p-SCHATTEN NORM HÖLDER'S TYPE INEQUALITIES OF OPIAL-LASOTA KIND FOR TWO SEQUENCES

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Abstract. An operator $A \in \mathcal{B}(H)$ is said to belong to the *von Neumann-Schatten class* $\mathcal{B}_p(H)$, $1 \leq p < \infty$ if the *p-Schatten norm* is finite

$$\|A\|_p := [\operatorname{tr}(|A|^p)]^{1/p} < \infty.$$

In this paper we show among others that, for $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, assume that $\{A_i\}_{i=0}^N \subset B_p(H)$, $\{B_i\}_{i=0}^N \subset B_q(H)$ are sequences of operators with $A_0 = B_N = 0$, then

$$\sum_{i=1}^{N-1} \|A_i B_i\|_1 \leq \frac{1}{2}\left(\sum_{i=0}^{N-1}(N-i-1)(N-i)\|\Delta A_i\|_p^\alpha\right)^{1/\alpha} \times \left(\sum_{i=0}^{N-1}(i+1)\,i\,\|\Delta B_i\|_q^\beta\right)^{1/\beta}$$
$$\leq \frac{1}{2}N(N-1) \times \begin{cases} \frac{1}{3}(N+1)\max_{i\in\{0,\ldots,N-1\}}\|\Delta A_i\|_p \\ \times \max_{i\in\{0,\ldots,N-1\}}\|\Delta B_i\|_q, \\ \\ \left(\sum_{i=0}^{N-1}\|\Delta A_i\|_p^\alpha\right)^{1/\alpha}\left(\sum_{i=0}^{N-1}\|\Delta B_i\|_q^\beta\right)^{1/\beta}, \end{cases}$$

where $\alpha, \beta > 1$ with $\frac{1}{\alpha} + \frac{1}{\beta} = 1$.

1. Introduction

For a sequence $\{x_i\}_{i=0}^N$, we consider the forward operator Δ defined by $\Delta x_i = x_{i+1} - x_i$, $i = 0, ..., N-1$. The summation by parts formula also holds

$$\sum_{k=m}^{n} a_k \Delta b_k = a_n b_{n+1} - a_m b_m - \sum_{k=m}^{n-1} b_{k+1}\Delta a_k, \tag{1.1}$$

where a_k and b_k are some sequences for which the products above exist.

In [12], Lasota provided discrete versions of Opial inequality [15] about the forward difference operator as follows:

Theorem 1. *Let $\{x_i\}_{i=0}^N$ be a sequence of real numbers with $x_0 = x_N = 0$. Then, the following inequality holds*

$$\sum_{i=1}^{N-1}|x_i \Delta x_i| \leq \frac{1}{2}\left\lfloor \frac{N+1}{2}\right\rfloor \sum_{i=0}^{N-1}|\Delta x_i|^2, \tag{1.2}$$

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where $\lfloor\cdot\rfloor$ *is the integer part function. If* N *is even, then the inequality (1.2) is sharp.*

For various Opial type inequalities, see [2]-[6] and [14].

Also, we have the following results, see [1]:

Theorem 2. *Let* $\{x_i\}_{i=0}^{N}$ *be a sequence of real numbers. If* $x_0=0$, *then*

$$\sum_{i=1}^{\tau-1}|x_i\Delta x_i|\le\frac{1}{2}(\tau-1)\sum_{i=0}^{\tau-1}|\Delta x_i|^2,\ \tau\in\{2,...,N\}. \tag{1.3}$$

If $x_N=0$, *then*

$$\sum_{i=\tau}^{N-1}|x_i\Delta x_i|\le\frac{1}{2}(N-\tau+1)\sum_{i=0}^{N-1}|\Delta x_i|^2,\ \tau\in\{1,...,N-1\}. \tag{1.4}$$

For other discrete Opial type inequalities, see [10], [11] and [16]-[20].

In the recent paper [9] we obtained the following result:

Theorem 3. *Assume that* $\{x_i\}_{i=0}^{N}$, $\{y_i\}_{i=0}^{N}$ *are sequences of complex numbers. If* $y_0=y_N=0$, *then for* $n\in\{2,...,N-1\}$, $p,\ q>1$, $\frac{1}{p}+\frac{1}{q}=1$

$$\begin{aligned}\sum_{i=1}^{N-1}|\Delta x_i||y_i| &\le\left(\sum_{i=0}^{N-1}p_i(n)|\Delta x_i|^p\right)^{1/p}\left(\sum_{i=0}^{N-1}q_i(n)|\Delta y_i|^q\right)^{1/q}\\ &\le\frac{1}{p}\sum_{i=0}^{N-1}p_i(n)|\Delta x_i|^p+\frac{1}{q}\sum_{i=0}^{N-1}q_i(n)|\Delta y_i|^q,\end{aligned} \tag{1.5}$$

where

$$p_i(n):=\begin{cases} i, & \text{if } 0\le i\le n-1,\\ N-i, & \text{if } n\le i\le N-1\end{cases}$$

and

$$q_i(n):=\begin{cases} n-i-1, & \text{if } 0\le i\le n-1,\\ i+1-n, & \text{if } n\le i\le N-1.\end{cases}$$

