ON SOME p-SCHATTEN NORM INEQUALITIES OF OPIAL-LASOTA'S TYPE FOR TWO SEQUENCES

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Abstract. An operator $A \in \mathcal{B}(H)$ is said to belong to the *von Neumann-Schatten class* $\mathcal{B}_p(H)$, $1 \leq p < \infty$ if the *p-Schatten norm* is finite

$$\|A\|_p := [\operatorname{tr}(|A|^p)]^{1/p} < \infty.$$

In this paper we show among others that, if $p,\ q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, and $\{A_i\}_{i=0}^N \subset B_p(H)$, $\{B_i\}_{i=0}^N \subset B_q(H)$ that are sequences of operators with $A_0 = B_N = 0$, then

$$\sum_{i=1}^{N-1} \|A_i B_i\|_1 \leq \frac{1}{2} N \left[\sum_{i=0}^{N-1} (N-i) \|\Delta A_i\|_p^2\right]^{1/2} \left[\sum_{i=0}^{N-1} (i+1) \|\Delta B_i\|_q^2\right]^{1/2}$$
$$\leq \frac{1}{4} N \sum_{i=0}^{N-1} \left[(N-i) \|\Delta A_i\|_p^2 + (i+1) \|\Delta B_i\|_q^2\right],$$

where $\Delta A_i := A_{i+1} - A_i$, $i \in \{0, ..., N-1\}$ is the forward difference.

1. Introduction

For a sequence $\{x_i\}_{i=0}^N$, we consider the forward operator Δ defined by $\Delta x_i = x_{i+1} - x_i$, $i = 0, ..., N-1$. Recall the summation by parts formula stated as

$$\sum_{k=m}^{n} a_k \Delta b_k = a_n b_{n+1} - a_m b_m - \sum_{k=m}^{n-1} b_{k+1} \Delta a_k, \tag{1.1}$$

where a_k and b_k are some sequences for which the products above exist.

In [12], Lasota provided discrete versions of Opial inequality [15] about the forward difference operator as follows:

Theorem 1. *Let $\{x_i\}_{i=0}^N$ be a sequence of real numbers with $x_0 = x_N = 0$. Then, the following inequality holds*

$$\sum_{i=1}^{N-1} |x_i \Delta x_i| \leq \frac{1}{2} \left\lfloor \frac{N+1}{2} \right\rfloor \sum_{i=0}^{N-1} |\Delta x_i|^2, \tag{1.2}$$

where $\lfloor \cdot \rfloor$ is the integer part function. If N is even, then the inequality (1.2) is sharp.

For various Opial type inequalities, see [2]-[6] and [14].

Also, we have the following results, see [1]:

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Theorem 2. *Let* $\{x_i\}_{i=0}^{N}$ *be a sequence of real numbers. If* $x_0 = 0$, *then*

$$\sum_{i=1}^{\tau-1} |x_i \Delta x_i| \le \frac{1}{2}(\tau - 1)\sum_{i=0}^{\tau-1} |\Delta x_i|^2, \ \tau \in \{2, ..., N\}. \tag{1.3}$$

If $x_N = 0$, *then*

$$\sum_{i=\tau}^{N-1} |x_i \Delta x_i| \le \frac{1}{2}(N - \tau + 1)\sum_{i=0}^{N-1} |\Delta x_i|^2, \ \tau \in \{1, ..., N-1\}. \tag{1.4}$$

For other discrete Opial type inequalities, see [10], [11] and [16]-[20].

In the recent paper [9] we obtained the following extension for two sequences:

Theorem 3. *Assume that* $\{x_i\}_{i=0}^{N}, \{y_i\}_{i=0}^{N}$ *are sequences of complex numbers. If* $y_0 = y_N = 0$, *then for* $n \in \{2, ..., N-1\}$,

$$\begin{aligned} \sum_{i=1}^{N-1} |\Delta x_i| |y_i| &\le \left(\sum_{i=0}^{N-1} p_i(n) |\Delta x_i|^2\right)^{1/2} \left(\sum_{i=0}^{N-1} q_i(n) |\Delta y_i|^2\right)^{1/2} \\ &\le \frac{1}{2}\sum_{i=0}^{N-1} \left(p_i(n) |\Delta x_i|^2 + q_i(n) |\Delta y_i|^2\right), \end{aligned} \tag{1.5}$$

where

$$p_i(n) := \begin{cases} i, & \text{if } 0 \le i \le n-1, \\ N - i, & \text{if } n \le i \le N-1 \end{cases}$$

and

$$q_i(n) := \begin{cases} n - i - 1, & \text{if } 0 \le i \le n-1, \\ i + 1 - n, & \text{if } n \le i \le N-1. \end{cases}$$

Corollary 1. *Assume that* $\{x_i\}_{i=0}^{N}$ *is a sequence of complex numbers with* $x_0 = x_N = 0$, *then*

$$\begin{aligned} \sum_{i=1}^{N-1} |\Delta x_i| |x_i| &\le \left(\sum_{i=0}^{N-1} p_i(n) |\Delta x_i|^2\right)^{1/2} \left(\sum_{i=0}^{N-1} q(n)_i |\Delta x_i|^2\right)^{1/2} \\ &\le \frac{1}{2}\sum_{i=0}^{N-1} s_i(n) |\Delta x_i|^2, \end{aligned} \tag{1.6}$$

where

$$s_i(n) := \begin{cases} n - 1, & \text{if } 0 \le i \le n-1, \\ N - n + 1, & \text{if } n \le i \le N-1. \end{cases}$$

