \text{if } 0\le i\le n-1,\\ N-n+1, & \text{if } n\le i\le N-1.\end{cases}$$

Remark 1. *If we take in (1.6) $n = \left\lfloor \frac{N+1}{2} \right\rfloor + 1$, then by (1.6) we get*

$$\begin{aligned}
(1.7)\quad & \sum_{i=1}^{N-1} |\Delta x_i| \, |x_i| \\
& \leq \left(\sum_{i=0}^{N-1} p_i \left(\left\lfloor \frac{N+1}{2} \right\rfloor + 1 \right) |\Delta x_i|^2 \right)^{1/2} \left(\sum_{i=0}^{N-1} q \left(\left\lfloor \frac{N+1}{2} \right\rfloor + 1 \right) |\Delta x_i|^2 \right)^{1/2} \\
& \leq \frac{1}{2} \sum_{i=0}^{N-1} s_i \left(\left\lfloor \frac{N+1}{2} \right\rfloor + 1 \right) |\Delta x_i|^2 \leq \frac{1}{2} \left\lfloor \frac{N+1}{2} \right\rfloor \sum_{i=0}^{N-1} |\Delta x_i|^2 ,
\end{aligned}$$

where

$$s_i(n) := \begin{cases} \left\lfloor \frac{N+1}{2} \right\rfloor, & \text{if } 0 \leq i \leq \left\lfloor \frac{N+1}{2} \right\rfloor, \\ N - \left\lfloor \frac{N+1}{2} \right\rfloor, & \text{if } \left\lfloor \frac{N+1}{2} \right\rfloor + 1 \leq i \leq N-1 \end{cases} \leq \left\lfloor \frac{N+1}{2} \right\rfloor .$$

The inequality (1.7) is a refinement of Lasota's result (1.2).

In order to extend these results for p-Schatten norms we need the following preparations.

Let $(H; \langle ., . \rangle)$ be a complex Hilbert space and $\mathcal{B}(H)$ the Banach algebra of all bounded linear operators on H. If $\{e_i\}_{i \in I}$ an orthonormal basis of H, we say that $A \in \mathcal{B}(H)$ is of *trace class* if

$$\|A\|_1 := \sum_{i \in I} \langle |A| e_i, e_i \rangle < \infty. \tag{1.8}$$

The definition of $\|A\|_1$ does not depend on the choice of the orthonormal basis $\{e_i\}_{i \in I}$. We denote by $\mathcal{B}_1(H)$ the set of trace class operators in $\mathcal{B}(H)$.

We define the *trace* of a trace class operator $A \in \mathcal{B}_1(H)$ to be

$$\operatorname{tr}(A) := \sum_{i \in I} \langle A e_i, e_i \rangle , \tag{1.9}$$

where $\{e_i\}_{i \in I}$ an orthonormal basis of H. Note that this coincides with the usual definition of the trace if H is finite-dimensional. We observe that the series (1.9) converges absolutely and it is independent from the choice of basis.

The following result collects some properties of the trace:

Theorem 4. *We have:*

(i) If $A \in \mathcal{B}_1(H)$ then $A^* \in \mathcal{B}_1(H)$ and

$$\operatorname{tr}(A^*) = \overline{\operatorname{tr}(A)}; \tag{1.10}$$

(ii) If $A \in \mathcal{B}_1(H)$ and $T \in \mathcal{B}(H)$, then AT, $TA \in \mathcal{B}_1(H)$ and

$$\operatorname{tr}(AT) = \operatorname{tr}(TA) \text{ and } |\operatorname{tr}(AT)| \leq \|A\|_1 \|T\| ; \tag{1.11}$$

(iii) $\operatorname{tr}(\cdot)$ is a bounded linear functional on $\mathcal{B}_1(H)$ with $\|\operatorname{tr}\| = 1$;

(iv) If A, $B \in \mathcal{B}_2(H)$ then AB, $BA \in \mathcal{B}_1(H)$ and $\operatorname{tr}(AB) = \operatorname{tr}(BA)$;

(v) $\mathcal{B}_{fin}(H)$ is a dense subspace of $\mathcal{B}_1(H)$.

An operator $A \in \mathcal{B}(H)$ is said to belong to the *von Neumann-Schatten class* $\mathcal{B}_p(H)$, $1 \leq p < \infty$ if the *p-Schatten norm* is finite [21, p. 60-64]

$$\|A\|_p := [\operatorname{tr}(|A|^p)]^{1/p} < \infty.$$

For $1 < p < q < \infty$ we have that

$$\mathcal{B}_1(H) \subset \mathcal{B}_p(H) \subset \mathcal{B}_q(H) \subset \mathcal{B}(H) \tag{1.12}$$

and

$$\|A\|_1 \geq \|A\|_p \geq \|A\|_q \geq \|A\|. \tag{1.13}$$

For $p \geq 1$ the functional $\|\cdot\|_p$ is a *norm* on the $*$-ideal $\mathcal{B}_p(H)$ and $\left(\mathcal{B}_p(H), \|\cdot\|_p\right)$ is a Banach space.

Also, see for instance [21, p. 60-64],

$$\|A\|_p = \|A^*\|_p, \ A \in \mathcal{B}_p(H) \tag{1.14}$$

$$\|AB\|_p \leq \|A\|_p \|B\|_p, \ A, B \in \mathcal{B}_p(H) \tag{1.15}$$

and

$$\|AB\|_p \leq \|A\|_p \|B\|, \ \|BA\|_p \leq \|B\| \|A\|_p, \ A \in \mathcal{B}_p(H), \ B \in \mathcal{B}(H). \tag{1.16}$$

This implies that

$$\|CAB\|_p \leq \|C\| \|A\|_p \|B\|, \ A \in \mathcal{B}_p(H), \ B, \ C \in \mathcal{B}(H). \tag{1.17}$$

In terms of *p-Schatten norm* we have the *Hölder inequality* for $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$

$$(|\operatorname{tr}(AB)| \leq) \|AB\|_1 \leq \|A\|_p \|B\|_q, \ A \in \mathcal{B}_p(H), \ B \in \mathcal{B}_q(H). \tag{1.18}$$

For the theory of trace functionals and their applications the reader is referred to [18] and [21].