In particular,

$$\begin{aligned}\sum_{i=1}^{N-1}|\Delta x_i||y_i| &\le\left(\sum_{i=0}^{N-1}p_i(n)|\Delta x_i|^2\right)^{1/2}\left(\sum_{i=0}^{N-1}q_i(n)|\Delta y_i|^2\right)^{1/2}\\ &\le\frac{1}{2}\sum_{i=0}^{N-1}\left[p_i(n)|\Delta x_i|^2+q_i(n)|\Delta y_i|^2\right].\end{aligned} \tag{1.6}$$

In order to extend these results for p-Schatten norms and two different sequences we need the following preparations.

Let $(H;\langle.,.\rangle)$ be a complex Hilbert space and $\mathcal{B}(H)$ the Banach algebra of all bounded linear operators on H. If $\{e_i\}_{i\in I}$ an orthonormal basis of H, we say that $A\in\mathcal{B}(H)$ is of *trace class* if

$$\|A\|_1:=\sum_{i\in I}\langle|A|e_i,e_i\rangle<\infty. \tag{1.7}$$

The definition of $\|A\|_1$ does not depend on the choice of the orthonormal basis $\{e_i\}_{i\in I}$. We denote by $\mathcal{B}_1(H)$ the set of trace class operators in $\mathcal{B}(H)$.

We define the *trace* of a trace class operator $A \in \mathcal{B}_1(H)$ to be

$$\operatorname{tr}(A) := \sum_{i\in I} \langle Ae_i, e_i\rangle, \tag{1.8}$$

where $\{e_i\}_{i\in I}$ an orthonormal basis of H. Note that this coincides with the usual definition of the trace if H is finite-dimensional. We observe that the series (1.8) converges absolutely and it is independent from the choice of basis.

The following result collects some properties of the trace:

Theorem 4. *We have:*

(i) If $A \in \mathcal{B}_1(H)$ *then* $A^* \in \mathcal{B}_1(H)$ *and*

$$\operatorname{tr}(A^*) = \overline{\operatorname{tr}(A)}; \tag{1.9}$$

(ii) If $A \in \mathcal{B}_1(H)$ *and* $T \in \mathcal{B}(H)$, *then* $AT, TA \in \mathcal{B}_1(H)$ *and*

$$\operatorname{tr}(AT) = \operatorname{tr}(TA) \text{ and } |\operatorname{tr}(AT)| \le \|A\|_1 \|T\|; \tag{1.10}$$

(iii) $\operatorname{tr}(\cdot)$ *is a bounded linear functional on* $\mathcal{B}_1(H)$ *with* $\|\operatorname{tr}\| = 1$;

(iv) If $A, B \in \mathcal{B}_2(H)$ *then* $AB, BA \in \mathcal{B}_1(H)$ *and* $\operatorname{tr}(AB) = \operatorname{tr}(BA)$;

(v) $\mathcal{B}_{fin}(H)$ *is a dense subspace of* $\mathcal{B}_1(H)$.

An operator $A \in \mathcal{B}(H)$ is said to belong to the *von Neumann-Schatten class* $\mathcal{B}_p(H)$, $1 \le p < \infty$ if the *p-Schatten norm* is finite [21, p. 60-64]

$$\|A\|_p := [\operatorname{tr}(|A|^p)]^{1/p} < \infty.$$

For $1 < p < q < \infty$ we have that

$$\mathcal{B}_1(H) \subset \mathcal{B}_p(H) \subset \mathcal{B}_q(H) \subset \mathcal{B}(H) \tag{1.11}$$

and

$$\|A\|_1 \ge \|A\|_p \ge \|A\|_q \ge \|A\|. \tag{1.12}$$

For $p \ge 1$ the functional $\|\cdot\|_p$ is a *norm* on the $*$-ideal $\mathcal{B}_p(H)$ and $\left(\mathcal{B}_p(H), \|\cdot\|_p\right)$ is a Banach space.

Also, see for instance [21, p. 60-64],

$$\|A\|_p = \|A^*\|_p, \ A \in \mathcal{B}_p(H) \tag{1.13}$$

$$\|AB\|_p \le \|A\|_p \|B\|_p, \ A, B \in \mathcal{B}_p(H) \tag{1.14}$$

and

$$\|AB\|_p \le \|A\|_p \|B\|, \ \|BA\|_p \le \|B\| \|A\|_p, \ A \in \mathcal{B}_p(H), \ B \in \mathcal{B}(H). \tag{1.15}$$

This implies that

$$\|CAB\|_p \le \|C\| \|A\|_p \|B\|, \ A \in \mathcal{B}_p(H), \ B, \ C \in \mathcal{B}(H). \tag{1.16}$$

In terms of *p-Schatten norm* we have the *Hölder inequality* for $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$

$$(|\operatorname{tr}(AB)| \le) \|AB\|_1 \le \|A\|_p \|B\|_q, \ A \in \mathcal{B}_p(H), \ B \in \mathcal{B}_q(H). \tag{1.17}$$

For the theory of trace functionals and their applications the reader is referred to [18] and [21].