Remark 1. *If we take in (1.6) $n = \left\lfloor \frac{N+1}{2} \right\rfloor + 1$, then by (1.6) we get*

$$
\begin{aligned}
(1.7)\quad & \sum_{i=1}^{N-1} |\Delta x_i| \, |x_i| \\
& \leq \left(\sum_{i=0}^{N-1} p_i \left(\left\lfloor \frac{N+1}{2} \right\rfloor + 1 \right) |\Delta x_i|^2 \right)^{1/2} \left(\sum_{i=0}^{N-1} q \left(\left\lfloor \frac{N+1}{2} \right\rfloor + 1 \right) |\Delta x_i|^2 \right)^{1/2} \\
& \leq \frac{1}{2} \sum_{i=0}^{N-1} s_i \left(\left\lfloor \frac{N+1}{2} \right\rfloor + 1 \right) |\Delta x_i|^2 \leq \frac{1}{2} \left\lfloor \frac{N+1}{2} \right\rfloor \sum_{i=0}^{N-1} |\Delta x_i|^2 ,
\end{aligned}
$$

where

$$
s_i(n) := \begin{cases} \left\lfloor \frac{N+1}{2} \right\rfloor, & \text{if } 0 \leq i \leq \left\lfloor \frac{N+1}{2} \right\rfloor, \\ N - \left\lfloor \frac{N+1}{2} \right\rfloor, & \text{if } \left\lfloor \frac{N+1}{2} \right\rfloor + 1 \leq i \leq N-1 \end{cases} \leq \left\lfloor \frac{N+1}{2} \right\rfloor .
$$

The inequality (1.7) is a refinement of Lasota's result (1.2).

In order to extend these results for p-Schatten norms we need the following preparations.

Let $(H; \langle \cdot, \cdot \rangle)$ be a complex Hilbert space and $\mathcal{B}(H)$ the Banach algebra of all bounded linear operators on H. If $\{e_i\}_{i \in I}$ an orthonormal basis of H, we say that $A \in \mathcal{B}(H)$ is of *trace class* if

$$
\|A\|_1 := \sum_{i \in I} \langle |A| e_i, e_i \rangle < \infty. \tag{1.8}
$$

The definition of $\|A\|_1$ does not depend on the choice of the orthonormal basis $\{e_i\}_{i \in I}$. We denote by $\mathcal{B}_1(H)$ the set of trace class operators in $\mathcal{B}(H)$.

We define the *trace* of a trace class operator $A \in \mathcal{B}_1(H)$ to be

$$
\operatorname{tr}(A) := \sum_{i \in I} \langle A e_i, e_i \rangle , \tag{1.9}
$$

where $\{e_i\}_{i \in I}$ an orthonormal basis of H. Note that this coincides with the usual definition of the trace if H is finite-dimensional. We observe that the series (1.9) converges absolutely and it is independent from the choice of basis.

The following result collects some properties of the trace:

Theorem 4. *We have:*

(i) If $A \in \mathcal{B}_1(H)$ then $A^* \in \mathcal{B}_1(H)$ and

$$
\operatorname{tr}(A^*) = \overline{\operatorname{tr}(A)}; \tag{1.10}
$$

(ii) If $A \in \mathcal{B}_1(H)$ and $T \in \mathcal{B}(H)$, then $AT, TA \in \mathcal{B}_1(H)$ and

$$
\operatorname{tr}(AT) = \operatorname{tr}(TA) \ \text{ and } \ |\operatorname{tr}(AT)| \leq \|A\|_1 \|T\| ; \tag{1.11}
$$

(iii) $\operatorname{tr}(\cdot)$ is a bounded linear functional on $\mathcal{B}_1(H)$ with $\|\operatorname{tr}\| = 1$;

(iv) If $A, B \in \mathcal{B}_2(H)$ then $AB, BA \in \mathcal{B}_1(H)$ and $\operatorname{tr}(AB) = \operatorname{tr}(BA)$;

(v) $\mathcal{B}_{fin}(H)$ is a dense subspace of $\mathcal{B}_1(H)$.

An operator $A \in \mathcal{B}(H)$ is said to belong to the *von Neumann-Schatten class* $\mathcal{B}_p(H)$, $1 \leq p < \infty$ if the *p-Schatten norm* is finite [21, p. 60-64]

$$
\|A\|_p := [\operatorname{tr}(|A|^p)]^{1/p} < \infty.
$$

For $1 < p < q < \infty$ we have that

$$\mathcal{B}_1(H) \subset \mathcal{B}_p(H) \subset \mathcal{B}_q(H) \subset \mathcal{B}(H) \tag{1.12}$$

and

$$\|A\|_1 \geq \|A\|_p \geq \|A\|_q \geq \|A\|. \tag{1.13}$$

For $p \geq 1$ the functional $\|\cdot\|_p$ is a *norm* on the $*$-ideal $\mathcal{B}_p(H)$ and $\left(\mathcal{B}_p(H), \|\cdot\|_p\right)$ is a Banach space.

Also, see for instance [21, p. 60-64],

$$\|A\|_p = \|A^*\|_p, \; A \in \mathcal{B}_p(H) \tag{1.14}$$

$$\|AB\|_p \leq \|A\|_p \|B\|_p, \; A, B \in \mathcal{B}_p(H) \tag{1.15}$$

and

$$\|AB\|_p \leq \|A\|_p \|B\|, \; \|BA\|_p \leq \|B\| \|A\|_p, \; A \in \mathcal{B}_p(H), \; B \in \mathcal{B}(H). \tag{1.16}$$

This implies that

$$\|CAB\|_p \leq \|C\| \|A\|_p \|B\|, \; A \in \mathcal{B}_p(H), \; B, \; C \in \mathcal{B}(H). \tag{1.17}$$

In terms of *p-Schatten norm* we have the *Hölder inequality* for $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$

$$(|\operatorname{tr}(AB)| \leq) \|AB\|_1 \leq \|A\|_p \|B\|_q, \; A \in \mathcal{B}_p(H), \; B \in \mathcal{B}_q(H). \tag{1.18}$$

For the theory of trace functionals and their applications the reader is referred to [18] and [21].