For some classical trace inequalities see [7], [8], and [13], which are continuations of the work of Bellman [2].

2. 1-Schatten Norm Inequalities

We have the following result for two sequences:

Theorem 5. *For $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, assume that $\{A_i\}_{i=0}^N \subset B_p(H)$, $\{B_i\}_{i=0}^N \subset B_q(H)$ are sequences of operators with $A_0 = B_0 = 0$, then*

$$\begin{aligned} &\sum_{i=1}^{N} \|A_i B_i\|_1 \\ &\leq \frac{1}{2} \left(\sum_{i=0}^{N-1} (N-i)(N+i+1) \|\Delta A_i\|_p^2 \right)^{1/2} \\ &\times \left(\sum_{i=0}^{N-1} (N-i)(N+i+1) \|\Delta B_i\|_q^2 \right)^{1/2} \\ &\leq \frac{1}{2} N(N+1) \\ &\times \begin{cases} \frac{1}{3}(2N+1) \max_{i \in \{0,\dots,N-1\}} \|\Delta A_i\|_p \max_{i \in \{0,\dots,N-1\}} \|\Delta B_i\|_q, \\ \\ \left(\sum_{i=0}^{N-1} \|\Delta A_i\|_p^2 \right)^{1/2} \left(\sum_{i=0}^{N-1} \|\Delta B_i\|_q^2 \right)^{1/2}. \end{cases} \end{aligned} \tag{2.1}$$

Also,

$$\sum_{i=1}^{N}\|A_iB_i\|_1 \le \frac{1}{2}\left(\sum_{i=0}^{N-1}(N-i)(N+i+1)\|\Delta A_i\|_p^2\right)^{1/2} \times\left(\sum_{i=0}^{N-1}(N-i)(N+i+1)\|\Delta B_i\|_q^2\right)^{1/2} \le \frac{1}{4}\sum_{i=0}^{N-1}(N-i)(N+i+1)\left(\|\Delta A_i\|_p^2+\|\Delta B_i\|_q^2\right). \tag{2.2}$$

Proof. Since $A_0 = B_0 = 0$, hence $A_i = \sum_{j=0}^{i-1}\Delta A_j$ and $B_i = \sum_{j=0}^{i-1}\Delta B_j$ for $i = 1, ..., N$. Then, by (1.18)

$$\begin{aligned}\sum_{i=1}^{N}\|A_iB_i\|_1 &\le \sum_{i=1}^{N}\|A_i\|_p\|B_i\|_q \\ &= \sum_{i=1}^{N}\left\|\sum_{j=0}^{i-1}\Delta A_j\right\|_p\left\|\sum_{j=0}^{i-1}\Delta B_j\right\|_q \\ &= \sum_{i=1}^{N} i\frac{1}{\sqrt{i}}\left\|\sum_{j=0}^{i-1}\Delta A_j\right\|_p\frac{1}{\sqrt{i}}\left\|\sum_{j=0}^{i-1}\Delta B_j\right\|_q =: \Gamma.\end{aligned}$$

By Cauchy-Bunyakowsky-Schwarz (CBS) inequality we have

$$\begin{aligned}\frac{1}{\sqrt{i}}\left\|\sum_{j=0}^{i-1}\Delta A_j\right\|_p &\le \left(\sum_{j=0}^{i-1}\|\Delta A_j\|_p^2\right)^{1/2}, \\ \frac{1}{\sqrt{i}}\left\|\sum_{j=0}^{i-1}\Delta B_j\right\|_q &\le \left(\sum_{j=0}^{i-1}\|\Delta B_j\|_q^2\right)^{1/2},\end{aligned}$$

which gives.

$$\Gamma \le \sum_{i=1}^{N} i\left(\sum_{j=0}^{i-1}\|\Delta A_j\|_p^2\right)^{1/2}\left(\sum_{j=0}^{i-1}\|\Delta B_j\|_q^2\right)^{1/2}.$$

By the weighted (CBS) inequality we have

$$\begin{aligned}&\sum_{i=1}^{N} i\left(\sum_{j=0}^{i-1}\|\Delta A_j\|_p^2\right)^{1/2}\left(\sum_{j=0}^{i-1}\|\Delta B_j\|_q^2\right)^{1/2} \\ &\le \left(\sum_{i=1}^{N} i\left[\left(\sum_{j=0}^{i-1}\|\Delta A_j\|_p^2\right)^{1/2}\right]^2\right)^{1/2}\left(\sum_{i=1}^{N} i\left[\left(\sum_{j=0}^{i-1}\|\Delta B_j\|_q^2\right)^{1/2}\right]^2\right)^{1/2} \\ &= \left(\sum_{i=1}^{N} i\left(\sum_{j=0}^{i-1}\|\Delta A_j\|_p^2\right)\right)^{1/2}\left(\sum_{i=1}^{N} i\left(\sum_{j=0}^{i-1}\|\Delta B_j\|_q^2\right)\right)^{1/2},\end{aligned}$$

which implies that

$$\Gamma \leq \left(\sum_{i=1}^{N} i\left(\sum_{j=0}^{i-1}\|\Delta A_j\|_p^2\right)\right)^{1/2}\left(\sum_{i=1}^{N} i\left(\sum_{j=0}^{i-1}\|\Delta B_j\|_q^2\right)\right)^{1/2} =: \Delta. \tag{2.3}$$

