p-SCHATTEN NORM HÖLDER'S TYPE INEQUALITIES OF OPIAL-LASOTA'S KIND

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Abstract. An operator $A \in \mathcal{B}(H)$ is said to belong to the *von Neumann-Schatten class* $\mathcal{B}_p(H)$, $1 \le p < \infty$ if the *p-Schatten norm* is finite

$$\|A\|_p := [\operatorname{tr}(|A|^p)]^{1/p} < \infty.$$

In this paper we show among others that, for $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, assume that $\{A_i\}_{i=0}^N \subset B_p(H)$, $\{B_i\}_{i=0}^N \subset B_q(H)$ are sequences of operators with $B_0 = B_N = 0$, then for $n \in \{2, ..., N-1\}$, $\alpha, \beta > 1$, $\frac{1}{\alpha} + \frac{1}{\beta} = 1$ we have that

$$\sum_{i=1}^{N-1} \|\Delta A_i B_i\|_1 \le \left(\sum_{i=0}^{N-1} \alpha_i(n) \|\Delta A_i\|_p^\alpha\right)^{1/\alpha} \left(\sum_{i=0}^{N-1} \beta_i(n) \|\Delta B_i\|_q^\beta\right)^{1/\beta}$$
$$\le \frac{1}{\alpha} \sum_{i=0}^{N-1} \alpha_i(n) \|\Delta A_i\|_p^\alpha + \frac{1}{\beta} \sum_{i=0}^{N-1} \beta_i(n) \|\Delta B_i\|_q^\beta,$$

where

$$\alpha_i(n) := \begin{cases} i, \text{ if } 0 \le i \le n-1, \\ N-i, \text{ if } n \le i \le N-1 \end{cases}, \ \beta_i(n) := \begin{cases} n-i-1, \text{ if } 0 \le i \le n-1, \\ i+1-n, \text{ if } n \le i \le N-1. \end{cases}$$

1. Introduction

We recall the following Opial type inequalities:

Theorem 1. *Assume that $u : [a, b] \subset \mathbb{R} \to \mathbb{R}$ is an absolutely continuous function on the interval $[a, b]$ and such that $u' \in L_2[a, b]$.*

(i) *If $u(a) = u(b) = 0$, then*

$$\int_a^b |u(t) u'(t)| \, dt \le \frac{1}{4} (b-a) \int_a^b |u'(t)|^2 \, dt, \tag{1.1}$$

with equality if and only if

$$u(t) = \begin{cases} c(t-a) \ \text{ if } a \le t \le \frac{a+b}{2}, \\ c(b-t) \ \text{ if } \frac{a+b}{2} < t \le b \end{cases}$$

where c is an arbitrary constant;

(ii) *If $u(a) = 0$, then*

$$\int_a^b |u(t) u'(t)| \, dt \le \frac{1}{2} (b-a) \int_a^b |u'(t)|^2 \, dt, \tag{1.2}$$

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with equality if and only if $u(t) = c(t-a)$ *for some constant* c;

(iii) *If* $\int_a^b u(t)\,dt = 0$, *then the inequality (1.1) holds with equality if and only if*

$$u(t) = c\left(t - \frac{a+b}{2}\right)$$

for any constant c.

The inequality (1.1) was obtained by Olech in [14] in which he gave a simplified proof of an inequality originally due in a slightly less general form to Zdzislaw Opial [15].

Embedded in Olech's proof is the half-interval form of Opial's inequality, also discovered by Beesack [2], which is satisfied by those u vanishing only at a.

The inequality (1.1) in the case (iii), namely in the case that u satisfies the condition $\int_a^b u(t)\,dt = 0$ was obtained by Brown and Plum in [6].

As mentioned in [6] the inequality (1.1) also holds if $u(a) + u(b) = 0$.

For a sequence $\{x_i\}_{i=0}^n$, we consider the forward operator Δ defined by $\Delta x_i = x_{i+1} - x_i$, $i = 0, ..., n-1$. The summation by parts formula also holds

$$\sum_{k=m}^{n} a_k \Delta b_k = a_n b_{n+1} - a_m b_m - \sum_{k=m}^{n-1} b_{k+1} \Delta a_k. \tag{1.3}$$

In [12], Lasota provided discrete versions of Opial inequality (1.1) about the forward difference operator as follows:

Theorem 2. *Let* $\{x_i\}_{i=0}^N$ *be a sequence of numbers real numbers with* $x_0 = x_N = 0$. *Then, the following inequality holds*

$$\sum_{i=1}^{N-1} |x_i \Delta x_i| \le \frac{1}{2}\left\lfloor \frac{N+1}{2} \right\rfloor \sum_{i=0}^{N-1} |\Delta x_i|^2, \tag{1.4}$$

where $\lfloor\cdot\rfloor$ *is the integer part function. If* N *is even, then the inequality (1.4) is sharp.*

Also, we have the following results, see [1]:

Theorem 3. *Let* $\{x_i\}_{i=0}^N$ *be a sequence of numbers real numbers. If* $x_0 = 0$, *then*

$$\sum_{i=1}^{\tau-1} |x_i \Delta x_i| \le \frac{1}{2}(\tau - 1) \sum_{i=0}^{\tau-1} |\Delta x_i|^2, \ \tau \in \{2, ..., N\}. \tag{1.5}$$

If $x_N = 0$, *then*

$$\sum_{i=\tau}^{N-1} |x_i \Delta x_i| \le \frac{1}{2}(N - \tau + 1) \sum_{i=0}^{N-1} |\Delta x_i|^2, \ \tau \in \{1, ..., N-1\}. \tag{1.6}$$

For other discrete Opial type inequalities, see [10], [11] and [16]-[20].

In order to extend these results for p-Schatten norms we need the following preparations.

Let $(H; \langle \cdot,\cdot \rangle)$ be a complex Hilbert space and $\mathcal{B}(H)$ the Banach algebra of all bounded linear operators on H. If $\{e_i\}_{i\in I}$ an orthonormal basis of H, we say that $A \in \mathcal{B}(H)$ is of *trace class* if

$$\|A\|_1 := \sum_{i\in I} \langle |A| e_i, e_i \rangle < \infty. \tag{1.7}$$

The definition of $\|A\|_1$ does not depend on the choice of the orthonormal basis $\{e_i\}_{i\in I}$. We denote by $\mathcal{B}_1(H)$ the set of trace class operators in $\mathcal{B}(H)$.

We define the *trace* of a trace class operator $A\in\mathcal{B}_1(H)$ to be

$$\operatorname{tr}(A):=\sum_{i\in I}\langle Ae_i,e_i\rangle, \tag{1.8}$$

where $\{e_i\}_{i\in I}$ an orthonormal basis of H. Note that this coincides with the usual definition of the trace if H is finite-dimensional. We observe that the series (1.8) converges absolutely and it is independent from the choice of basis.