For some classical trace inequalities see [7], [8], and [13], which are continuations of the work of Bellman [2].

2. 1-Schatten Norm Inequalities

The first main result is as follows:

Theorem 5. *For $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, assume that $\{A_i\}_{i=0}^N \subset B_p(H)$, $\{B_i\}_{i=0}^N \subset B_q(H)$ are sequences of operators with $A_0 = B_N = 0$, then*

$$\begin{aligned} &\sum_{i=1}^{N-1} \|A_i B_i\|_1 \\ &\leq \frac{1}{2} \left[\sum_{i=0}^{N-1} (N-i-1)(N-i) \|\Delta A_i\|_p^2\right]^{1/2} \left[\sum_{i=0}^{N-1} (i+1) i \|\Delta B_i\|_q^2\right]^{1/2} \\ &\leq \frac{1}{2} N (N-1) \\ &\times \begin{cases} \frac{1}{3}(N+1) \max_{i \in \{0,\dots,N-1\}} \|\Delta A_i\|_p \max_{i \in \{0,\dots,N-1\}} \|\Delta B_i\|_q, \\ \left(\sum_{i=0}^{N-1} \|\Delta A_i\|_p^2\right)^{1/2} \left(\sum_{i=0}^{N-1} \|\Delta B_i\|_q^2\right)^{1/2}. \end{cases} \end{aligned} \tag{2.1}$$

Also,

$$\sum_{i=1}^{N-1}\|A_iB_i\|_1 \tag{2.2}$$
$$\le \frac{1}{2}\left[\sum_{i=0}^{N-1}(N-i-1)(N-i)\|\Delta A_i\|_p^2\right]^{1/2}\left[\sum_{i=0}^{N-1}(i+1)\,i\,\|\Delta B_i\|_q^2\right]^{1/2}$$
$$\le \frac{1}{4}\sum_{i=0}^{N-1}\left[(N-i-1)(N-i)\|\Delta A_i\|_p^2+(i+1)\,i\,\|\Delta B_i\|_q^2\right].$$

Proof. Since $A_0=0$, and $B_N=0$, hence $A_i=\sum_{j=0}^{i-1}\Delta A_j$ and $B_i=-\sum_{j=i}^{N-1}\Delta B_j$. Then by Hölder inequality (1.18) and (CBS) inequality we have

$$\sum_{i=1}^{N-1}\|A_iB_i\|_1 \le \sum_{i=1}^{N-1}\|A_i\|_p\|B_i\|_q = \sum_{i=1}^{N-1}\left\|\sum_{j=0}^{i-1}\Delta A_j\right\|_p\left\|\sum_{j=i}^{N-1}\Delta B_j\right\|_q \tag{2.3}$$
$$\le \sum_{i=1}^{N-1}\sqrt{i}\left(\sum_{j=0}^{i-1}\|\Delta A_j\|_p^2\right)^{1/2}\sqrt{N-i}\left(\sum_{j=i}^{N-1}\|\Delta B_j\|_q^2\right)^{1/2}$$
$$= \sum_{i=1}^{N-1}\sqrt{N-i}\left(\sum_{j=0}^{i-1}\|\Delta A_j\|_p^2\right)^{1/2}\sqrt{i}\left(\sum_{j=i}^{N-1}\|\Delta B_j\|_q^2\right)^{1/2}$$
$$\le \left[\sum_{i=1}^{N-1}(N-i)\left(\sum_{j=0}^{i-1}\|\Delta A_j\|_p^2\right)\right]^{1/2}\left[\sum_{i=1}^{N-1}i\left(\sum_{j=i}^{N-1}\|\Delta B_j\|_q^2\right)\right]^{1/2} =: F.$$

We use the identity

$$\sum_{i=1}^{N}c_i\Delta d_i = c_Nd_{N+1}-c_1d_1-\sum_{i=1}^{N-1}d_{i+1}\Delta c_i. \tag{2.4}$$

Take $c_i=\sum_{j=0}^{i-1}\|\Delta A_j\|_p^2$ and $d_i=-\frac{1}{2}(N-i)(N-i+1)$, then

$$\Delta d_i = d_{i+1}-d_i = -\frac{1}{2}(N-i-1)(N-i)+\frac{1}{2}(N-i)(N-i+1)$$
$$= \frac{1}{2}(N-i)\left[N-i+1-(N-i-1)\right]=(N-i),$$

$$\Delta c_i = c_{i+1}-c_i = \sum_{j=0}^{i}\|\Delta A_j\|_p^2-\sum_{j=0}^{i-1}\|\Delta A_j\|_p^2=\|\Delta A_i\|_p^2$$

and by (2.4)

$$\sum_{i=1}^{N-1}(N-i)\left(\sum_{j=0}^{i-1}\|\Delta A_j\|_p^2\right) = 0\times\sum_{j=0}^{N-1}\|\Delta A_j\|_p^2+\frac{1}{2}N(N-1)\|\Delta A_0\|_p^2 + \frac{1}{2}\sum_{i=1}^{N-1}(N-i-1)(N-i)\|\Delta A_i\|_p^2 = \frac{1}{2}\sum_{i=0}^{N-1}(N-i-1)(N-i)\|\Delta A_i\|_p^2. \tag{2.5}$$