From the formula (1.1), we get

$$\sum_{i=1}^{N} c_i \Delta d_i = c_N d_{N+1} - c_1 d_1 - \sum_{i=1}^{N-1} d_{i+1}\Delta c_i. \tag{2.4}$$

Now, if we take $c_i = \sum_{j=0}^{i-1}\|\Delta A_j\|_p^2,\ i = 1,...,N,\ d_i = \frac{1}{2}i\,(i-1)\,,$ then $c_N = \sum_{j=0}^{N-1}\|\Delta A_j\|_p^2\,,$

$$\Delta d_i = d_{i+1} - d_i = \frac{1}{2}i\,(i+1) - \frac{1}{2}i\,(i-1) = i,$$

and

$$\Delta c_i = c_{i+1} - c_i = \sum_{j=0}^{i}\|\Delta A_j\|_p^2 - \sum_{j=0}^{i-1}\|\Delta A_j\|_p^2 = \|\Delta A_i\|_p^2\,.$$

By (2.4) we derive

$$\begin{aligned}
&\sum_{i=1}^{N} i\left(\sum_{j=0}^{i-1}\|\Delta A_j\|_p^2\right) \\
&= \frac{1}{2}N\,(N+1)\sum_{j=0}^{N-1}\|\Delta A_j\|_p^2 - \sum_{k=1}^{N-1}\frac{1}{2}i\,(i+1)\,\|\Delta A_i\|_p^2 \\
&= \sum_{i=0}^{N-1}\left[\frac{1}{2}N\,(N+1) - \frac{1}{2}i\,(i+1)\right]\|\Delta A_i\|_p^2 \\
&= \frac{1}{2}\sum_{i=0}^{N-1}(N-i)\,(N+i+1)\,\|\Delta A_i\|_p^2
\end{aligned} \tag{2.5}$$

and, similarly

$$\sum_{i=1}^{N} i\left(\sum_{j=0}^{i-1}\|\Delta B_j\|_q^2\right) = \frac{1}{2}\sum_{i=0}^{N-1}(N-i)\,(N+i+1)\,\|\Delta B_i\|_q^2\,. \tag{2.6}$$

Therefore

$$\begin{aligned}
\Delta = \frac{1}{2}&\left(\sum_{i=0}^{N-1}(N-i)\,(N+i+1)\,\|\Delta A_i\|_p^2\right)^{1/2} \\
&\times\left(\sum_{i=0}^{N-1}(N-i)\,(N+i+1)\,\|\Delta B_i\|_q^2\right)^{1/2}
\end{aligned}$$

and by (2.3) we derive the first inequality in (2.1).

Now observe that

$$\sum_{i=0}^{N-1}(N-i)(N+i+1)\|\Delta A_i\|_p^2 \le \max_{i\in\{0,\dots,N-1\}}\|\Delta A_i\|_p^2\sum_{i=0}^{N-1}(N-i)(N+i+1)$$
$$=\frac{1}{3}N(N+1)(2N+1)\max_{i\in\{0,\dots,N-1\}}\|\Delta A_i\|_p^2,$$

which proves the first branch in (2.1).

Observe also that

$$\sum_{i=0}^{N-1}(N-i)(N+i+1)\|\Delta A_i\|_p^2 \le \max_{i\in\{0,\dots,N-1\}}[(N-i)(N+i+1)]\sum_{i=0}^{N-1}\|\Delta A_i\|_p^2$$
$$=\max_{i\in\{0,\dots,N-1\}}[N(N+1)-i(i+1)]\sum_{i=0}^{N-1}\|\Delta A_i\|_p^2$$
$$=N(N+1)\sum_{i=0}^{N-1}\|\Delta A_i\|_p^2,$$

which proves the second branch in (2.1).

The last inequality in (2.2) follows by the *A-G-means inequality*

$$\sqrt{\alpha\beta}\le\frac{1}{2}(\alpha+\beta),\ \alpha,\ \beta\ge 0.$$

□

Corollary 2. *For* $p,\, q>1$ *with* $\frac{1}{p}+\frac{1}{q}=1$, *assume that* $\{A_i\}_{i=0}^N\subset B_p(H)\cap B_q(H)$ *is a sequence of operators with* $A_0=0$, *then*

$$\begin{aligned}\sum_{i=1}^{N}\|A_i\|_2^2 &\le\frac{1}{2}\left(\sum_{i=0}^{N-1}(N-i)(N+i+1)\|\Delta A_i\|_p^2\right)^{1/2} \\ &\times\left(\sum_{i=0}^{N-1}(N-i)(N+i+1)\|\Delta A_i\|_q^2\right)^{1/2} \\ &\le\frac{1}{2}N(N+1) \\ &\times\begin{cases}\frac{1}{3}(2N+1) \\ \times\max_{i\in\{0,\dots,N-1\}}\|\Delta A_i\|_p\max_{i\in\{0,\dots,N-1\}}\|\Delta A_i\|_q, \\ \\ \left(\sum_{i=0}^{N-1}\|\Delta A_i\|_p^2\right)^{1/2}\left(\sum_{i=0}^{N-1}\|\Delta A_i\|_q^2\right)^{1/2}.\end{cases}\end{aligned}\tag{2.7}$$

In particular, for $p = q = 2$, we derive

$$\sum_{i=1}^{N}\|A_i\|_2^2 \leq \frac{1}{2}\sum_{i=0}^{N-1}(N-i)(N+i+1)\|\Delta A_i\|_2^2 \tag{2.8}$$
$$\leq \frac{1}{2}N(N+1)\times\begin{cases}\frac{1}{3}(2N+1)\max_{i\in\{0,\dots,N-1\}}\|\Delta A_i\|_2^2, \\ \sum_{i=0}^{N-1}\|\Delta A_i\|_2^2.\end{cases}$$

The proof follows by taking $B_i = A_i^*$, $i = 0, ..., N$ in (2.1) and observing that $\|A_i A_i^*,\|_1 = \|A_i\|_2^2$ and $\|\Delta A_i^*\|_q^2 = \|\Delta A_i\|_q^2$ for $i = 0, ..., N$.