The following result collects some properties of the trace:

Theorem 4. *We have:*

(i) If $A\in\mathcal{B}_1(H)$ then $A^*\in\mathcal{B}_1(H)$ and

$$\operatorname{tr}(A^*)=\overline{\operatorname{tr}(A)}; \tag{1.9}$$

(ii) If $A\in\mathcal{B}_1(H)$ and $T\in\mathcal{B}(H)$, then AT, $TA\in\mathcal{B}_1(H)$ and

$$\operatorname{tr}(AT)=\operatorname{tr}(TA)\ \text{ and }\ |\operatorname{tr}(AT)|\le\|A\|_1\|T\|; \tag{1.10}$$

(iii) $\operatorname{tr}(\cdot)$ *is a bounded linear functional on $\mathcal{B}_1(H)$ with $\|\operatorname{tr}\|=1$;*

(iv) If A, $B\in\mathcal{B}_2(H)$ then AB, $BA\in\mathcal{B}_1(H)$ and $\operatorname{tr}(AB)=\operatorname{tr}(BA)$;

(v) $\mathcal{B}_{fin}(H)$ is a dense subspace of $\mathcal{B}_1(H)$.

An operator $A\in\mathcal{B}(H)$ is said to belong to the *von Neumann-Schatten class* $\mathcal{B}_p(H)$, $1\le p<\infty$ if the *p-Schatten norm* is finite [21, p. 60-64]

$$\|A\|_p:=[\operatorname{tr}(|A|^p)]^{1/p}<\infty.$$

For $1<p<q<\infty$ we have that

$$\mathcal{B}_1(H)\subset\mathcal{B}_p(H)\subset\mathcal{B}_q(H)\subset\mathcal{B}(H) \tag{1.11}$$

and

$$\|A\|_1\ge\|A\|_p\ge\|A\|_q\ge\|A\|. \tag{1.12}$$

For $p\ge1$ the functional $\|\cdot\|_p$ is a *norm* on the $*$-ideal $\mathcal{B}_p(H)$ and $\left(\mathcal{B}_p(H),\|\cdot\|_p\right)$ is a Banach space.

Also, see for instance [21, p. 60-64],

$$\|A\|_p=\|A^*\|_p,\ A\in\mathcal{B}_p(H) \tag{1.13}$$

$$\|AB\|_p\le\|A\|_p\|B\|_p,\ A,B\in\mathcal{B}_p(H) \tag{1.14}$$

and

$$\|AB\|_p\le\|A\|_p\|B\|,\ \ \|BA\|_p\le\|B\|\|A\|_p,\ A\in\mathcal{B}_p(H),\ B\in\mathcal{B}(H). \tag{1.15}$$

This implies that

$$\|CAB\|_p\le\|C\|\|A\|_p\|B\|,\ A\in\mathcal{B}_p(H),\ B,\ C\in\mathcal{B}(H). \tag{1.16}$$

In terms of *p-Schatten norm* we have the *Hölder inequality* for p, $q>1$ with $\frac{1}{p}+\frac{1}{q}=1$

$$(|\operatorname{tr}(AB)|\le)\,\|AB\|_1\le\|A\|_p\|B\|_q,\ A\in\mathcal{B}_p(H),\ B\in\mathcal{B}_q(H). \tag{1.17}$$

For the theory of trace functionals and their applications the reader is referred to [18] and [21].

For some classical trace inequalities see [7], [8], and [13], which are continuations of the work of Bellman [2].

2. 1-Schatten Norm Inequalities

We have the following result for two sequences:

Theorem 5. *For $p,\ q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, assume that $\{A_i\}_{i=0}^{N}\subset B_p(H)$, $\{B_i\}_{i=0}^{N}\subset B_q(H)$ are sequences of operators with $B_0=0$, then for $n\in\{2,...,N\}$,*

$$\sum_{i=1}^{n-1}\|\Delta A_iB_i\|_1\leq\left(\sum_{i=0}^{n-1}i\|\Delta A_i\|_p^{\alpha}\right)^{1/\alpha}\left(\sum_{i=0}^{n-1}(n-i-1)\|\Delta B_i\|_q^{\beta}\right)^{1/\beta}\tag{2.1}$$
$$\leq\frac{1}{\alpha}\sum_{i=0}^{n-1}i\|\Delta A_i\|_p^{\alpha}+\frac{1}{\beta}\sum_{i=0}^{n-1}(n-i-1)\|\Delta B_i\|_q^{\beta},$$

where $\alpha,\ \beta>1$, $\frac{1}{\alpha}+\frac{1}{\beta}=1$.

In particular,

$$\sum_{i=1}^{n-1}\|\Delta A_iB_i\|_1\leq\left(\sum_{i=0}^{n-1}i\|\Delta A_i\|_p^{2}\right)^{1/2}\left(\sum_{i=0}^{n-1}(n-i-1)\|\Delta B_i\|_q^{2}\right)^{1/2}\tag{2.2}$$
$$\leq\frac{1}{2}\sum_{i=0}^{n-1}\left(i\|\Delta A_i\|_p^{2}+(n-i-1)\|\Delta B_i\|_q^{2}\right).$$

Proof. Let $n\in\{2,...,N\}$. Since $B_0=0$, hence $B_i=\sum_{j=0}^{i-1}\Delta B_j$ for $i=1,...,n-1$. Then by (1.17)

$$\begin{aligned}\sum_{i=1}^{n-1}\|\Delta A_iB_i\|_1 &\leq \sum_{i=1}^{n-1}\|\Delta A_i\|_p\|B_i\|_q=\sum_{i=1}^{n-1}\|\Delta A_i\|_p\left\|\sum_{j=0}^{i-1}\Delta B_j\right\|_q\\ &= \sum_{i=1}^{n-1}i^{1/\alpha}\|\Delta A_i\|_p\,i^{-1/\alpha}\left\|\sum_{j=0}^{i-1}\Delta B_j\right\|_q=:\Gamma.\end{aligned}$$

Using the discrete Hölder inequality for $\alpha,\ \beta>1$, $\frac{1}{\alpha}+\frac{1}{\beta}=1$, we have

$$\Gamma\leq\left(\sum_{i=1}^{n-1}i\|\Delta A_i\|_p^{\alpha}\right)^{1/\alpha}\left[\sum_{i=1}^{n-1}i^{-\beta/\alpha}\left\|\sum_{j=0}^{i-1}\Delta B_j\right\|_q^{\beta}\right]^{1/\beta}=:\Delta.$$