For some classical trace inequalities see [7], [8], and [13], which are continuations of the work of Bellman [2].

2. 1-Schatten Norm Inequalities

We have the following result for two sequences:

Theorem 5. *For $p,\ q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, assume that $\{A_i\}_{i=0}^{N}\subset B_p(H)$, $\{B_i\}_{i=0}^{N}\subset B_q(H)$ are sequences of operators with $A_0=B_0=0$, then*

$$\sum_{i=1}^{N}\|A_iB_i\|_1 \le \frac{1}{2}\left(\sum_{i=0}^{N-1}(N-i)(N+i+1)\|\Delta A_i\|_p^{\alpha}\right)^{1/\alpha} \times\left(\sum_{i=0}^{N-1}(N-i)(N+i+1)\|\Delta B_i\|_q^{\beta}\right)^{1/\beta} \le \frac{1}{2}N(N+1) \times\begin{cases}\frac{1}{3}(2N+1) \\ \times\max_{i\in\{0,\dots,N-1\}}\|\Delta A_i\|_p\max_{i\in\{0,\dots,N-1\}}\|\Delta B_i\|_q, \\ \\ \left(\sum_{i=0}^{N-1}\|\Delta A_i\|_p^{\alpha}\right)^{1/\alpha}\left(\sum_{i=0}^{N-1}\|\Delta B_i\|_q^{\beta}\right)^{1/\beta},\end{cases} \tag{2.1}$$

for $\alpha,\ \beta>1$ with $\frac{1}{\alpha}+\frac{1}{\beta}=1$.

Also,

$$\sum_{i=1}^{N}\|A_iB_i\|_1 \le \frac{1}{2}\left(\sum_{i=0}^{N-1}(N-i)(N+i+1)\|\Delta A_i\|_p^{\alpha}\right)^{1/\alpha} \times\left(\sum_{i=0}^{N-1}(N-i)(N+i+1)\|\Delta B_i\|_q^{\beta}\right)^{1/\beta} \le \frac{1}{2}\sum_{i=0}^{N-1}(N-i)(N+i+1)\left(\frac{1}{\alpha}\|\Delta A_i\|_p^{\alpha}+\frac{1}{\beta}\|\Delta B_i\|_q^{\beta}\right). \tag{2.2}$$

Proof. Since $A_0=B_0=0$, hence $A_i=\sum_{j=0}^{i-1}\Delta A_j$ and $B_i=\sum_{j=0}^{i-1}\Delta B_j$ for $i=1,\dots,N$. Then by (1.17)

$$\begin{aligned}\sum_{i=1}^{N}\|A_iB_i\|_1 &\le \sum_{i=1}^{N}\|A_i\|_p\|B_i\|_q \\ &= \sum_{i=1}^{N}\left\|\sum_{j=0}^{i-1}\Delta A_j\right\|_p\left\|\sum_{j=0}^{i-1}\Delta B_j\right\|_q \\ &= \sum_{i=1}^{N}i\frac{1}{i^{1/\beta}}\left\|\sum_{j=0}^{i-1}\Delta A_j\right\|_p\frac{1}{i^{1/\alpha}}\left\|\sum_{j=0}^{i-1}\Delta B_j\right\|_q =:\Gamma.\end{aligned}$$

By Hölder's inequality we have

$$\frac{1}{i^{1/\beta}}\left\|\sum_{j=0}^{i-1}\Delta A_j\right\|_p \leq \left(\sum_{j=0}^{i-1}\|\Delta A_j\|_p^{\alpha}\right)^{1/\alpha},$$

$$\frac{1}{i^{1/\alpha}}\left\|\sum_{j=0}^{i-1}\Delta B_j\right\|_q \leq \left(\sum_{j=0}^{i-1}\|\Delta B_j\|_q^{\beta}\right)^{1/\beta},$$

which gives.

$$\Gamma \leq \sum_{i=1}^{N} i\left(\sum_{j=0}^{i-1}\|\Delta A_j\|_p^{\alpha}\right)^{1/\alpha}\left(\sum_{j=0}^{i-1}\|\Delta B_j\|_q^{\beta}\right)^{1/\beta}.$$