Also, we have:

Theorem 6. *For p, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, assume that $\{A_i\}_{i=0}^N \subset B_p(H)$, $\{B_i\}_{i=0}^N \subset B_q(H)$ are sequences of operators with $A_N = B_N = 0$, then*

$$\sum_{i=0}^{N-1}\|A_iB_i\|_1 \tag{2.9}$$
$$\leq \frac{1}{2}\left(\sum_{i=0}^{N-1}(i+1)(2N-i)\|\Delta A_i\|_p^2\right)^{1/2}\left(\sum_{i=0}^{N-1}(i+1)(2N-i)\|\Delta B_i\|_q^2\right)^{1/2}$$
$$\leq \frac{1}{2}N(N+1)$$
$$\times\begin{cases}\frac{1}{3}(2N+1)\max_{i\in\{0,\dots,N-1\}}\|\Delta A_i\|_p\max_{i\in\{0,\dots,N-1\}}\|\Delta B_i\|_q, \\ \left(\sum_{i=0}^{N-1}\|\Delta A_i\|_p^2\right)^{1/2}\left(\sum_{i=0}^{N-1}\|\Delta B_i\|_q^2\right)^{1/2}.\end{cases}$$

Also, we have

$$\sum_{i=0}^{N-1}\|A_iB_i\|_1 \leq \frac{1}{2}\left(\sum_{i=0}^{N-1}(i+1)(2N-i)\|\Delta A_i\|_p^2\right)^{1/2} \tag{2.10}$$
$$\times\left(\sum_{i=0}^{N-1}(i+1)(2N-i)\|\Delta B_i\|_q^2\right)^{1/2}$$
$$\leq \frac{1}{4}\sum_{i=0}^{N-1}(i+1)(2N-i)\left(\|\Delta A_i\|_p^2 + \|\Delta B_i\|_q^2\right).$$

Proof. If $A_N = B_N = 0$, then $A_i = -\sum_{j=i}^{N-1} \Delta A_j$ and $B_i = -\sum_{j=i}^{N-1} \Delta B_j$, $i \in \{0, ..., N-1\}$. Then by (1.18) we have

$$\begin{aligned}
\sum_{i=0}^{N-1} \|A_i B_i\|_1 &\le \sum_{i=1}^{N} \|A_i\|_p \|B_i\|_q \\
&= \sum_{i=0}^{N-1} \left\| \sum_{j=i}^{N-1} \Delta A_j \right\|_p \left\| \sum_{j=i}^{N-1} \Delta B_j \right\|_q \\
&= \sum_{i=0}^{N-1} (N-i) \frac{1}{\sqrt{N-i}} \left\| \sum_{j=i}^{N-1} \Delta A_j \right\|_p \frac{1}{\sqrt{N-i}} \left\| \sum_{j=i}^{N-1} \Delta B_j \right\|_q =: \Theta.
\end{aligned}$$

By Cauchy-Bunyakowsky-Schwarz (CBS) inequality we have

$$\frac{1}{\sqrt{N-i}} \left\| \sum_{j=i}^{N-1} \Delta A_j \right\|_p \le \left(\sum_{j=i}^{N-1} \|\Delta A_j\|_p^2 \right)^{1/2}$$

and

$$\frac{1}{\sqrt{N-i}} \left\| \sum_{j=i}^{N-1} \Delta B_j \right\|_q \le \left(\sum_{j=i}^{N-1} \|\Delta B_j\|_q^2 \right)^{1/2},$$

which gives

$$\Theta \le \sum_{i=0}^{N-1} (N-i) \left(\sum_{j=i}^{N-1} \|\Delta A_j\|_q^2 \right)^{1/2} \left(\sum_{j=i}^{N-1} \|\Delta B_j\|_q^2 \right)^{1/2}.$$

By the weighted (CBS) inequality we have

$$\begin{aligned}
&\sum_{i=0}^{N-1} (N-i) \left(\sum_{j=i}^{N-1} \|\Delta A_j\|_p^2 \right)^{1/2} \left(\sum_{j=i}^{N-1} \|\Delta B_j\|_q^2 \right)^{1/2} \\
&\le \left(\sum_{i=0}^{N-1} (N-i) \left(\sum_{j=i}^{N-1} \|\Delta A_j\|_p^2 \right) \right)^{1/2} \left(\sum_{i=0}^{N-1} (N-i) \left(\sum_{j=i}^{N-1} \|\Delta B_j\|_q^2 \right) \right)^{1/2},
\end{aligned}$$

which gives

(2.11)

$$\begin{aligned}
\Theta &\le \left(\sum_{i=0}^{N-1} (N-i) \left(\sum_{j=i}^{N-1} \|\Delta A_j\|_p^2 \right) \right)^{1/2} \left(\sum_{i=0}^{N-1} (N-i) \left(\sum_{j=i}^{N-1} \|\Delta B_j\|_q^2 \right) \right)^{1/2} \\
&=: \Lambda.
\end{aligned}$$

From (1.1) we get for $m = 0$ and $n = N-1$ that

$$\sum_{i=0}^{N-1} c_i \Delta d_i = c_{N-1} d_N - c_0 d_0 - \sum_{i=0}^{N-2} d_{i+1} \Delta c_i. \tag{2.12}$$