By Hölder's inequality we have

$$i^{-1/\alpha}\left\|\sum_{j=0}^{i-1}\Delta B_j\right\|_q\leq\left(\sum_{j=0}^{i-1}\left|\|\Delta B_j\|_q\right|^{\beta}\right)^{1/\beta},$$

which implies that

$$i^{-\beta/\alpha}\left\|\sum_{j=0}^{i-1}\Delta B_j\right\|_q^{\beta}\leq\sum_{j=0}^{i-1}\|\Delta B_j\|_q^{\beta},$$

which implies that

$$\Delta \le \left(\sum_{i=1}^{n-1} i \left\|\Delta A_i\right\|_p^{\alpha}\right)^{1/\alpha} \left[\sum_{i=1}^{n-1}\left(\sum_{j=0}^{i-1}\left\|\Delta B_j\right\|_q^{\beta}\right)\right]^{1/\beta}. \tag{2.3}$$

From (1.3), we have for $m=1$ and n is replaced by $n-1$ that

$$\sum_{i=1}^{n-1} a_i \Delta b_i = a_{n-1}b_n - a_1 b_1 - \sum_{i=1}^{n-2} b_{i+1}\Delta a_i,$$

which by taking $a_i = \sum_{j=0}^{i-1}\left\|\Delta B_j\right\|_q^{\beta}$, $b_i = i$, produces that

$$\begin{aligned}
&\sum_{i=1}^{n-1}\left(\sum_{j=0}^{i-1}\left\|\Delta B_j\right\|_q^{\beta}\right)\\
&= n\sum_{j=0}^{n-2}\left\|\Delta B_j\right\|_q^{\beta} - \left\|\Delta B_0\right\|_q^{\beta} - \sum_{i=1}^{n-2}(i+1)\left\|\Delta B_i\right\|_q^{\beta}\\
&= (n-1)\left\|\Delta B_0\right\|_q^{\beta} + \sum_{i=1}^{n-2}(n-i-1)\left\|\Delta B_i\right\|_q^{\beta} = \sum_{i=0}^{n-2}(n-i-1)\left\|\Delta B_i\right\|_q^{\beta}.
\end{aligned}$$

Now, it is obvious that

$$\sum_{i=1}^{n-1} i\left\|\Delta A_i\right\|_p^{\beta} = \sum_{i=0}^{n-1} i\left\|\Delta A_i\right\|_p^{\beta}$$

and

$$\sum_{i=0}^{n-2}(n-i-1)\left\|\Delta B_i\right\|_q^{\beta} = \sum_{i=0}^{n-1}(n-i-1)\left\|\Delta B_i\right\|_q^{\beta}.$$

By making use of (2.3), we derive the first inequality in (2.1).

The second inequality follows by the elementary inequality

$$a^{1/\alpha}b^{1/\beta} \le \frac{1}{\alpha}a + \frac{1}{\beta}b,\ \ a,\ \ b \ge 0. \tag{2.4}$$

□

From (2.1) we derive for $\alpha = p$ and $\beta = q$ that

$$\begin{aligned}
\sum_{i=1}^{n-1}\left\|\Delta A_i B_i\right\|_1 &\le \left(\sum_{i=0}^{n-1} i\left\|\Delta A_i\right\|_p^{p}\right)^{1/p}\left(\sum_{i=0}^{n-1}(n-i-1)\left\|\Delta B_i\right\|_q^{q}\right)^{1/q}\\
&\le \frac{1}{p}\sum_{i=0}^{n-1} i\left\|\Delta A_i\right\|_p^{p} + \frac{1}{q}\sum_{i=0}^{n-1}(n-i-1)\left\|\Delta B_i\right\|_q^{q}.
\end{aligned} \tag{2.5}$$

By the use of trace properties we derive from (2.5) the following result

$$\begin{aligned}
&\left|\operatorname{tr}\left(\sum_{i=1}^{n-1}\Delta A_i B_i\right)\right| \\
&\leq \left[\operatorname{tr}\left(\sum_{i=0}^{n-1} i\left|\Delta A_i\right|^p\right)\right]^{1/p}\left[\operatorname{tr}\left(\sum_{i=0}^{n-1}(n-i-1)\left|\Delta B_i\right|^q\right)\right]^{1/q} \\
&\leq \operatorname{tr}\left[\sum_{i=0}^{n-1}\left(i\frac{1}{p}\left|\Delta A_i\right|^p+(n-i-1)\frac{1}{q}\left|\Delta B_i\right|^q\right)\right],
\end{aligned} \tag{2.6}$$

for $p,\ q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, where $\{A_i\}_{i=0}^N\subset B_p(H)$, $\{B_i\}_{i=0}^N\subset B_q(H)$ are sequences of operators with $B_0=0$ and $n\in\{2,...,N\}$.

Corollary 1. *With the assumptions of Theorem 5 we have*

$$\begin{aligned}
&\sum_{i=1}^{n-1}\left\|\Delta A_i B_i\right\|_1 \\
&\leq \left(\sum_{i=0}^{n-1} i\left\|\Delta A_i\right\|_p^\alpha\right)^{1/\alpha}\left(\sum_{i=0}^{n-1}(n-i-1)\left\|\Delta B_i\right\|_q^\beta\right)^{1/\beta} \\
&\leq \begin{cases}
\frac{(n-1)n}{2}\max_{i\in\{0,\ldots,n-1\}}\left\|\Delta A_i\right\|_p\max_{i\in\{0,\ldots,n-1\}}\left\|\Delta B_i\right\|_q, \\
\\
(n-1)\left(\sum_{i=1}^{n-1}\left\|\Delta A_i\right\|_p^\alpha\right)^{1/\alpha}\left(\sum_{i=1}^{n-1}\left\|\Delta B_i\right\|_q^\beta\right)^{1/\beta},
\end{cases}
\end{aligned} \tag{2.7}$$

where $\alpha,\ \beta>1,\ \frac{1}{\alpha}+\frac{1}{\beta}=1.$

In particular,

$$\begin{aligned}
\sum_{i=1}^{n-1}\left\|\Delta A_i B_i\right\|_1 &\leq \left(\sum_{i=0}^{n-1} i\left\|\Delta A_i\right\|_p^2\right)^{1/2}\left(\sum_{i=0}^{n-1}(n-i-1)\left\|\Delta B_i\right\|_q^2\right)^{1/2} \\
&\leq (n-1)\left(\sum_{i=0}^{n-1}\left\|\Delta A_i\right\|_p^2\right)^{1/2}\left(\sum_{i=0}^{n-1}\left\|\Delta B_i\right\|_q^2\right)^{1/2}.
\end{aligned} \tag{2.8}$$