By the weighted Hölder inequality we have

$$\begin{aligned}
&\sum_{i=1}^{N} i\left(\sum_{j=0}^{i-1}\|\Delta A_j\|_p^{\alpha}\right)^{1/\alpha}\left(\sum_{j=0}^{i-1}\|\Delta B_j\|_q^{\beta}\right)^{1/\beta}\\
&\leq \left(\sum_{i=1}^{N} i\left[\left(\sum_{j=0}^{i-1}\|\Delta A_j\|_p^{\alpha}\right)^{1/\alpha}\right]^{\alpha}\right)^{1/\alpha}\\
&\times \left(\sum_{i=1}^{N} i\left[\left(\sum_{j=0}^{i-1}\|\Delta B_j\|_q^{\beta}\right)^{1/\beta}\right]^{\beta}\right)^{1/\beta}\\
&= \left(\sum_{i=1}^{N} i\left(\sum_{j=0}^{i-1}\|\Delta A_j\|_p^{\alpha}\right)\right)^{1/\alpha}\left(\sum_{i=1}^{N} i\left(\sum_{j=0}^{i-1}\|\Delta B_j\|_q^{\beta}\right)\right)^{1/\beta},
\end{aligned}$$

which implies that

$$\Gamma \leq \left(\sum_{i=1}^{N} i\left(\sum_{j=0}^{i-1}\|\Delta A_j\|_p^{\alpha}\right)\right)^{1/\alpha}\left(\sum_{i=1}^{N} i\left(\sum_{j=0}^{i-1}\|\Delta B_j\|_q^{\beta}\right)\right)^{1/\beta} =: \Delta. \tag{2.3}$$

From the formula (1.1), we get

$$\sum_{i=1}^{N} a_i\Delta b_i = a_N b_{N+1} - a_1 b_1 - \sum_{k=1}^{N-1} b_{i+1}\Delta a_i. \tag{2.4}$$

Now, if we take $a_i = \sum_{j=0}^{i-1}\|\Delta A_j\|_p^{\alpha}$, $i = 1, ..., N$, $b_i = \frac{1}{2}i\,(i-1)$, then $a_N = \sum_{j=0}^{N-1}\|\Delta A_j\|_p^{\alpha}$,

$$\Delta b_i = b_{i+1} - b_i = \frac{1}{2}i\,(i+1) - \frac{1}{2}i\,(i-1) = i,$$

and

$$\Delta a_i = a_{i+1} - a_i = \sum_{j=0}^{i}\|\Delta A_j\|_p^{\alpha} - \sum_{j=0}^{i-1}\|\Delta A_j\|_p^{\alpha} = \|\Delta A_i\|_p^{\alpha}.$$

By (2.4) we derive

$$\sum_{i=1}^{N} i\left(\sum_{j=0}^{i-1}\|\Delta A_j\|_p^{\alpha}\right) = \frac{1}{2}N(N+1)\sum_{j=0}^{N-1}\|\Delta A_j\|_p^{\alpha} - \sum_{k=1}^{N-1}\frac{1}{2}i(i+1)\|\Delta A_i\|_p^{\alpha} \tag{2.5}$$
$$= \sum_{i=0}^{N-1}\left[\frac{1}{2}N(N+1) - \frac{1}{2}i(i+1)\right]\|\Delta A_i\|_p^{\alpha}$$
$$= \frac{1}{2}\sum_{i=0}^{N-1}(N-i)(N+i+1)\|\Delta A_i\|_p^{\alpha}$$

and, similarly

$$\sum_{i=1}^{N} i\left(\sum_{j=0}^{i-1}\|\Delta B_j\|_q^{\beta}\right) = \frac{1}{2}\sum_{i=0}^{N-1}(N-i)(N+i+1)\|\Delta B_i\|_q^{\beta}. \tag{2.6}$$

Therefore

$$B = \frac{1}{2}\left(\sum_{i=0}^{N-1}(N-i)(N+i+1)\|\Delta A_i\|_p^{\alpha}\right)^{1/\alpha}$$
$$\times\left(\sum_{i=0}^{N-1}(N-i)(N+i+1)\|\Delta B_i\|_q^{\beta}\right)^{1/\beta}$$

and by (2.3) we derive the first inequality in (2.1).

Now observe that

$$\sum_{i=0}^{N-1}(N-i)(N+i+1)\|\Delta A_i\|_p^{\alpha} \le \max_{i\in\{0,\dots,N-1\}}\|\Delta A_i\|_p^{\alpha}\sum_{i=0}^{N-1}(N-i)(N+i+1)$$
$$= \frac{1}{3}N(N+1)(2N+1)\max_{i\in\{0,\dots,N-1\}}\|\Delta A_i\|_p^{\alpha},$$

which proves the first branch in (2.1).

Observe also that

$$\sum_{i=0}^{N-1}(N-i)(N+i+1)\|\Delta A_i\|_p^{\alpha} \le \max_{i\in\{0,\dots,N-1\}}[(N-i)(N+i+1)]\sum_{i=0}^{N-1}\|\Delta A_i\|_p^{\alpha}$$
$$= \max_{i\in\{0,\dots,N-1\}}[N(N+1) - i(i+1)]\sum_{i=0}^{N-1}\|\Delta A_i\|_p^{\alpha}$$
$$= N(N+1)\sum_{i=0}^{N-1}\|\Delta A_i\|_p^{2},$$

which proves the second branch in (2.1).