Take $c_i=\sum_{j=i}^{N-1}\|\Delta B_j\|_q^2$ and $d_i=\frac{1}{2}i(i-1)$, then

$$\Delta d_i=d_{i+1}-d_i=\frac{1}{2}(i+1)i-\frac{1}{2}i(i-1)=\frac{1}{2}i(i+1-i+1)=i,$$

$$\Delta c_i=c_{i+1}-c_i=\sum_{j=i+1}^{N-1}\|\Delta B_j\|_q^2-\sum_{j=i}^{N-1}\|\Delta B_j\|_q^2=-\|\Delta B_i\|_q^2$$

and by the identity

$$\sum_{i=1}^{N-1}c_i\Delta d_i=c_{N-1}d_N-c_1d_1-\sum_{i=1}^{N-2}d_{i+1}\Delta c_i$$

we get

$$\sum_{i=1}^{N-1}i\left(\sum_{j=i}^{N-1}\|\Delta B_j\|_q^2\right)=\frac{1}{2}N(N-1)\|\Delta B_{N-1}\|_q^2+\frac{1}{2}\sum_{i=1}^{N-2}(i+1)i\|\Delta B_i\|_q^2 = \frac{1}{2}\sum_{i=1}^{N-1}(i+1)i\|\Delta B_i\|_q^2=\frac{1}{2}\sum_{i=0}^{N-1}(i+1)i\|\Delta B_i\|_q^2. \tag{2.6}$$

Therefore

$$F=\left[\frac{1}{2}\sum_{i=0}^{N-1}(N-i-1)(N-i)\|\Delta A_i\|_p^2\right]^{1/2}\left[\frac{1}{2}\sum_{i=0}^{N-1}(i+1)i\|\Delta B_i\|_q^2\right]^{1/2} = \frac{1}{2}\left[\sum_{i=0}^{N-1}(N-i-1)(N-i)\|\Delta A_i\|_p^2\right]^{1/2}\left[\sum_{i=0}^{N-1}(i+1)i\|\Delta B_i\|_q^2\right]^{1/2},$$

which proves the first inequality in (2.1).

Now, observe that

$$\sum_{i=0}^{N-1}(N-i-1)(N-i)\|\Delta A_i\|_p^2\le\max_{i\in\{0,\dots,N-1\}}\|\Delta A_i\|_p^2\sum_{i=0}^{N-1}(N-i-1)(N-i) = \max_{i\in\{0,\dots,N-1\}}\|\Delta A_i\|_p^2\,\frac{1}{3}(N-1)N(N+1)$$

and

$$\sum_{i=0}^{N-1}(i+1)\,i\,\|\Delta B_i\|_q^2 \le \max_{i\in\{0,\dots,N-1\}}\|\Delta B_i\|_q^2\sum_{i=0}^{N-1}(i+1)\,i$$
$$= \max_{i\in\{0,\dots,N-1\}}\|\Delta B_i\|_q^2\,\frac{1}{3}(N-1)\,N\,(N+1)\,,$$

which proves the first branch in (2.1).

Observe also that

$$\sum_{i=0}^{N-1}(N-i-1)(N-i)\,\|\Delta A_i\|_p^2 \le \max_{i\in\{0,\dots,N-1\}}\left[(N-i-1)(N-i)\right]\sum_{i=0}^{N-1}\|\Delta A_i\|_p^2$$
$$= N(N-1)\sum_{i=0}^{N-1}\|\Delta A_i\|_p^2$$

and

$$\sum_{i=0}^{N-1}(i+1)\,i\,\|\Delta B_i\|_q^2 \le \max_{i\in\{0,\dots,N-1\}}\left[i\,(i+1)\right]\sum_{i=0}^{N-1}\|\Delta B_i\|_q^2$$
$$\le N(N-1)\sum_{i=0}^{N-1}\|\Delta B_i\|_q^2\,,$$

which proves the second branch in (2.1). □

Remark 2. *Assume that* $\{A_i\}_{i=0}^N,\ \{B_i\}_{i=0}^N \subset B_2(H)$ *are sequences of operators with* $A_0 = B_N = 0$, *then*

$$(2.7)\qquad \sum_{i=1}^{N-1}\|A_iB_i\|_1$$
$$\le \frac{1}{2}\left[\sum_{i=0}^{N-1}(N-i-1)(N-i)\,\|\Delta A_i\|_2^2\right]^{1/2}\left[\sum_{i=0}^{N-1}(i+1)\,i\,\|\Delta B_i\|_2^2\right]^{1/2}$$
$$\le \frac{1}{2}N(N-1)$$
$$\times\begin{cases}\frac{1}{3}(N+1)\max_{i\in\{0,\dots,N-1\}}\|\Delta A_i\|_2\max_{i\in\{0,\dots,N-1\}}\|\Delta B_i\|_2\,,\\[2ex] \left(\sum_{i=0}^{N-1}\|\Delta A_i\|_2^2\right)^{1/2}\left(\sum_{i=0}^{N-1}\|\Delta B_i\|_2^2\right)^{1/2}.\end{cases}$$

Also,

$$(2.8)\qquad \sum_{i=1}^{N-1}\|A_iB_i\|_1$$
$$\le \frac{1}{2}\left[\sum_{i=0}^{N-1}(N-i-1)(N-i)\,\|\Delta A_i\|_2^2\right]^{1/2}\left[\sum_{i=0}^{N-1}(i+1)\,i\,\|\Delta B_i\|_2^2\right]^{1/2}$$
$$\le \frac{1}{4}\sum_{i=0}^{N-1}\left[(N-i-1)(N-i)\,\|\Delta A_i\|_2^2+(i+1)\,i\,\|\Delta B_i\|_2^2\right].$$