Proof. We have

$$\sum_{i=0}^{n-1} i\left\|\Delta A_i\right\|_p^\alpha \leq \max_{i\in\{1,\ldots,n-1\}}\left\|\Delta A_i\right\|_p^\alpha\sum_{i=0}^{n-1} i=\frac{(n-1)\,n}{2}\max_{i\in\{0,\ldots,n-1\}}\left\|\Delta A_i\right\|_p^\alpha$$

and

$$\begin{aligned}
\sum_{i=0}^{n-1}(n-i-1)\left\|\Delta B_i\right\|_q^\beta &\leq \max_{i\in\{0,\ldots,n-1\}}\left\|\Delta B_i\right\|_q^\beta\sum_{i=0}^{n-1}(n-i-1) \\
&=\frac{(n-1)\,n}{2}\max_{i\in\{0,\ldots,n-1\}}\left\|\Delta B_i\right\|_q^\beta.
\end{aligned}$$

Also,

$$\sum_{i=0}^{n-1} i \left\|\Delta A_i\right\|_p^{\alpha} \leq (n-1) \sum_{i=0}^{n-1} \left\|\Delta A_i\right\|_p^{\alpha},$$

$$\sum_{i=0}^{n-1} (n-i-1) \left\|\Delta B_i\right\|_q^{\beta} \leq (n-1) \sum_{i=0}^{n-1} \left\|\Delta B_i\right\|_q^{\beta}.$$

These complete the proof. □

Remark 1. *Assume that* $\{A_i\}_{i=0}^{N} \subset B_2(H)$ *with* $A_0 = 0$, *then for* $n \in \{2, ..., N\}$,

$$\sum_{i=1}^{n-1} \left\|\Delta A_i A_i\right\|_1 \leq \left(\sum_{i=0}^{n-1} i \left\|\Delta A_i\right\|_2^2\right)^{1/2} \left(\sum_{i=0}^{n-1} (n-i-1) \left\|\Delta A_i\right\|_2^2\right)^{1/2} \leq \frac{1}{2}(n-1) \sum_{i=0}^{n-1} \left\|\Delta A_i\right\|_2^2. \tag{2.9}$$

Also, we have the complementary result:

Theorem 6. *For* $p,\ q > 1$ *with* $\frac{1}{p} + \frac{1}{q} = 1$, *assume that* $\{A_i\}_{i=0}^{N} \subset B_p(H)$, $\{B_i\}_{i=0}^{N} \subset B_q(H)$ *are sequences of operators with* $B_N = 0$, *then for* $n \in \{1, ..., N-1\}$,

$$\sum_{i=n}^{N-1} \left\|\Delta A_i B_i\right\|_1 \leq \left(\sum_{i=n}^{N-1} (N-i) \left\|\Delta A_i\right\|_p^{\alpha}\right)^{1/\alpha} \left(\sum_{i=n}^{N-1} (i+1-n) \left\|\Delta B_i\right\|_q^{\beta}\right)^{1/\beta} \leq \frac{1}{\alpha} \sum_{i=n}^{N-1} (N-i) \left\|\Delta A_i\right\|_p^{\alpha} + \frac{1}{\beta} \sum_{i=n}^{N-1} (i+1-n) \left\|\Delta B_i\right\|_q^{\beta}, \tag{2.10}$$

where $\alpha,\ \beta > 1$, $\frac{1}{\alpha} + \frac{1}{\beta} = 1$.

In particular,

$$\sum_{i=n}^{N-1} \left\|\Delta A_i B_i\right\|_1 \leq \left(\sum_{i=n}^{N-1} (N-i) \left\|\Delta A_i\right\|_p^2\right)^{1/2} \left(\sum_{i=n}^{N-1} (i+1-n) \left\|\Delta B_i\right\|_q^2\right)^{1/2} \leq \frac{1}{2} \sum_{i=n}^{N-1} \left[(N-i) \left\|\Delta A_i\right\|_p^2 + (i+1-n) \left\|\Delta B_i\right\|_q^2\right]. \tag{2.11}$$

Proof. If $B_N = 0$, then $B_i = -\sum_{j=i}^{N-1} \Delta B_j$ for $i = n+1, ..., N-1$. Then

$$\sum_{i=n}^{N-1} \left\|\Delta A_i B_i\right\|_1 \leq \sum_{i=n}^{N-1} \left\|\Delta A_i\right\|_p \left\|B_i\right\|_q = \sum_{i=n}^{N-1} \left\|\Delta A_i\right\|_p \left\|\sum_{j=i}^{N-1} \Delta B_j\right\|_q = \sum_{i=n}^{N-1} (N-i)^{1/\alpha} \left\|\Delta A_i\right\|_p \frac{1}{(N-i)^{1/\alpha}} \left\|\sum_{j=i}^{N-1} \Delta B_j\right\|_q =: \Theta.$$

By the discrete Hölder's inequality, we have

$$
\begin{aligned}
\Theta &\le \left(\sum_{i=n}^{N-1}(N-i)\left\|\Delta A_i\right\|_p^{\alpha}\right)^{1/\alpha}\left(\sum_{i=n}^{N-1}\frac{1}{(N-i)^{\beta/\alpha}}\left\|\sum_{j=i}^{N-1}\Delta B_j\right\|_q^{\beta}\right)^{1/\beta} \qquad (2.12)\\
&= \left(\sum_{i=n}^{N-1}(N-i)\left\|\Delta A_i\right\|_p^{\alpha}\right)^{1/\alpha}\left[\sum_{i=n}^{N-1}\left(\frac{1}{(N-i)^{1/\alpha}}\left\|\sum_{j=i}^{N-1}\Delta B_j\right\|_q\right)^{\beta}\right]^{1/\beta}\\
&\le \left(\sum_{i=n}^{N-1}(N-i)\left\|\Delta A_i\right\|_p^{\alpha}\right)^{1/\alpha}\left[\sum_{i=n}^{N-1}\left(\left(\sum_{j=i}^{N-1}\left\|\Delta B_j\right\|_q^{\beta}\right)^{1/\beta}\right)^{\beta}\right]^{1/\beta}\\
&= \left(\sum_{i=n}^{N-1}(N-i)\left\|\Delta A_i\right\|_p^{\alpha}\right)^{1/\alpha}\left[\sum_{i=n}^{N-1}\left(\sum_{j=i}^{N-1}\left\|\Delta B_j\right\|_q^{\beta}\right)\right]^{1/\beta} =: \Lambda.
\end{aligned}
$$