The last inequality in (2.2) follows by Young's inequality

$$a^{1/\alpha}b^{1/\beta} \le \frac{1}{\alpha}a + \frac{1}{\beta}b,\ a,\ b \ge 0.$$

□

We observe that for $\alpha = p$ and $\beta = q$ in (2.1), we get

$$\sum_{i=1}^{N} \|A_i B_i\|_1 \le \frac{1}{2} \left(\sum_{i=0}^{N-1} (N-i)(N+i+1) \|\Delta A_i\|_p^p \right)^{1/p} \times \left(\sum_{i=0}^{N-1} (N-i)(N+i+1) \|\Delta B_i\|_q^q \right)^{1/q} \le \frac{1}{2} N(N+1) \times \begin{cases} \frac{1}{3}(2N+1) \\ \times \max_{i\in\{0,\dots,N-1\}} \|\Delta A_i\|_p \max_{i\in\{0,\dots,N-1\}} \|\Delta B_i\|_q, \\ \\ \left(\sum_{i=0}^{N-1} \|\Delta A_i\|_p^p \right)^{1/p} \left(\sum_{i=0}^{N-1} \|\Delta B_i\|_q^q \right)^{1/q}. \end{cases} \tag{2.7}$$

By the use of trace properties, this implies the following result

$$\left| \operatorname{tr}\left(\sum_{i=1}^{N} A_i B_i \right) \right| \le \frac{1}{2} \left[\operatorname{tr}\left(\sum_{i=0}^{N-1} (N-i)(N+i+1) |\Delta A_i|^p \right) \right]^{1/p} \times \left[\operatorname{tr}\left(\sum_{i=0}^{N-1} (N-i)(N+i+1) |\Delta B_i|^q \right) \right]^{1/q} \le \frac{1}{2} N(N+1) \times \begin{cases} \frac{1}{3}(2N+1) \\ \times \max_{i\in\{0,\dots,N-1\}} \|\Delta A_i\|_p \max_{i\in\{0,\dots,N-1\}} \|\Delta B_i\|_q, \\ \\ \left[\operatorname{tr}\left(\sum_{i=0}^{N-1} |\Delta A_i|^p \right) \right]^{1/p} \left[\operatorname{tr}\left(\sum_{i=0}^{N-1} |\Delta B_i|^q \right) \right]^{1/q}, \end{cases} \tag{2.8}$$

for $p,\ q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ and $\{A_i\}_{i=0}^N \subset B_p(H)$, $\{B_i\}_{i=0}^N \subset B_q(H)$ with $A_0 = B_0 = 0$.

Corollary 1. *For $p,\ q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, assume that $\{A_i\}_{i=0}^N \subset B_p(H) \cap B_q(H)$ with $A_0 = 0$, then for $\alpha,\ \beta > 1$ with $\frac{1}{\alpha} + \frac{1}{\beta} = 1$,*

$$\sum_{i=1}^{N} \|A_i\|_2^2 \le \frac{1}{2} \left(\sum_{i=0}^{N-1} (N-i)(N+i+1) \|\Delta A_i\|_p^\alpha \right)^{1/\alpha} \times \left(\sum_{i=0}^{N-1} (N-i)(N+i+1) \|\Delta A_i\|_q^\beta \right)^{1/\beta} \le \frac{1}{2} N(N+1) \times \begin{cases} \frac{1}{3}(2N+1) \max_{i\in\{0,\dots,N-1\}} \|\Delta A_i\|_p \\ \times \max_{i\in\{0,\dots,N-1\}} \|\Delta A_i\|_q, \\ \\ \left(\sum_{i=0}^{N-1} \|\Delta A_i\|_p^\alpha \right)^{1/\alpha} \left(\sum_{i=0}^{N-1} \|\Delta A_i\|_q^\beta \right)^{1/\beta}. \end{cases} \tag{2.9}$$

For $\alpha = p$ and $\beta = q$ we derive from (2.9) that

$$\sum_{i=1}^{N} \|A_i\|_2^2 \leq \frac{1}{2} \left(\sum_{i=0}^{N-1} (N-i)(N+i+1) \|\Delta A_i\|_p^p \right)^{1/p} \times \left(\sum_{i=0}^{N-1} (N-i)(N+i+1) \|\Delta A_i\|_q^q \right)^{1/q} \leq \frac{1}{2} N(N+1) \times \begin{cases} \frac{1}{3}(2N+1) \max_{i\in\{0,\dots,N-1\}} \|\Delta A_i\|_p \\ \times \max_{i\in\{0,\dots,N-1\}} \|\Delta A_i\|_q , \\ \\ \left(\sum_{i=0}^{N-1} \|\Delta A_i\|_p^p\right)^{1/p} \left(\sum_{i=0}^{N-1} \|\Delta A_i\|_q^q\right)^{1/q} . \end{cases} \tag{2.10}$$

This can be written as

$$\operatorname{tr}\left(\sum_{i=1}^{N} |A_i|^2\right) \leq \frac{1}{2}\left[\operatorname{tr}\left(\sum_{i=0}^{N-1} (N-i)(N+i+1) |\Delta A_i|^p\right)\right]^{1/p} \times \left[\operatorname{tr}\left(\sum_{i=0}^{N-1} (N-i)(N+i+1) |\Delta A_i|^q\right)\right]^{1/q} \leq \frac{1}{2} N(N+1) \left[\operatorname{tr}\left(\sum_{i=0}^{N-1} |\Delta A_i|^p\right)\right]^{1/p} \left[\operatorname{tr}\left(\sum_{i=0}^{N-1} |\Delta A_i|^q\right)\right]^{1/q}, \tag{2.11}$$

for $p,\, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ and $\{A_i\}_{i=0}^{N} \subset B_p(H) \cap B_q(H)$ with $A_0 = 0$.