Take $c_i = \sum_{j=i}^{N-1} \|\Delta A_j\|_p^2$ and $d_i = -\frac{1}{2}(N-i)(N-i+1)$, then we get

$$\Delta d_i = d_{i+1} - d_i = -\frac{1}{2}(N-i-1)(N-i) + \frac{1}{2}(N-i)(N-i+1)$$
$$= \frac{1}{2}(N-i)(-N+i+1+N-i+1) = N-i$$

and

$$\Delta c_i = c_{i+1} - c_i = \sum_{j=i+1}^{N-1} \|\Delta A_j\|_p^2 - \sum_{j=i}^{N-1} \|\Delta A_j\|_p^2 = -\|\Delta A_i\|_p^2 .$$

Then

$$\sum_{i=0}^{N-1} (N-i)\left(\sum_{j=i}^{N-1} \|\Delta A_j\|_p^2\right)$$
$$= \frac{1}{2}N(N+1)\sum_{j=0}^{N-1} \|\Delta A_j\|_p^2 - \frac{1}{2}\sum_{i=0}^{N-2}(N-i-1)(N-i)\|\Delta A_i\|_p^2$$
$$= \frac{1}{2}N(N+1)\sum_{i=0}^{N-1} \|\Delta A_j\|_p^2 - \frac{1}{2}\sum_{i=0}^{N-1}(N-i-1)(N-i)\|\Delta A_i\|_p^2$$
$$= \frac{1}{2}\sum_{i=0}^{N-1}[N(N+1)-(N-i-1)(N-i)]\|\Delta A_i\|_p^2$$
$$= \frac{1}{2}\sum_{i=0}^{N-1}(i+1)(2N-i)\|\Delta A_i\|_p^2$$

and

$$\sum_{i=0}^{N-1} (N-i)\left(\sum_{j=i}^{N-1} \|\Delta B_j\|_q^2\right) = \frac{1}{2}\sum_{i=0}^{N-1}(i+1)(2N-i)\|\Delta B_i\|_q^2 .$$

Therefore

$$\Lambda = \frac{1}{2}\left(\sum_{i=0}^{N-1}(i+1)(2N-i)\|\Delta A_i\|_p^2\right)^{1/2}\left(\sum_{i=0}^{N-1}(i+1)(2N-i)\|\Delta B_i\|_q^2\right)^{1/2},$$

and by (2.11) we derive the first inequality in (2.9).

Now, observe that

$$\sum_{i=0}^{N-1}(i+1)(2N-i)\|\Delta A_i\|_p^2 \le \max_{i\in\{0,\dots,N-1\}} |\Delta A_i|^2 \sum_{i=0}^{N-1}(i+1)(2N-i)$$
$$= \frac{1}{3}N(N+1)(2N+1)\max_{i\in\{0,\dots,N-1\}} \|\Delta A_i\|_p^2$$

and

$$\sum_{i=0}^{N-1}(i+1)(2N-i)\|\Delta B_i\|_q^2 \le \frac{1}{3}N(N+1)(2N+1)\max_{i\in\{0,\dots,N-1\}} \|\Delta B_i\|_q^2 ,$$

which proves the first branch of (2.9).

Also

$$
\begin{aligned}
&\sum_{i=0}^{N-1}(i+1)(2N-i)\|\Delta A_i\|_p^2 \\
&=\sum_{i=0}^{N-1}[N(N+1)-(N-i-1)(N-i)]\|\Delta A_i\|_p^2 \\
&\le \max_{i\in\{0,\dots,N-1\}}[N(N+1)-(N-i-1)(N-i)]\sum_{i=0}^{N-1}\|\Delta A_i\|_p^2 \\
&=N(N+1)\sum_{i=0}^{N-1}\|\Delta A_i\|_p^2,
\end{aligned}
$$

which proves the second branch of (2.9). □

Corollary 3. *For $p, q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, assume that $\{A_i\}_{i=0}^N \subset B_p(H)\cap B_q(H)$ with $A_N=0$, then*

$$
\begin{aligned}
&\sum_{i=0}^{N-1}\|A_iB_i\|_1 \\
&\le \frac{1}{2}\left(\sum_{i=0}^{N-1}(i+1)(2N-i)\|\Delta A_i\|_p^2\right)^{1/2} \\
&\times\left(\sum_{i=0}^{N-1}(i+1)(2N-i)\|\Delta B_i\|_q^2\right)^{1/2} \\
&\le \frac{1}{2}N(N+1) \\
&\times\begin{cases} \frac{1}{3}(2N+1)\max_{i\in\{0,\dots,N-1\}}\|\Delta A_i\|_p\max_{i\in\{0,\dots,N-1\}}\|\Delta B_i\|_q, \\ \left(\sum_{i=0}^{N-1}\|\Delta A_i\|_p^2\right)^{1/2}\left(\sum_{i=0}^{N-1}\|\Delta B_i\|_q^2\right)^{1/2}. \end{cases}
\end{aligned} \tag{2.13}
$$

We also have:

Theorem 7. *For $p,\ q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, assume that $\{A_i\}_{i=0}^{N}\subset B_p(H)$, $\{B_i\}_{i=0}^{N}\subset B_q(H)$ are sequences of operators with $A_0=B_N=0$, then*

$$\begin{aligned}
&\sum_{i=1}^{N-1}\left\|A_iB_i\right\|_1 \qquad (2.14)\\
&\leq \frac{1}{2}\left(\sum_{i=0}^{N-1}(N-i)(N+i+1)\left\|\Delta A_i\right\|_p^2\right)^{1/2}\\
&\times\left(\sum_{i=0}^{N-1}(i+1)(2N-i)\left\|\Delta B_i\right\|_q^2\right)^{1/2}\\
&\leq \frac{1}{2}N(N+1)\\
&\times\begin{cases}\frac{1}{3}(2N+1)\max_{i\in\{0,\dots,N-1\}}\left\|\Delta A_i\right\|_p\max_{i\in\{0,\dots,N-1\}}\left\|\Delta B_i\right\|_q,\\ \\ \left(\sum_{i=0}^{N-1}\left\|\Delta A_i\right\|_p^2\right)^{1/2}\left(\sum_{i=0}^{N-1}\left\|\Delta B_i\right\|_q^2\right)^{1/2}.\end{cases}
\end{aligned}$$