From (1.3) we have

$$\sum_{i=n}^{N-1} a_i\Delta b_i = a_{N-1}b_N - a_n b_n - \sum_{i=n}^{N-2} b_{i+1}\Delta a_i,$$

and by $a_i = \sum_{j=i}^{N-1}\left\|\Delta B_i\right\|_q^{\beta}$ and $b_i = i$, we have

$$
\begin{aligned}
&\sum_{i=n}^{N-1}\left(\sum_{j=i}^{N-1}\left\|\Delta B_i\right\|_q^{\beta}\right)\\
&= N\left\|\Delta B_{N-1}\right\|_q^{\beta} - n\sum_{j=n}^{N-1}\left\|\Delta B_i\right\|_q^{\beta} - \sum_{i=n}^{N-2}(i+1)\left(\sum_{j=i+1}^{N-1}\left\|\Delta B_i\right\|_q^{\beta} - \sum_{j=i}^{N-1}\left\|\Delta B_j\right\|_q^{\beta}\right)\\
&= N\left\|\Delta B_{N-1}\right\|_q^{\beta} + \sum_{i=n}^{N-2}(i+1)\left\|\Delta B_i\right\|_q^{\beta} - n\sum_{j=n}^{N-1}\left\|\Delta B_i\right\|_q^{\beta}\\
&= \sum_{i=n}^{N-2}(i+1)\left\|\Delta B_i\right\|_q^{\beta} - n\sum_{j=n}^{N-2}\left\|\Delta B_i\right\|_q^{\beta}\\
&= \sum_{i=n}^{N-2}(i+1-n)\left\|\Delta B_i\right\|_q^{\beta} + (N-n)\left\|\Delta B_{N-1}\right\|_q^{\beta} = \sum_{i=n}^{N-1}(i+1-n)\left\|\Delta B_i\right\|_q^{\beta}.
\end{aligned}
$$

Then

$$\Lambda = \left(\sum_{i=n}^{N-1}(N-i)\left\|\Delta A_i\right\|_p^{\alpha}\right)^{1/\alpha}\left(\sum_{i=n}^{N-1}(i+1-n)\left\|\Delta B_i\right\|_q^{\beta}\right)^{1/\beta},$$

which, by (2.12), proves the first inequality in (2.10).

The second part follows by Young's inequality (2.4). The last part is obvious. □

Corollary 2. *With the assumptions of Theorem 6 we have*

$$\sum_{i=n}^{N-1}\left\|\Delta A_i B_i\right\|_1 \tag{2.13}$$

$$\leq \left(\sum_{i=n}^{N-1}(N-i)\left\|\Delta A_i\right\|_p^{\alpha}\right)^{1/\alpha}\left(\sum_{i=n}^{N-1}(i+1-n)\left\|\Delta B_i\right\|_q^{\beta}\right)^{1/\beta}$$

$$\leq \begin{cases} \frac{(N-n)(N-n+1)}{2}\max_{i\in\{n,...,N-1\}}\left\|\Delta A_i\right\|_p \max_{i\in\{n,...,N-1\}}\left\|\Delta B_i\right\|_q, \\ \\ (N-n)\left(\sum_{i=n}^{N-1}\left\|\Delta A_i\right\|_p^{\alpha}\right)^{1/\alpha}\left(\sum_{i=n}^{N-1}\left\|\Delta B_i\right\|_q^{\beta}\right)^{1/\beta}, \end{cases}$$

where $\alpha,\ \beta>1,\ \frac{1}{\alpha}+\frac{1}{\beta}=1.$

In particular,

$$\sum_{i=n}^{N-1}\left\|\Delta A_i B_i\right\|_1 \leq (N-n)\left(\sum_{i=n}^{N-1}\left\|\Delta A_i\right\|_p^2\right)^{1/2}\left(\sum_{i=n}^{N-1}\left\|\Delta B_i\right\|_q^2\right)^{1/2}. \tag{2.14}$$

Proof. We have

$$\sum_{i=n}^{N-1}(N-i)\left\|\Delta A_i\right\|_p^{\alpha} \leq \max_{i\in\{n,...,N-1\}}\left\|\Delta A_i\right\|_p^{\alpha}\sum_{i=n}^{N-1}(N-i) = \frac{(N-n)(N-n+1)}{2}\max_{i\in\{n,...,N-1\}}\left\|\Delta A_i\right\|_p^{\alpha}$$

and

$$\sum_{i=n}^{N-1}(i+1-n)\left\|\Delta B_i\right\|_q^{\beta} \leq \max_{i\in\{n,...,N-1\}}\left\|\Delta B_i\right\|_q^{\beta}\sum_{i=n}^{N-1}(i+1-n) = \frac{(N-n)(N-n+1)}{2}\max_{i\in\{n,...,N-1\}}\left\|\Delta B_i\right\|_q^{\beta}.$$

Also

$$\sum_{i=n}^{N-1}(N-i)\left\|\Delta A_i\right\|_p^{\alpha} \leq (N-n)\sum_{i=n}^{N-1}\left\|\Delta A_i\right\|_p^{\alpha},\ \ \sum_{i=n}^{N-1}(i+1-n)\left\|\Delta B_i\right\|_q^{\beta} \leq (N-n)\sum_{i=n}^{N-1}\left\|\Delta B_i\right\|_q^{\beta}$$

and the inequality is proved. □

Remark 2. *Assume that* $\{A_i\}_{i=0}^{N}\subset B_2(H)$ *with* $A_N=0$, *then for* $n\in\{1,...,N-1\}$,

$$\sum_{i=n}^{N-1}\left\|\Delta A_i A_i\right\|_1 \leq \left(\sum_{i=n}^{N-1}(N-i)\left\|\Delta A_i\right\|_2^2\right)^{1/2}\left(\sum_{i=n}^{N-1}(i+1-n)\left\|\Delta A_i\right\|_2^2\right)^{1/2} \leq \frac{1}{2}(N-n)\sum_{i=n}^{N-1}\left\|\Delta A_i\right\|_2^2. \tag{2.15}$$

We also have the following result that incorporates both cases:

Theorem 7. *For $p,\ q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, assume that $\{A_i\}_{i=0}^{N}\subset B_p(H)$, $\{B_i\}_{i=0}^{N}\subset B_q(H)$ are sequences of operators with $B_0=B_N=0$, then for $n\in\{2,...,N-1\}$, $\alpha,\ \beta>1$, $\frac{1}{\alpha}+\frac{1}{\beta}=1$*

$$\begin{aligned}(2.16)\qquad \sum_{i=1}^{N-1}\|\Delta A_iB_i\|_1 &\leq\left(\sum_{i=0}^{N-1}\alpha_i(n)\|\Delta A_i\|_p^{\alpha}\right)^{1/\alpha}\left(\sum_{i=0}^{N-1}\beta_i(n)\|\Delta B_i\|_q^{\beta}\right)^{1/\beta}\\ &\leq\frac{1}{\alpha}\sum_{i=0}^{N-1}\alpha_i(n)\|\Delta A_i\|_p^{\alpha}+\frac{1}{\beta}\sum_{i=0}^{N-1}\beta_i(n)\|\Delta B_i\|_q^{\beta},\end{aligned}$$

where

$$\alpha_i(n):=\begin{cases} i, & \text{if } 0\leq i\leq n-1,\\ N-i, & \text{if } n\leq i\leq N-1\end{cases}$$

and

$$\beta_i(n):=\begin{cases} n-i-1, & \text{if } 0\leq i\leq n-1,\\ i+1-n, & \text{if } n\leq i\leq N-1.\end{cases}$$