By using the trace properties and (2.7) we get

$$
\begin{aligned}
&\left|\operatorname{tr}\left(\sum_{i=1}^{N-1} A_i B_i\right)\right| \\
&\leq \frac{1}{2}\left[\operatorname{tr}\left(\sum_{i=0}^{N-1}(N-i-1)(N-i)\left|\Delta A_i\right|^2\right)\right]^{1/2} \\
&\times\left[\operatorname{tr}\left(\sum_{i=0}^{N-1}(i+1)\, i\left|\Delta B_i\right|^2\right)\right]^{1/2} \\
&\leq \frac{1}{2} N(N-1)\left[\operatorname{tr}\left(\sum_{i=0}^{N-1}\left|\Delta A_i\right|^2\right)\right]^{1/2}\left[\operatorname{tr}\left(\sum_{i=0}^{N-1}\left|\Delta B_i\right|^2\right)\right]^{1/2},
\end{aligned}
\tag{2.9}
$$

where $\{A_i\}_{i=0}^{N}, \{B_i\}_{i=0}^{N} \subset B_2(H)$ are sequences of operators with $A_0 = B_N = 0$.

Corollary 2. *For $p, q > 1$ with $\frac{1}{p}+\frac{1}{q}=1$, assume that $\{A_i\}_{i=0}^{N} \subset B_p(H) \cap B_q(H)$ is a sequence of operators with $A_0 = A_N = 0$, then*

$$
\begin{aligned}
&\sum_{i=1}^{N}\left\|A_i\right\|_2^2 \\
&\leq \frac{1}{2}\left[\sum_{i=0}^{N-1}(N-i-1)(N-i)\left\|\Delta A_i\right\|_p^2\right]^{1/2}\left[\sum_{i=0}^{N-1}(i+1)\, i\left\|\Delta A_i\right\|_q^2\right]^{1/2} \\
&\leq \frac{1}{2} N(N-1) \\
&\times\begin{cases}
\frac{1}{3}(N+1) \max_{i\in\{0,\dots,N-1\}}\left\|\Delta A_i\right\|_p \max_{i\in\{0,\dots,N-1\}}\left\|\Delta A_i\right\|_q, \\
\left(\sum_{i=0}^{N-1}\left\|\Delta A_i\right\|_p^2\right)^{1/2}\left(\sum_{i=0}^{N-1}\left\|\Delta A_i\right\|_q^2\right)^{1/2}.
\end{cases}
\end{aligned}
\tag{2.10}
$$

Also,

$$
\begin{aligned}
&\sum_{i=1}^{N}\left\|A_i\right\|_2^2 \\
&\leq \frac{1}{2}\left[\sum_{i=0}^{N-1}(N-i-1)(N-i)\left\|\Delta A_i\right\|_p^2\right]^{1/2}\left[\sum_{i=0}^{N-1}(i+1)\, i\left\|\Delta A_i\right\|_q^2\right]^{1/2} \\
&\leq \frac{1}{4} \sum_{i=0}^{N-1}\left[(N-i-1)(N-i)\left\|\Delta A_i\right\|_p^2+(i+1)\, i\left\|\Delta A_i\right\|_q^2\right].
\end{aligned}
\tag{2.11}
$$

The proof follows by taking $B_i = A_i^*$, $i = 0, ..., N$ in (**??**) and observing that $\|A_i A_i^*, \|_1 = \|A_i\|_2^2$ and $\|\Delta A_i^*\|_q^2 = \|\Delta A_i\|_q^2$ for $i = 0, ..., N$.

Remark 3. *If we take $p=q=2$ in (2.10), then we get*

$$\sum_{i=1}^{N}\|A_i\|_2^2 \le \frac{1}{2}\left[\sum_{i=0}^{N-1}(N-i-1)(N-i)\|\Delta A_i\|_2^2\right]^{1/2}\left[\sum_{i=0}^{N-1}(i+1)\,i\,\|\Delta A_i\|_2^2\right]^{1/2}. \tag{2.12}$$

We also have the following simpler result:

Theorem 6. *For $p,\ q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, assume that $\{A_i\}_{i=0}^{N}\subset B_p(H)$, $\{B_i\}_{i=0}^{N}\subset B_q(H)$ are sequences of operators with $A_0=B_N=0$, then*

$$\sum_{i=1}^{N-1}\|A_iB_i\|_1 \le \frac{1}{2}N\left[\sum_{i=0}^{N-1}(N-i)\|\Delta A_i\|_p^2\right]^{1/2}\left[\sum_{i=0}^{N-1}(i+1)\|\Delta B_i\|_q^2\right]^{1/2} \le \frac{1}{4}N\sum_{i=0}^{N-1}\left[(N-i)\|\Delta A_i\|_p^2+(i+1)\|\Delta B_i\|_q^2\right]. \tag{2.13}$$