Also,

$$\begin{aligned}
&\sum_{i=1}^{N-1}\left\|A_iB_i\right\|_1 \qquad (2.15)\\
&\leq \frac{1}{2}\left(\sum_{i=0}^{N-1}(N-i)(N+i+1)\left\|\Delta A_i\right\|_p^2\right)^{1/2}\\
&\times\left(\sum_{i=0}^{N-1}(i+1)(2N-i)\left\|\Delta B_i\right\|_q^2\right)^{1/2}\\
&\leq \frac{1}{4}\left[\sum_{i=0}^{N-1}(N-i)(N+i+1)\left\|\Delta A_i\right\|_p^2+\sum_{i=0}^{N-1}(i+1)(2N-i)\left\|\Delta B_i\right\|_q^2\right].
\end{aligned}$$

Proof. Since $A_0=0$, and $B_N=0$, hence $A_i=\sum_{j=0}^{i-1}\Delta A_j$ and $B_i=-\sum_{j=i}^{N-1}\Delta B_j$. Then by Hölder and (CBS) inequalities

$$\begin{aligned}
\sum_{i=1}^{N-1}\left\|A_iB_i\right\|_1 &\leq \sum_{i=1}^{N-1}\left\|A_i\right\|_p\left\|B_i\right\|_q \qquad (2.16)\\
&=\sum_{i=1}^{N-1}\left\|\sum_{j=0}^{i-1}\Delta A_j\right\|_p\left\|\sum_{j=i}^{N-1}\Delta B_j\right\|_q\\
&\leq \sum_{i=1}^{N-1}\sqrt{i}\left(\sum_{j=0}^{i-1}\left\|\Delta A_j\right\|_p^2\right)^{1/2}\sqrt{N-i}\left(\sum_{j=i}^{N-1}\left\|\Delta B_j\right\|_q^2\right)^{1/2}
\end{aligned}$$

$$\leq \left(\sum_{i=1}^{N-1} \left[\sqrt{i} \left(\sum_{j=0}^{i-1} \|\Delta A_j\|_p^2 \right)^{1/2} \right]^2 \right)^{1/2}$$

$$\times \left(\sum_{i=1}^{N-1} \left[\sqrt{N-i} \left(\sum_{j=i}^{N-1} \|\Delta B_j\|_q^2 \right)^{1/2} \right]^2 \right)^{1/2}$$

$$= \left(\sum_{i=1}^{N-1} i \left(\sum_{j=0}^{i-1} \|\Delta A_j\|_p^2 \right) \right)^{1/2} \left(\sum_{i=1}^{N-1} (N-i) \left(\sum_{j=i}^{N-1} \|\Delta B_j\|_q^2 \right) \right)^{1/2} =: E.$$

Since

$$\sum_{i=1}^{N} i \left(\sum_{j=0}^{i-1} \|\Delta A_j\|_p^2 \right) = \frac{1}{2} \sum_{i=0}^{N-1} (N-i)(N+i+1) \|\Delta A_i\|_p^2$$

and

$$\sum_{i=0}^{N-1} (N-i) \left(\sum_{j=i}^{N-1} \|\Delta B_j\|_q^2 \right) = \frac{1}{2} \sum_{i=0}^{N-1} (i+1)(2N-i) \|\Delta B_i\|_q^2 ,$$

hence

$$E = \frac{1}{2} \left(\sum_{i=0}^{N-1} (N-i)(N+i+1) \|\Delta A_i\|_p^2 \right)^{1/2} \times \left(\sum_{i=0}^{N-1} (i+1)(2N-i) \|\Delta B_i\|_q^2 \right)^{1/2} .$$

By employing (2.16) we get the first part of (2.14).

The rest follows as above. □

Corollary 4. *For* $p, q > 1$ *with* $\frac{1}{p} + \frac{1}{q} = 1$*, assume that* $\{A_i\}_{i=0}^{N} \subset B_p(H) \cap B_q(H)$ *with* $A_0 = A_N = 0$*, then*

(2.17)

$$\begin{aligned} \sum_{i=1}^{N-1} \|A_i\|_2^2 &\leq \frac{1}{2} \left(\sum_{i=0}^{N-1} (N-i)(N+i+1) \|\Delta A_i\|_p^2 \right)^{1/2} \\ &\times \left(\sum_{i=0}^{N-1} (i+1)(2N-i) \|\Delta A_i\|_q^2 \right)^{1/2} \\ &\leq \frac{1}{2} N(N+1) \\ &\times \begin{cases} \frac{1}{3}(2N+1) \max_{i \in \{0,\dots,N-1\}} \|\Delta A_i\|_p \max_{i \in \{0,\dots,N-1\}} \|\Delta A_i\|_q , \\ \\ \left(\sum_{i=0}^{N-1} \|\Delta A_i\|_p^2 \right)^{1/2} \left(\sum_{i=0}^{N-1} \|\Delta A_i\|_q^2 \right)^{1/2} . \end{cases} \end{aligned}$$