In particular,

$$\begin{aligned}(2.17)\qquad \sum_{i=1}^{N-1}\|\Delta A_iB_i\|_1 &\leq\left(\sum_{i=0}^{N-1}\alpha_i(n)\|\Delta A_i\|_p^{2}\right)^{1/2}\left(\sum_{i=0}^{N-1}\beta_i(n)\|\Delta B_i\|_q^{2}\right)^{1/2}\\ &\leq\frac{1}{2}\sum_{i=0}^{N-1}\left[\alpha_i(n)\|\Delta A_i\|_p^{2}+\beta_i(n)\|\Delta B_i\|_q^{2}\right],\end{aligned}$$

Proof. We have for $n\in\{2,...,N-1\}$ that

$$\sum_{i=1}^{n-1}\|\Delta A_iB_i\|_1\leq\left(\sum_{i=0}^{n-1}i\|\Delta A_i\|_p^{\alpha}\right)^{1/\alpha}\left(\sum_{i=0}^{n-1}(n-i-1)\|\Delta B_i\|_q^{\beta}\right)^{1/\beta}$$

and

$$\sum_{i=n}^{N-1}\|\Delta A_iB_i\|_1\leq\left(\sum_{i=n}^{N-1}(N-i)\|\Delta A_i\|_p^{\alpha}\right)^{1/\alpha}\left(\sum_{i=n}^{N-1}(i+1-n)\|\Delta B_i\|_q^{\beta}\right)^{1/\beta}.$$

If we add these inequalities, then we get, by the elementary inequality

$$ab+cd\leq(a^{\alpha}+c^{\alpha})^{1/\alpha}\left(b^{\beta}+d^{\beta}\right)^{1/\beta},\ a,\ b,\ c,\ d\geq 0$$

that

$$\begin{aligned}\sum_{i=1}^{N-1}\left\|\Delta A_i B_i\right\|_1 &\leq \left(\sum_{i=0}^{n-1} i\left\|\Delta A_i\right\|_p^\alpha\right)^{1/\alpha}\left(\sum_{i=0}^{n-1}(n-i-1)\left\|\Delta B_i\right\|_q^\beta\right)^{1/\beta} \\ &+\left(\sum_{i=n}^{N-1}(N-i)\left\|\Delta A_i\right\|_p^\alpha\right)^{1/\alpha}\left(\sum_{i=n}^{N-1}(i+1-n)\left\|\Delta B_i\right\|_q^2\right)^{1/2} \\ &\leq\left(\sum_{i=0}^{n-1} i\left\|\Delta A_i\right\|_p^\alpha+\sum_{i=n}^{N-1}(N-i)\left\|\Delta A_i\right\|_p^\alpha\right)^{1/\alpha} \\ &\times\left(\sum_{i=0}^{n-1}(n-i-1)\left\|\Delta B_i\right\|_q^\beta+\sum_{i=n}^{N-1}(i+1-n)\left\|\Delta B_i\right\|_q^\beta\right)^{1/\beta} \\ &=\left(\sum_{i=0}^{N-1}\alpha_i\left\|\Delta A_i\right\|_p^\alpha\right)^{1/\alpha}\left(\sum_{i=0}^{N-1}\beta_i\left\|\Delta B_i\right\|_q^\beta\right)^{1/\beta},\end{aligned}$$

which proves the first inequality in (2.16). □

Corollary 3. *With the assumptions of Theorem 7, we have*

$$\begin{aligned}\sum_{i=1}^{N-1}\left\|\Delta A_i B_i\right\|_1 &\leq\left(\sum_{i=0}^{N-1}\alpha_i(n)\left\|\Delta A_i\right\|_p^\alpha\right)^{1/\alpha}\left(\sum_{i=0}^{N-1}\beta_i(n)\left\|\Delta B_i\right\|_q^\beta\right)^{1/\beta} \\ &\leq\left[\left(n-\frac{N+1}{2}\right)^2+\frac{1}{4}\left(N^2-1\right)\right] \\ &\times\max_{i\in\{0,\ldots,N-1\}}\left\|\Delta A_i\right\|_p\max_{i\in\{0,\ldots,N-1\}}\left\|\Delta B_i\right\|_q\end{aligned}\tag{2.18}$$

for $n\in\{2,\ldots,N-1\}$, $\alpha,\ \beta>1,\ \frac{1}{\alpha}+\frac{1}{\beta}=1.$

For $n=\left\lfloor\frac{N+1}{2}\right\rfloor$, *where* $\lfloor\cdot\rfloor$ *is the integer part function, we get*

$$\begin{aligned}&\sum_{i=1}^{N-1}\left\|\Delta A_i B_i\right\|_1 \\ &\leq\left(\sum_{i=0}^{N-1}\alpha_i\left(\left\lfloor\frac{N+1}{2}\right\rfloor\right)\left\|\Delta A_i\right\|_p^\alpha\right)^{1/\alpha}\left(\sum_{i=0}^{N-1}\beta_i\left(\left\lfloor\frac{N+1}{2}\right\rfloor\right)\left\|\Delta B_i\right\|_q^\beta\right)^{1/\beta} \\ &\leq\left[\left(\left\lfloor\frac{N+1}{2}\right\rfloor-\frac{N+1}{2}\right)^2+\frac{1}{4}\left(N^2-1\right)\right] \\ &\times\max_{i\in\{0,\ldots,N-1\}}\left\|\Delta A_i\right\|_p\max_{i\in\{0,\ldots,N-1\}}\left\|\Delta B_i\right\|_q.\end{aligned}\tag{2.19}$$