Remark 1. *If $\{A_i\}_{i=0}^{N} \subset B_2(H)$ with $A_0 = 0$, then for $\alpha,\, \beta > 1$ with $\frac{1}{\alpha} + \frac{1}{\beta} = 1$,*

$$\sum_{i=1}^{N} \|A_i\|_2^2 \leq \frac{1}{2} \left(\sum_{i=0}^{N-1} (N-i)(N+i+1) \|\Delta A_i\|_2^\alpha \right)^{1/\alpha} \times \left(\sum_{i=0}^{N-1} (N-i)(N+i+1) \|\Delta A_i\|_2^\beta \right)^{1/\beta} \leq \frac{1}{2} N(N+1) \times \begin{cases} \frac{1}{3}(2N+1) \max_{i\in\{0,\dots,N-1\}} \|\Delta A_i\|_2^2 \\ \\ \left(\sum_{i=0}^{N-1} \|\Delta A_i\|_2^\alpha\right)^{1/\alpha} \left(\sum_{i=0}^{N-1} \|\Delta A_i\|_2^\beta\right)^{1/\beta} . \end{cases} \tag{2.12}$$

In particular, for $\alpha = \beta = 2$ we derive

$$\sum_{i=1}^{N} \|A_i\|_2^2 \leq \frac{1}{2} \sum_{i=0}^{N-1} (N-i)(N+i+1) \|\Delta A_i\|_2^2 .$$

Also, we have:

Theorem 6. *For* $p,\ q>1$ *with* $\frac{1}{p}+\frac{1}{q}=1$, *assume that* $\{A_i\}_{i=0}^{N}\subset B_p(H)$, $\{B_i\}_{i=0}^{N}\subset B_q(H)$ *are sequences of operators with* $A_N=B_N=0$, *then*

$$\sum_{i=0}^{N-1}\|A_iB_i\|_1\le\frac{1}{2}\left(\sum_{i=0}^{N-1}(i+1)(2N-i)\|\Delta A_i\|_p^{\alpha}\right)^{1/\alpha}\times\left(\sum_{i=0}^{N-1}(i+1)(2N-i)\|\Delta B_i\|_q^{\beta}\right)^{1/\beta}\le\frac{1}{2}N(N+1)\times\begin{cases}\frac{1}{3}(2N+1)\max_{i\in\{0,\dots,N-1\}}\|\Delta A_i\|_p\\ \times\max_{i\in\{0,\dots,N-1\}}\|\Delta B_i\|_q,\\ \\ \left(\sum_{i=0}^{N-1}\|\Delta A_i\|_p^{\alpha}\right)^{1/\alpha}\left(\sum_{i=0}^{N-1}\|\Delta B_i\|_q^{\beta}\right)^{1/\beta},\end{cases}\tag{2.13}$$

for $\alpha,\ \beta>1$ *with* $\frac{1}{\alpha}+\frac{1}{\beta}=1$.

Also, we have

$$\sum_{i=0}^{N-1}\|A_iB_i\|_1\le\frac{1}{2}\left(\sum_{i=0}^{N-1}(i+1)(2N-i)\|\Delta A_i\|_p^{\alpha}\right)^{1/\alpha}\times\left(\sum_{i=0}^{N-1}(i+1)(2N-i)\|\Delta B_i\|_q^{\beta}\right)^{1/\beta}\le\frac{1}{2}\sum_{i=0}^{N-1}(i+1)(2N-i)\left(\frac{1}{\alpha}\|\Delta A_i\|_p^{\alpha}+\frac{1}{\beta}\|\Delta B_i\|_q^{\beta}\right).\tag{2.14}$$

Proof. If $A_N=B_N=0$, then $A_i=-\sum_{j=i}^{N-1}\Delta A_j$ and $B_i=-\sum_{j=i}^{N-1}\Delta B_j$, $i\in\{0,...,N-1\}$. Then by (1.17)

$$\sum_{i=0}^{N-1}\|A_iB_i\|_1\le\sum_{i=0}^{N-1}\|A_i\|_p\|B_i\|_q=\sum_{i=0}^{N-1}\left\|\sum_{j=i}^{N-1}\Delta A_j\right\|_p\left\|\sum_{j=i}^{N-1}\Delta B_j\right\|_q$$
$$=\sum_{i=0}^{N-1}(N-i)\frac{1}{(N-i)^{1/\beta}}\left\|\sum_{j=i}^{N-1}\Delta A_j\right\|_p\frac{1}{(N-i)^{1/\alpha}}\left\|\sum_{j=i}^{N-1}\Delta B_j\right\|_q$$
$$=:\Theta.$$