Also,

$$\begin{aligned}(2.18)\quad &\sum_{i=1}^{N-1}\|A_i\|_2^2 \\ &\leq \frac{1}{2}\left(\sum_{i=0}^{N-1}(N-i)(N+i+1)\|\Delta A_i\|_p^2\right)^{1/2} \\ &\times\left(\sum_{i=0}^{N-1}(i+1)(2N-i)\|\Delta B_i\|_q^2\right)^{1/2} \\ &\leq \frac{1}{4}\left[\sum_{i=0}^{N-1}(N-i)(N+i+1)\|\Delta A_i\|_p^2+\sum_{i=0}^{N-1}(i+1)(2N-i)\|\Delta B_i\|_q^2\right].\end{aligned}$$

Remark 2. *For $p=q=2$ we derive from Corollary 4 for $\{A_i\}_{i=0}^N \subset B_2(H)$ with $A_0=A_N=0$ that*

$$\begin{aligned}(2.19)\quad \sum_{i=1}^{N-1}\|A_i\|_2^2 &\leq \frac{1}{2}\left(\sum_{i=0}^{N-1}(N-i)(N+i+1)\|\Delta A_i\|_2^2\right)^{1/2} \\ &\times\left(\sum_{i=0}^{N-1}(i+1)(2N-i)\|\Delta A_i\|_2^2\right)^{1/2} \\ &\leq \frac{1}{2}N(N+1)\times\begin{cases}\frac{1}{3}(2N+1)\max_{i\in\{0,\dots,N-1\}}\|\Delta A_i\|_2^2, \\ \sum_{i=0}^{N-1}\|\Delta A_i\|_2^2.\end{cases}\end{aligned}$$

From (2.19) we derive the trace inequality

$$\begin{aligned}(2.20)\quad \operatorname{tr}\left(\sum_{i=1}^{N-1}|A_i|^2\right) &\leq \frac{1}{2}\left[\operatorname{tr}\left(\sum_{i=0}^{N-1}(N-i)(N+i+1)|\Delta A_i|^2\right)\right]^{1/2} \\ &\times\left[\operatorname{tr}\left(\sum_{i=0}^{N-1}(i+1)(2N-i)|\Delta A_i|^2\right)\right]^{1/2}\end{aligned}$$

for $\{A_i\}_{i=0}^N \subset B_2(H)$ with $A_0=A_N=0$.

3. p-Schatten Norm Inequalities

From an alternative perspective, we have:

Theorem 8. *For $p \geq 1$, assume that $\{A_i\}_{i=0}^N, \{B_i\}_{i=0}^N \subset B_p(H)$ are sequences of operators with $A_0 = B_0 = 0$, then*

$$\begin{aligned}
(3.1)\qquad & \sum_{i=1}^{N} \|A_i B_i\|_p \\
& \leq \frac{1}{2} \left(\sum_{i=0}^{N-1} (N-i)(N+i+1) \|\Delta A_i\|_p^2 \right)^{1/2} \\
& \times \left(\sum_{i=0}^{N-1} (N-i)(N+i+1) \|\Delta B_i\|_p^2 \right)^{1/2} \\
& \leq \frac{1}{2} N (N+1) \\
& \times \begin{cases} \frac{1}{3}(2N+1) \max_{i\in\{0,...,N-1\}} \|\Delta A_i\|_p \max_{i\in\{0,...,N-1\}} \|\Delta B_i\|_p, \\ \\ \left(\sum_{i=0}^{N-1} \|\Delta A_i\|_p^2\right)^{1/2} \left(\sum_{i=0}^{N-1} \|\Delta B_i\|_p^2\right)^{1/2}. \end{cases}
\end{aligned}$$

Also,

$$\begin{aligned}
(3.2)\qquad \sum_{i=1}^{N} \|A_i B_i\|_p & \leq \frac{1}{2} \left(\sum_{i=0}^{N-1} (N-i)(N+i+1) \|\Delta A_i\|_p^2 \right)^{1/2} \\
& \times \left(\sum_{i=0}^{N-1} (N-i)(N+i+1) \|\Delta B_i\|_p^2 \right)^{1/2} \\
& \leq \frac{1}{4} \sum_{i=0}^{N-1} (N-i)(N+i+1) \left(\|\Delta A_i\|_p^2 + \|\Delta B_i\|_p^2 \right).
\end{aligned}$$

Proof. Since $A_0 = B_0 = 0$, hence $A_i = \sum_{j=0}^{i-1} \Delta A_j$ and $B_i = \sum_{j=0}^{i-1} \Delta B_j$ for $i = 1, ..., N$. Then, by (1.15)

$$\begin{aligned}
\sum_{i=1}^{N} \|A_i B_i\|_p & \leq \sum_{i=1}^{N} \|A_i\|_p \|B_i\|_p \\
& = \sum_{i=1}^{N} \left\| \sum_{j=0}^{i-1} \Delta A_j \right\|_p \left\| \sum_{j=0}^{i-1} \Delta B_j \right\|_p \\
& = \sum_{i=1}^{N} i \frac{1}{\sqrt{i}} \left\| \sum_{j=0}^{i-1} \Delta A_j \right\|_p \frac{1}{\sqrt{i}} \left\| \sum_{j=0}^{i-1} \Delta B_j \right\|_p =: \Psi.
\end{aligned}$$

By Cauchy-Bunyakowsky-Schwarz (CBS) inequality we have

$$\begin{aligned} \frac{1}{\sqrt{i}}\left\|\sum_{j=0}^{i-1}\Delta A_j\right\|_p &\leq \left(\sum_{j=0}^{i-1}\|\Delta A_j\|_p^2\right)^{1/2}, \\ \frac{1}{\sqrt{i}}\left\|\sum_{j=0}^{i-1}\Delta B_j\right\|_p &\leq \left(\sum_{j=0}^{i-1}\|\Delta B_j\|_p^2\right)^{1/2}, \end{aligned}$$

which gives.