Proof. We have

$$\begin{aligned}
&\sum_{i=0}^{N-1}\alpha_i(n)\|\Delta A_i\|_p^\alpha \\
&\leq \max_{i\in\{0,\dots,N-1\}}\|\Delta A_i\|_p^\alpha\sum_{i=0}^{N-1}\alpha_i(n)=\max_{i\in\{0,\dots,N-1\}}\|\Delta A_i\|_p^\alpha\left[\sum_{i=0}^{n-1}i+\sum_{i=n}^{N-1}(N-i)\right] \\
&=\max_{i\in\{0,\dots,N-1\}}\|\Delta A_i\|_p^\alpha\left[\left(n-\frac{N+1}{2}\right)^2+\frac{1}{4}\left(N^2-1\right)\right]
\end{aligned}$$

and

$$\begin{aligned}
\sum_{i=0}^{N-1}\beta_i(n)\|\Delta B_i\|_q^\beta &\leq \max_{i\in\{0,\dots,N-1\}}\|\Delta B_i\|_q^\beta\sum_{i=0}^{N-1}\beta_i(n) \\
&= \max_{i\in\{0,\dots,N-1\}}\|\Delta B_i\|_q^\beta\left[\sum_{i=0}^{n-1}(n-i-1)+\sum_{i=n}^{N-1}(i+1-n)\right] \\
&= \max_{i\in\{0,\dots,N-1\}}\|\Delta B_i\|_q^\beta\left[\left(n-\frac{N+1}{2}\right)^2+\frac{1}{4}\left(N^2-1\right)\right],
\end{aligned}$$

which proves the desired result. □

Remark 3. *Assume that* $\{A_i\}_{i=0}^N\subset B_2(H)$ *with* $A_0=A_N=0$, *then*

$$\begin{aligned}
\sum_{i=1}^{N-1}\|\Delta A_iA_i\|_1 &\leq \left(\sum_{i=0}^{N-1}\alpha_i(n)\|\Delta A_i\|_2^2\right)^{1/2}\left(\sum_{i=0}^{N-1}\beta_i(n)\|\Delta A_i\|_2^2\right)^{1/2} \\
&\leq \frac{1}{2}\sum_{i=0}^{N-1}s_i(n)\|\Delta A_i\|_2^2
\end{aligned}\tag{2.20}$$

where

$$s_i(n):=\begin{cases} n-1, & \text{if } 0\leq i\leq n-1, \\ N-n+1, & \text{if } n\leq i\leq N-1.\end{cases}$$

If we take in (2.20) $n=\left\lfloor\frac{N+1}{2}\right\rfloor+1$, *then by (2.20) we get*

$$\begin{aligned}
\sum_{i=1}^{N-1}\|\Delta A_iA_i\|_1 &\leq \left(\sum_{i=0}^{N-1}\alpha_i\left(\left\lfloor\frac{N+1}{2}\right\rfloor+1\right)\|\Delta A_i\|_2^2\right)^{1/2} \\
&\quad\times\left(\sum_{i=0}^{N-1}\beta\left(\left\lfloor\frac{N+1}{2}\right\rfloor+1\right)\|\Delta A_i\|_2^2\right)^{1/2} \\
&\leq \frac{1}{2}\sum_{i=0}^{N-1}s_i\left(\left\lfloor\frac{N+1}{2}\right\rfloor+1\right)\|\Delta A_i\|_2^2 \\
&\leq \frac{1}{2}\left\lfloor\frac{N+1}{2}\right\rfloor\sum_{i=0}^{N-1}\|\Delta A_i\|_2^2,
\end{aligned}\tag{2.21}$$

where

$$s_i(n) := \begin{cases} \left\lfloor \frac{N+1}{2} \right\rfloor, & \text{if } 0 \le i \le \left\lfloor \frac{N+1}{2} \right\rfloor, \\ N - \left\lfloor \frac{N+1}{2} \right\rfloor, & \text{if } \left\lfloor \frac{N+1}{2} \right\rfloor + 1 \le i \le N-1 \end{cases} \le \left\lfloor \frac{N+1}{2} \right\rfloor.$$

3. p-Schatten Norm Inequalities

We have the following result as well:

Theorem 8. *For $p \ge 1$, assume that $\{A_i\}_{i=0}^N, \{B_i\}_{i=0}^N \subset B_p(H)$ are sequences of operators with $B_0 = 0$, then for $n \in \{2, ..., N\}$,*

$$\sum_{i=1}^{n-1} \|\Delta A_i B_i\|_p \le \left(\sum_{i=0}^{n-1} i \|\Delta A_i\|_p^\alpha\right)^{1/\alpha} \left(\sum_{i=0}^{n-1} (n-i-1) \|\Delta B_i\|_p^\beta\right)^{1/\beta} \le \frac{1}{\alpha} \sum_{i=0}^{n-1} i \|\Delta A_i\|_p^\alpha + \frac{1}{\beta} \sum_{i=0}^{n-1} (n-i-1) \|\Delta B_i\|_p^\beta, \tag{3.1}$$

where $\alpha, \beta > 1$, $\frac{1}{\alpha} + \frac{1}{\beta} = 1$.

In particular,

$$\sum_{i=1}^{n-1} \|\Delta A_i B_i\|_p \le \left(\sum_{i=0}^{n-1} i \|\Delta A_i\|_p^2\right)^{1/2} \left(\sum_{i=0}^{n-1} (n-i-1) \|\Delta B_i\|_p^2\right)^{1/2} \le \frac{1}{2} \sum_{i=0}^{n-1} \left(i \|\Delta A_i\|_p^2 + (n-i-1) \|\Delta B_i\|_p^2\right). \tag{3.2}$$

Proof. Let $n \in \{2, ..., N\}$. Since $B_0 = 0$, hence $B_i = \sum_{j=0}^{i-1} \Delta B_j$ for $i = 1, ..., n-1$. Then by (1.14) we derive

$$\sum_{i=1}^{n-1} \|\Delta A_i B_i\|_p \le \sum_{i=1}^{n-1} \|\Delta A_i\|_p \|B_i\|_p = \sum_{i=1}^{n-1} \|\Delta A_i\|_p \left\| \sum_{j=0}^{i-1} \Delta B_j \right\|_p = \sum_{i=1}^{n-1} i^{1/\alpha} \|\Delta A_i\|_p i^{-1/\alpha} \left\| \sum_{j=0}^{i-1} \Delta B_j \right\|_p.$$

Now, by utilising a similar argument to the one in the proof of Theorem 5 we derive the desired result (3.1). $\square$