Proof. Since $A_0=0$, and $B_N=0$, hence $A_i=\sum_{j=0}^{i-1}\Delta A_j$ and $B_i=-\sum_{j=i}^{N-1}\Delta B_j$. Then by Hölder and (CBS) inequalities

$$\begin{aligned}\sum_{i=1}^{N-1}\|A_iB_i\|_1 &\le \sum_{i=1}^{N-1}\|A_i\|_p\|B_i\|_q \\ &= \sum_{i=1}^{N-1}\left\|\sum_{j=0}^{i-1}\Delta A_j\right\|_p\left\|\sum_{j=i}^{N-1}\Delta B_j\right\|_q \\ &\le \sum_{i=1}^{N-1}\sqrt{i}\left(\sum_{j=0}^{i-1}\|\Delta A_j\|_p^2\right)^{1/2}\sqrt{N-i}\left(\sum_{j=i}^{N-1}\|\Delta B_j\|_q^2\right)^{1/2} \\ &\le \left(\sum_{i=1}^{N-1}\left[\sqrt{i}\left(\sum_{j=0}^{i-1}\|\Delta A_j\|_p^2\right)^{1/2}\right]^2\right)^{1/2} \\ &\quad\times\left(\sum_{i=1}^{N-1}\left[\sqrt{N-i}\left(\sum_{j=i}^{N-1}\|\Delta B_j\|_q^2\right)^{1/2}\right]^2\right)^{1/2} \\ &= \left(\sum_{i=1}^{N-1}i\left(\sum_{j=0}^{i-1}\|\Delta A_j\|_p^2\right)\right)^{1/2}\left(\sum_{i=1}^{N-1}(N-i)\left(\sum_{j=i}^{N-1}\|\Delta B_j\|_q^2\right)\right)^{1/2} =: E.\end{aligned} \tag{2.14}$$

Since

$$\sum_{i=1}^{N}i\left(\sum_{j=0}^{i-1}\|\Delta A_j\|_p^2\right) = \frac{1}{2}\sum_{i=0}^{N-1}(N-i)(N+i+1)\|\Delta A_i\|_p^2$$

and

$$\sum_{i=0}^{N-1}(N-i)\left(\sum_{j=i}^{N-1}\|\Delta B_j\|_q^2\right)=\frac{1}{2}\sum_{i=0}^{N-1}(i+1)(2N-i)\|\Delta B_i\|_q^2,$$

hence

$$E=\frac{1}{2}\left(\sum_{i=0}^{N-1}(N-i)(N+i+1)\|\Delta A_i\|_p^2\right)^{1/2}\times\left(\sum_{i=0}^{N-1}(i+1)(2N-i)\|\Delta B_i\|_q^2\right)^{1/2}.$$

Therefore we have

$$\sum_{i=1}^{N-1}\|A_iB_i\|_1\leq\frac{1}{2}\left(\sum_{i=0}^{N-1}(N-i)(N+i+1)\|\Delta A_i\|_p^2\right)^{1/2}\times\left(\sum_{i=0}^{N-1}(i+1)(2N-i)\|\Delta B_i\|_q^2\right)^{1/2}$$

and from (2.1)

$$\sum_{i=1}^{N-1}\|A_iB_i\|_1\leq\frac{1}{2}\left(\sum_{i=0}^{N-1}(N-i-1)(N-i)\|\Delta A_i\|_p^2\right)^{1/2}\left(\sum_{i=0}^{N-1}(i+1)i\|\Delta B_i\|_q^2\right)^{1/2}.$$

If we add these two inequalities, then we get by the elementary inequality

$$cb+cd\leq\left(a^2+c^2\right)^{1/2}\left(b^2+d^2\right)^{1/2}$$

that

$$2\sum_{i=1}^{N-1}\|A_iB_i\|_1\leq\frac{1}{2}\left(\sum_{i=0}^{N-1}(N-i)(N+i+1)\|\Delta A_i\|_p^2\right)^{1/2}\left(\sum_{i=0}^{N-1}(i+1)(2N-i)\|\Delta B_i\|_q^2\right)^{1/2}+\frac{1}{2}\left(\sum_{i=0}^{N-1}(N-i-1)(N-i)\|\Delta A_i\|_p^2\right)^{1/2}\left(\sum_{i=0}^{N-1}(i+1)i\|\Delta B_i\|_q^2\right)^{1/2}$$

$$\leq \frac{1}{2}\left[\sum_{i=0}^{N-1}(N-i)(N+i+1)\|\Delta A_i\|_p^2+\sum_{i=0}^{N-1}(N-i-1)(N-i)\|\Delta A_i\|_p^2\right]^{1/2}$$
$$\times\left[\sum_{i=0}^{N-1}(i+1)(2N-i)\|\Delta B_i\|_q^2+\sum_{i=0}^{N-1}(i+1)\,i\,\|\Delta B_i\|_q^2\right]^{1/2}$$
$$=N\left[\sum_{i=0}^{N-1}(N-i)\|\Delta A_i\|_p^2\right]^{1/2}\left[\sum_{i=0}^{N-1}(i+1)\|\Delta B_i\|_q^2\right]^{1/2},$$

which proves (2.13). □

Corollary 3. *For $p,\, q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, assume that $\{A_i\}_{i=0}^{N}\subset B_p(H)\cap B_q(H)$ is a sequence of operators with $A_0=A_N=0$, then*

$$\sum_{i=1}^{N-1}\|A_i\|_2^2\leq\frac{1}{2}N\left[\sum_{i=0}^{N-1}(N-i)\|\Delta A_i\|_p^2\right]^{1/2}\left[\sum_{i=0}^{N-1}(i+1)\|\Delta A_i\|_q^2\right]^{1/2}$$
$$\leq\frac{1}{4}N\sum_{i=0}^{N-1}\left[(N-i)\|\Delta A_i\|_p^2+(i+1)\|\Delta A_i\|_q^2\right]. \tag{2.15}$$