$$\Psi \leq \sum_{i=1}^{N} i\left(\sum_{j=0}^{i-1}\|\Delta A_j\|_p^2\right)^{1/2}\left(\sum_{j=0}^{i-1}\|\Delta B_j\|_p^2\right)^{1/2}.$$

By the weighted (CBS) inequality we have

$$\begin{aligned} &\sum_{i=1}^{N} i\left(\sum_{j=0}^{i-1}\|\Delta A_j\|_p^2\right)^{1/2}\left(\sum_{j=0}^{i-1}\|\Delta B_j\|_p^2\right)^{1/2} \\ &\leq \left(\sum_{i=1}^{N} i\left[\left(\sum_{j=0}^{i-1}\|\Delta A_j\|_p^2\right)^{1/2}\right]^2\right)^{1/2}\left(\sum_{i=1}^{N} i\left[\left(\sum_{j=0}^{i-1}\|\Delta B_j\|_p^2\right)^{1/2}\right]^2\right)^{1/2} \\ &= \left(\sum_{i=1}^{N} i\left(\sum_{j=0}^{i-1}\|\Delta A_j\|_p^2\right)\right)^{1/2}\left(\sum_{i=1}^{N} i\left(\sum_{j=0}^{i-1}\|\Delta B_j\|_p^2\right)\right)^{1/2}, \end{aligned}$$

which implies that

$$\Gamma \leq \left(\sum_{i=1}^{N} i\left(\sum_{j=0}^{i-1}\|\Delta A_j\|_p^2\right)\right)^{1/2}\left(\sum_{i=1}^{N} i\left(\sum_{j=0}^{i-1}\|\Delta B_j\|_p^2\right)\right)^{1/2} =: \Phi.$$

Now, by utilising a similar argument to the one in the proof of Theorem 5 we deduce the desired result (3.2). □

We can also prove the following results as well:

Theorem 9. *For $p \geq 1$, assume that $\{A_i\}_{i=0}^{N}, \{B_i\}_{i=0}^{N} \subset B_p(H)$ are sequences of operators with $A_N = B_N = 0$, then*

$$\begin{aligned}(3.3)\quad & \sum_{i=0}^{N-1} \|A_i B_i\|_p \\ & \leq \frac{1}{2}\left(\sum_{i=0}^{N-1}(i+1)(2N-i)\|\Delta A_i\|_p^2\right)^{1/2}\left(\sum_{i=0}^{N-1}(i+1)(2N-i)\|\Delta B_i\|_{pq}^2\right)^{1/2} \\ & \leq \frac{1}{2}N(N+1) \\ & \times \begin{cases} \frac{1}{3}(2N+1)\max_{i\in\{0,\dots,N-1\}}\|\Delta A_i\|_p \max_{i\in\{0,\dots,N-1\}}\|\Delta B_i\|_p, \\ \left(\sum_{i=0}^{N-1}\|\Delta A_i\|_p^2\right)^{1/2}\left(\sum_{i=0}^{N-1}\|\Delta B_i\|_q^2\right)^{1/2}. \end{cases}\end{aligned}$$

Also, we have

$$\begin{aligned}(3.4)\quad \sum_{i=0}^{N-1}\|A_i B_i\|_1 & \leq \frac{1}{2}\left(\sum_{i=0}^{N-1}(i+1)(2N-i)\|\Delta A_i\|_p^2\right)^{1/2} \\ & \times \left(\sum_{i=0}^{N-1}(i+1)(2N-i)\|\Delta B_i\|_q^2\right)^{1/2} \\ & \leq \frac{1}{4}\sum_{i=0}^{N-1}(i+1)(2N-i)\left(\|\Delta A_i\|_p^2 + \|\Delta B_i\|_q^2\right).\end{aligned}$$

Finally, we have

Theorem 10. *For $p \geq 1$, assume that $\{A_i\}_{i=0}^{N}, \{B_i\}_{i=0}^{N} \subset B_p(H)$ are sequences of operators with $A_0 = B_N = 0$, then*

$$\begin{aligned}(3.5)\quad & \sum_{i=1}^{N-1}\|A_i B_i\|_p \\ & \leq \frac{1}{2}\left(\sum_{i=0}^{N-1}(N-i)(N+i+1)\|\Delta A_i\|_p^2\right)^{1/2} \\ & \times \left(\sum_{i=0}^{N-1}(i+1)(2N-i)\|\Delta B_i\|_p^2\right)^{1/2} \\ & \leq \frac{1}{2}N(N+1) \\ & \times \begin{cases} \frac{1}{3}(2N+1)\max_{i\in\{0,\dots,N-1\}}\|\Delta A_i\|_p \max_{i\in\{0,\dots,N-1\}}\|\Delta B_i\|_p, \\ \left(\sum_{i=0}^{N-1}\|\Delta A_i\|_p^2\right)^{1/2}\left(\sum_{i=0}^{N-1}\|\Delta B_i\|_p^2\right)^{1/2}. \end{cases}\end{aligned}$$

Also,

$$\sum_{i=1}^{N-1}\left\|A_iB_i\right\|_p \tag{3.6}$$
$$\leq\frac{1}{2}\left(\sum_{i=0}^{N-1}\left(N-i\right)\left(N+i+1\right)\left\|\Delta A_i\right\|_p^2\right)^{1/2}$$
$$\times\left(\sum_{i=0}^{N-1}\left(i+1\right)\left(2N-i\right)\left\|\Delta B_i\right\|_p^2\right)^{1/2}$$
$$\leq\frac{1}{4}\left[\sum_{i=0}^{N-1}\left(N-i\right)\left(N+i+1\right)\left\|\Delta A_i\right\|_p^2+\sum_{i=0}^{N-1}\left(i+1\right)\left(2N-i\right)\left\|\Delta B_i\right\|_p^2\right].$$

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