Corollary 4. *With the assumptions of Theorem 8 we have*

$$\sum_{i=1}^{n-1} \|\Delta A_i B_i\|_p \le \left(\sum_{i=0}^{n-1} i \|\Delta A_i\|_p^\alpha\right)^{1/\alpha} \left(\sum_{i=0}^{n-1} (n-i-1) \|\Delta B_i\|_p^\beta\right)^{1/\beta} \le \begin{cases} \frac{(n-1)n}{2} \max_{i \in \{0,...,n-1\}} \|\Delta A_i\|_p \max_{i \in \{0,...,n-1\}} \|\Delta B_i\|_p, \\ (n-1) \left(\sum_{i=1}^{n-1} \|\Delta A_i\|_p^\alpha\right)^{1/\alpha} \left(\sum_{i=1}^{n-1} \|\Delta B_i\|_p^\beta\right)^{1/\beta}, \end{cases} \tag{3.3}$$

where $\alpha,\ \beta > 1,\ \frac{1}{\alpha} + \frac{1}{\beta} = 1.$

In particular,

$$\sum_{i=1}^{n-1} \|\Delta A_i B_i\|_p \le \left(\sum_{i=0}^{n-1} i\, \|\Delta A_i\|_p^2\right)^{1/2} \left(\sum_{i=0}^{n-1} (n-i-1)\, \|\Delta B_i\|_q^2\right)^{1/2} \le (n-1) \left(\sum_{i=0}^{n-1} \|\Delta A_i\|_p^2\right)^{1/2} \left(\sum_{i=0}^{n-1} \|\Delta B_i\|_q^2\right)^{1/2}. \tag{3.4}$$

We also have

Theorem 9. *For* $p \ge 1,$ *assume that* $\{A_i\}_{i=0}^N, \{B_i\}_{i=0}^N \subset B_p(H)$ *are sequences of operators with* $B_N = 0,$ *then for* $n \in \{1, ..., N-1\},$

$$\sum_{i=n}^{N-1} \|\Delta A_i B_i\|_p \le \left(\sum_{i=n}^{N-1} (N-i)\, \|\Delta A_i\|_p^\alpha\right)^{1/\alpha} \left(\sum_{i=n}^{N-1} (i+1-n)\, \|\Delta B_i\|_p^\beta\right)^{1/\beta} \le \frac{1}{\alpha} \sum_{i=n}^{N-1} (N-i)\, \|\Delta A_i\|_p^\alpha + \frac{1}{\beta} \sum_{i=n}^{N-1} (i+1-n)\, \|\Delta B_i\|_p^\beta, \tag{3.5}$$

where $\alpha,\ \beta > 1,\ \frac{1}{\alpha} + \frac{1}{\beta} = 1.$

In particular,

$$\sum_{i=n}^{N-1} \|\Delta A_i B_i\|_p \le \left(\sum_{i=n}^{N-1} (N-i)\, \|\Delta A_i\|_p^2\right)^{1/2} \left(\sum_{i=n}^{N-1} (i+1-n)\, \|\Delta B_i\|_p^2\right)^{1/2} \le \frac{1}{2} \sum_{i=n}^{N-1} \left[(N-i)\, \|\Delta A_i\|_p^2 + (i+1-n)\, \|\Delta B_i\|_p^2\right]. \tag{3.6}$$

Corollary 5. *With the assumptions of Theorem 9 we have*

$$\begin{aligned} \sum_{i=n}^{N-1} \|\Delta A_i B_i\|_p &\le \left(\sum_{i=n}^{N-1} (N-i)\, \|\Delta A_i\|_p^\alpha\right)^{1/\alpha} \left(\sum_{i=n}^{N-1} (i+1-n)\, \|\Delta B_i\|_p^\beta\right)^{1/\beta} \\ &\le \begin{cases} \frac{(N-n)(N-n+1)}{2} \max_{i\in\{n,\dots,N-1\}} \|\Delta A_i\|_p \max_{i\in\{n,\dots,N-1\}} \|\Delta B_i\|_p, \\ \\ (N-n) \left(\sum_{i=n}^{N-1} \|\Delta A_i\|_p^\alpha\right)^{1/\alpha} \left(\sum_{i=n}^{N-1} \|\Delta B_i\|_p^\beta\right)^{1/\beta}, \end{cases} \end{aligned} \tag{3.7}$$

where $\alpha,\ \beta > 1,\ \frac{1}{\alpha} + \frac{1}{\beta} = 1.$

In particular,

$$\sum_{i=n}^{N-1} \|\Delta A_i B_i\|_p \le (N-n) \left(\sum_{i=n}^{N-1} \|\Delta A_i\|_p^2\right)^{1/2} \left(\sum_{i=n}^{N-1} \|\Delta B_i\|_p^2\right)^{1/2}. \tag{3.8}$$

Theorem 10. *For $p \geq 1$, assume that $\{A_i\}_{i=0}^{N}, \{B_i\}_{i=0}^{N} \subset B_p(H)$ are sequences of operators with $B_0 = B_N = 0$, then for $n \in \{2, ..., N-1\}$, $\alpha, \beta > 1$, $\frac{1}{\alpha} + \frac{1}{\beta} = 1$*

$$\sum_{i=1}^{N-1} \|\Delta A_i B_i\|_p \leq \left(\sum_{i=0}^{N-1} \alpha_i(n) \|\Delta A_i\|_p^{\alpha}\right)^{1/\alpha} \left(\sum_{i=0}^{N-1} \beta_i(n) \|\Delta B_i\|_q^{\beta}\right)^{1/\beta} \leq \frac{1}{\alpha} \sum_{i=0}^{N-1} \alpha_i(n) \|\Delta A_i\|_p^{\alpha} + \frac{1}{\beta} \sum_{i=0}^{N-1} \beta_i(n) \|\Delta B_i\|_q^{\beta}, \tag{3.9}$$

where $\alpha_i(n)$ and $\beta_i(n)$ are defined in Theorem 7.

In particular,

$$\sum_{i=1}^{N-1} \|\Delta A_i B_i\|_p \leq \left(\sum_{i=0}^{N-1} \alpha_i(n) \|\Delta A_i\|_p^{2}\right)^{1/2} \left(\sum_{i=0}^{N-1} \beta_i(n) \|\Delta B_i\|_q^{2}\right)^{1/2} \leq \frac{1}{2} \sum_{i=0}^{N-1} \left[\alpha_i(n) \|\Delta A_i\|_p^{2} + \beta_i(n) \|\Delta B_i\|_q^{2}\right], \tag{3.10}$$

Corollary 6. *With the assumptions of Theorem 10, we have*

$$\sum_{i=1}^{N-1} \|\Delta A_i B_i\|_p \leq \left(\sum_{i=0}^{N-1} \alpha_i(n) \|\Delta A_i\|_p^{\alpha}\right)^{1/\alpha} \left(\sum_{i=0}^{N-1} \beta_i(n) \|\Delta B_i\|_p^{\beta}\right)^{1/\beta} \leq \left[\left(n - \frac{N+1}{2}\right)^2 + \frac{1}{4}\left(N^2 - 1\right)\right] \times \max_{i \in \{0,...,N-1\}} \|\Delta A_i\|_p \max_{i \in \{0,...,N-1\}} \|\Delta B_i\|_p \tag{3.11}$$

for $n \in \{2, ..., N-1\}$, $\alpha, \beta > 1$, $\frac{1}{\alpha} + \frac{1}{\beta} = 1$.

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