In particular, if $\{A_i\}_{i=0}^{N}\subset B_2(H)$, then

$$\sum_{i=1}^{N-1}\|A_i\|_2^2\leq\frac{1}{2}N\left[\sum_{i=0}^{N-1}(N-i)\|\Delta A_i\|_2^2\right]^{1/2}\left[\sum_{i=0}^{N-1}(i+1)\|\Delta A_i\|_2^2\right]^{1/2}$$
$$\leq\frac{1}{4}N(N+1)\sum_{i=0}^{N-1}\|\Delta A_i\|_2^2. \tag{2.16}$$

From (2.16) we obtain the following trace inequalities

$$\operatorname{tr}\left(\sum_{i=1}^{N-1}|A_i|^2\right) \tag{2.17}$$
$$\leq\frac{1}{2}N\left[\operatorname{tr}\left(\sum_{i=0}^{N-1}(N-i)|\Delta A_i|^2\right)\right]^{1/2}\left[\operatorname{tr}\left(\sum_{i=0}^{N-1}(i+1)|\Delta A_i|^2\right)\right]^{1/2}$$
$$\leq\frac{1}{4}N(N+1)\operatorname{tr}\left(\sum_{i=0}^{N-1}|\Delta A_i|^2\right),$$

provided $\{A_i\}_{i=0}^{N}\subset B_2(H)$ with $A_0=A_N=0$.

3. p-Schatten Norm Inequalities

We also have:

Theorem 7. *For $p \geq 1$, assume that $\{A_i\}_{i=0}^N, \{B_i\}_{i=0}^N \subset B_p(H)$ are sequences of operators with $A_0 = B_N = 0$, then*

$$\begin{aligned}
&\sum_{i=1}^{N-1} \|A_i B_i\|_p \\
&\leq \frac{1}{2}\left[\sum_{i=0}^{N-1} (N-i-1)(N-i)\|\Delta A_i\|_p^2\right]^{1/2}\left[\sum_{i=0}^{N-1}(i+1)\,i\,\|\Delta B_i\|_p^2\right]^{1/2} \\
&\leq \frac{1}{2}N(N-1) \\
&\times\begin{cases} \frac{1}{3}(N+1)\max_{i\in\{0,\dots,N-1\}}\|\Delta A_i\|_p \max_{i\in\{0,\dots,N-1\}}\|\Delta B_i\|_p, \\ \\ \left(\sum_{i=0}^{N-1}\|\Delta A_i\|_p^2\right)^{1/2}\left(\sum_{i=0}^{N-1}\|\Delta B_i\|_p^2\right)^{1/2}. \end{cases}
\end{aligned} \tag{3.1}$$

Also,

$$\begin{aligned}
&\sum_{i=1}^{N-1} \|A_i B_i\|_p \\
&\leq \frac{1}{2}\left[\sum_{i=0}^{N-1} (N-i-1)(N-i)\|\Delta A_i\|_p^2\right]^{1/2}\left[\sum_{i=0}^{N-1}(i+1)\,i\,\|\Delta B_i\|_p^2\right]^{1/2} \\
&\leq \frac{1}{4}\sum_{i=0}^{N-1}\left[(N-i-1)(N-i)\|\Delta A_i\|_p^2 + (i+1)\,i\,\|\Delta B_i\|_p^2\right].
\end{aligned} \tag{3.2}$$

Proof. Since $A_0 = 0$, and $B_N = 0$, hence $A_i = \sum_{j=0}^{i-1}\Delta A_j$ and $B_i = -\sum_{j=i}^{N-1}\Delta B_j$. Then by (1.15) and (CBS) inequality we have

$$\begin{aligned}
&\sum_{i=1}^{N-1}\|A_i B_i\|_p \leq \sum_{i=1}^{N-1}\|A_i\|_p\|B_i\|_p = \sum_{i=1}^{N-1}\left\|\sum_{j=0}^{i-1}\Delta A_j\right\|_p\left\|\sum_{j=i}^{N-1}\Delta B_j\right\|_p \\
&\leq \sum_{i=1}^{N-1}\sqrt{i}\left(\sum_{j=0}^{i-1}\|\Delta A_j\|_p^2\right)^{1/2}\sqrt{N-i}\left(\sum_{j=i}^{N-1}\|\Delta B_j\|_p^2\right)^{1/2} \\
&= \sum_{i=1}^{N-1}\sqrt{N-i}\left(\sum_{j=0}^{i-1}\|\Delta A_j\|_p^2\right)^{1/2}\sqrt{i}\left(\sum_{j=i}^{N-1}\|\Delta B_j\|_p^2\right)^{1/2} \\
&\leq \left[\sum_{i=1}^{N-1}(N-i)\left(\sum_{j=0}^{i-1}\|\Delta A_j\|_p^2\right)\right]^{1/2}\left[\sum_{i=1}^{N-1} i\left(\sum_{j=i}^{N-1}\|\Delta B_j\|_q^2\right)\right]^{1/2}.
\end{aligned}$$

Now, by the use of a similar argument to the one utilized in the proof of Theorem 5 we deduce the desired result (3.1). □