By Hölder's inequality we have

$$\frac{1}{(N-i)^{1/\beta}}\left\|\sum_{j=i}^{N-1}\Delta A_j\right\|_p\le\left(\sum_{j=i}^{N-1}\|\Delta A_j\|_p^{\alpha}\right)^{1/\alpha}$$

and

$$\frac{1}{(N-i)^{1/\alpha}}\left\|\sum_{j=i}^{N-1}\Delta B_j\right\|_q \leq \left(\sum_{j=i}^{N-1}\|\Delta B_j\|_q^{\beta}\right)^{1/\beta},$$

which gives

$$\Theta \leq \sum_{i=0}^{N-1}(N-i)\left(\sum_{j=i}^{N-1}\|\Delta A_j\|_p^{\alpha}\right)^{1/\alpha}\left(\sum_{j=i}^{N-1}\|\Delta B_j\|_q^{\beta}\right)^{1/\beta}.$$

By the weighted Hölder inequality we have

$$\begin{aligned}&\sum_{i=0}^{N-1}(N-i)\left(\sum_{j=i}^{N-1}\|\Delta A_j\|_p^{\alpha}\right)^{1/\alpha}\left(\sum_{j=i}^{N-1}\|\Delta B_j\|_q^{\beta}\right)^{1/\beta}\\&\leq\left(\sum_{i=0}^{N-1}(N-i)\left(\sum_{j=i}^{N-1}\|\Delta A_j\|_p^{\alpha}\right)\right)^{1/\alpha}\left(\sum_{i=0}^{N-1}(N-i)\left(\sum_{j=i}^{N-1}\|\Delta B_j\|_q^{\beta}\right)\right)^{1/\beta},\end{aligned}$$

which gives

$$\begin{aligned}\Theta &\leq \left(\sum_{i=0}^{N-1}(N-i)\left(\sum_{j=i}^{N-1}\|\Delta A_j\|_p^{\alpha}\right)\right)^{1/\alpha}\left(\sum_{i=0}^{N-1}(N-i)\left(\sum_{j=i}^{N-1}\|\Delta B_j\|_q^{\beta}\right)\right)^{1/\beta}\\&=: \Lambda.\end{aligned} \tag{2.15}$$

From (1.1) we get for $m=0$ and $n=N-1$ that

$$\sum_{i=0}^{N-1}a_i\Delta b_i = a_{N-1}b_N - a_0b_0 - \sum_{k=0}^{N-2}b_{i+1}\Delta a_i. \tag{2.16}$$

Take $a_i=\sum_{j=i}^{N-1}\|\Delta A_j\|_p^{\alpha}$ and $b_i=-\frac{1}{2}(N-i)(N-i+1)$, then we get

$$\begin{aligned}\Delta b_i = b_{i+1}-b_i &= -\frac{1}{2}(N-i-1)(N-i)+\frac{1}{2}(N-i)(N-i+1)\\&=\frac{1}{2}(N-i)(-N+i+1+N-i+1) = N-i\end{aligned}$$

and

$$\Delta a_i = a_{i+1}-a_i = \sum_{j=i+1}^{N-1}\|\Delta A_j\|_p^{\alpha} - \sum_{j=i}^{N-1}\|\Delta A_j\|_p^{\alpha} = -\|\Delta A_i\|_p^{\alpha}.$$

Then

$$\begin{aligned}
&\sum_{i=0}^{N-1}(N-i)\left(\sum_{j=i}^{N-1}\|\Delta A_j\|_p^\alpha\right)\\
&=\frac{1}{2}N(N+1)\sum_{j=0}^{N-1}\|\Delta A_j\|_p^\alpha-\frac{1}{2}\sum_{i=0}^{N-2}(N-i-1)(N-i)\|\Delta A_i\|_p^\alpha\\
&=\frac{1}{2}N(N+1)\sum_{j=0}^{N-1}\|\Delta A_j\|_p^\alpha-\frac{1}{2}\sum_{i=0}^{N-1}(N-i-1)(N-i)\|\Delta A_i\|_p^\alpha\\
&=\frac{1}{2}\sum_{i=0}^{N-1}[N(N+1)-(N-i-1)(N-i)]\|\Delta A_i\|_p^\alpha\\
&=\frac{1}{2}\sum_{i=0}^{N-1}(i+1)(2N-i)\|\Delta A_i\|_p^\alpha
\end{aligned}$$

and

$$\sum_{i=0}^{N-1}(N-i)\left(\sum_{j=i}^{N-1}\|\Delta B_j\|_q^\beta\right)=\frac{1}{2}\sum_{i=0}^{N-1}(i+1)(2N-i)\|\Delta B_i\|_q^\beta.$$

Therefore

$$\begin{aligned}
\Lambda=\frac{1}{2}&\left(\sum_{i=0}^{N-1}(i+1)(2N-i)\|\Delta A_i\|_p^\alpha\right)^{1/\alpha}\\
&\times\left(\sum_{i=0}^{N-1}(i+1)(2N-i)\|\Delta B_i\|_q^\beta\right)^{1/\beta},
\end{aligned}$$

and by (2.15) we derive the first inequality in (2.13).