Remark 4. *If we take* $B_i = A_i^*$, $i \in \{0, ..., N-1\}$ *and observe that for* $p \geq 1$

$$\|A_i B_i\|_p = \|A_i A_i^*\|_p = \|A_i^* A_i\|_p = \left\| |A_i|^2 \right\|_p = \left[\operatorname{tr}\left(\left| |A_i|^2 \right|^p \right) \right]^{1/p}$$
$$= \left[\operatorname{tr}\left(|A_i|^{2p} \right) \right]^{1/p} = \left(\left[\operatorname{tr}\left(|A_i|^{2p} \right) \right]^{1/(2p)} \right)^2 = \|A_i\|_{2p}^2 ,$$

then by (3.1) we derive

$$\sum_{i=1}^{N-1} \|A_i\|_{2p}^2 \tag{3.3}$$
$$\leq \frac{1}{2} \left[\sum_{i=0}^{N-1} (N-i-1)(N-i) \|\Delta A_i\|_p^2 \right]^{1/2} \left[\sum_{i=0}^{N-1} (i+1) i \|\Delta A_i\|_p^2 \right]^{1/2}$$
$$\leq \frac{1}{2} N(N-1) \times \begin{cases} \frac{1}{3}(N+1) \max_{i \in \{0,...,N-1\}} \|\Delta A_i\|_p^2 , \\ \sum_{i=0}^{N-1} \|\Delta A_i\|_p^2 . \end{cases}$$

We also have the simpler result:

Theorem 8. *For* $p \geq 1$, *assume that* $\{A_i\}_{i=0}^N$, $\{B_i\}_{i=0}^N \subset B_p(H)$ *are sequences of operators with* $A_0 = B_N = 0$, *then*

$$\sum_{i=1}^{N-1} \|A_i B_i\|_p \leq \frac{1}{2} N \left[\sum_{i=0}^{N-1} (N-i) \|\Delta A_i\|_p^2 \right]^{1/2} \left[\sum_{i=0}^{N-1} (i+1) \|\Delta B_i\|_p^2 \right]^{1/2} \tag{3.4}$$
$$\leq \frac{1}{4} N \sum_{i=0}^{N-1} \left[(N-i) \|\Delta A_i\|_p^2 + (i+1) \|\Delta B_i\|_p^2 \right] .$$

Remark 5. *If we take* $B_i = A_i^*$, $i \in \{0, ..., N-1\}$, *then by (3.1) we derive*

$$\sum_{i=1}^{N-1} \|A_i\|_{2p}^2 \leq \frac{1}{2} N \left[\sum_{i=0}^{N-1} (N-i) \|\Delta A_i\|_p^2 \right]^{1/2} \left[\sum_{i=0}^{N-1} (i+1) \|\Delta A_i\|_p^2 \right]^{1/2} \tag{3.5}$$
$$\leq \frac{1}{4} N (N+1) \sum_{i=0}^{N-1} \|\Delta A_i\|_p^2 .$$

References

[1] R. P. Agarwal and P. Y. H. Pang . Opial inequalities with application in differential and difference equations, Kluwer Academic Publishers. 1995.

[2] P. R. Beesack, On an integral inequality of Z. Opial. *Trans. Am. Math. Soc.* **104** (1962), 470–475.

[3] R. Bellman, Some inequalities for positive definite matrices, in: E.F. Beckenbach (Ed.), *General Inequalities 2*, Proceedings of the 2nd International Conference on General Inequalities, Birkhäuser, Basel, 1980, pp. 89–90.

[4] R. Bhatia, *Matrix Analysis.* Graduate Texts in Mathematics, 169. Springer-Verlag, New York, 1997. xii+347 pp.

[5] R. C. Brown Aand D. B. Hinton, Opial's inequality and oscilation of 2nd order equations, *Proc. Amer. Math. Soc.* **125** (1997), Number 4, 1123-1129.

[6] R. C. Brown and M. Plum, An Opial-type inequality with an integral boundary condition, *Proc. R. Soc. A* **461** (2005), 2635–2651. doi:10.1098/rspa.2005.1449.

[7] D. Chang, A matrix trace inequality for products of Hermitian matrices, *J. Math. Anal. Appl.* **237** (1999) 721–725.
[8] I. D. Coop, On matrix trace inequalities and related topics for products of Hermitian matrix, *J. Math. Anal. Appl.* **188** (1994) 999–1001.3.
[9] S. S. Dragomir, Discrete inequalities for two sequences related to Lasota-Opial's result, Preprint *RGMIA Res. Rep. Coll.* **24** (2021), Art.
[10] K.-C. Hsu and K.-L. Tseng, Some new discrete inequalities of Opial and Lasota's type, *Journal of Progressive Research in Mathematics*, Published online: June 17, 2015
[11] L.-K. Hua . On an inequality of Opial, *Scientia Sinica* **14** (1965), 789-790.
[12] A. Lasota, A discrete boundary value problem, *Ann. Polon. Math.* 20 (1968), 183-190.
[13] H. Neudecker, A matrix trace inequality, *J. Math. Anal. Appl.* **166** (1992) 302–303.
[14] C. Olech, A simple proof of a certain result of Z. Opial. *Ann. Polon. Math.* **8** (1960), 61–63.
[15] Z. Opial, Sur une inégalité. *Ann. Polon. Math.* **8** (1960), 29–32.
[16] B. G. Pachpatte . On Opial like discrete inequalities, *An. Sti. Univ. A1. I. Cuza Iasi, Mat.* **36** (1990), 237-240.
[17] B. G. Pachpatte, A note on Opial type finite difference inequalities, *Tamsui Oxford J. Math. Sci.* **21** (1)(2005), 33-39.
[18] B. Simon, *Trace Ideals and Their Applications,* Cambridge University Press, Cambridge, 1979.
[19] J. S. W. Wong . A discrete analogue of Opial's inequality, *Canadian Math. Bull.* **10** (1967), 115-118.
[20] G.-S. Yang . On a certain result of Z. Opial, *Proc. Japan Acad.* **42** (1966), 78-83.
[21] V. A. Zagrebnov, *Gibbs Semigroups,* Operator Theory: Advances and Applications, Volume 273, Birkhäuser, 2019

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