Now, observe that

$$\begin{aligned}
\sum_{i=0}^{N-1}(i+1)(2N-i)\|\Delta A_i\|_p^\alpha&\le\max_{i\in\{0,\dots,N-1\}}\|\Delta A_i\|_p^2\sum_{i=0}^{N-1}(i+1)(2N-i)\\
&=\frac{1}{3}N(N+1)(2N+1)\max_{i\in\{0,\dots,N-1\}}\|\Delta A_i\|_p^\alpha
\end{aligned}$$

and

$$\sum_{i=0}^{N-1}(i+1)(2N-i)\|\Delta B_i\|_q^\beta\le\frac{1}{3}N(N+1)(2N+1)\max_{i\in\{0,\dots,N-1\}}\|\Delta B_i\|_q^\beta,$$

which proves the first branch of (2.13).

Also

$$\begin{aligned}
&\sum_{i=0}^{N-1}(i+1)(2N-i)\|\Delta A_i\|_p^{\alpha}\\
&=\sum_{i=0}^{N-1}[N(N+1)-(N-i-1)(N-i)]\|\Delta A_i\|_p^{\alpha}\\
&\le \max_{i\in\{0,\dots,N-1\}}[N(N+1)-(N-i-1)(N-i)]\sum_{i=0}^{N-1}\|\Delta A_i\|_p^{\alpha}\\
&=N(N+1)\sum_{i=0}^{N-1}\|\Delta A_i\|_p^{\alpha},
\end{aligned}$$

which proves the second branch of (2.13). □

Corollary 2. *For $p, q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, assume that $\{A_i\}_{i=0}^{N}\subset B_p(H)\cap B_q(H)$ with $A_N=0$, then*

$$\begin{aligned}
\sum_{i=0}^{N-1}\|A_i\|_2^2 &\le \frac{1}{2}\left(\sum_{i=0}^{N-1}(i+1)(2N-i)\|\Delta A_i\|_p^{\alpha}\right)^{1/\alpha}\\
&\quad\times\left(\sum_{i=0}^{N-1}(i+1)(2N-i)\|\Delta A_i\|_q^{\beta}\right)^{1/\beta}\\
&\le \frac{1}{2}N(N+1)\\
&\quad\times\begin{cases}\frac{1}{3}(2N+1)\max_{i\in\{0,\dots,N-1\}}\|\Delta A_i\|_p\\ \times\max_{i\in\{0,\dots,N-1\}}\|\Delta A_i\|_q,\\ \\ \left(\sum_{i=0}^{N-1}\|\Delta A_i\|_p^{\alpha}\right)^{1/\alpha}\left(\sum_{i=0}^{N-1}\|\Delta A_i\|_q^{\beta}\right)^{1/\beta}.\end{cases}
\end{aligned}\tag{2.17}$$

Remark 2. *If $\{A_i\}_{i=0}^{N}\subset B_2(H)$ with $A_N=0$ then for $\alpha,\beta>1$ with $\frac{1}{\alpha}+\frac{1}{\beta}=1$,*

$$\begin{aligned}
\sum_{i=0}^{N-1}\|A_i\|_2^2 &\le \frac{1}{2}\left(\sum_{i=0}^{N-1}(i+1)(2N-i)\|\Delta A_i\|_2^{\alpha}\right)^{1/\alpha}\\
&\quad\times\left(\sum_{i=0}^{N-1}(i+1)(2N-i)\|\Delta A_i\|_2^{\beta}\right)^{1/\beta}\\
&\le \frac{1}{2}N(N+1)\\
&\quad\times\begin{cases}\frac{1}{3}(2N+1)\max_{i\in\{0,\dots,N-1\}}\|\Delta A_i\|_2^2,\\ \\ \left(\sum_{i=0}^{N-1}\|\Delta A_i\|_p^{\alpha}\right)^{1/\alpha}\left(\sum_{i=0}^{N-1}\|\Delta A_i\|_2^{\beta}\right)^{1/\beta}.\end{cases}
\end{aligned}\tag{2.18}$$

In particular, for $\alpha=\beta=2$ we derive

$$\sum_{i=0}^{N-1}\|A_i\|_2^2\le\frac{1}{2}\sum_{i=0}^{N-1}(i+1)(2N-i)\|\Delta A_i\|_2^2.\tag{2.19}$$

We also have:

Theorem 7. *For $p,\ q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, assume that $\{A_i\}_{i=0}^{N}\subset B_p(H)$, $\{B_i\}_{i=0}^{N}\subset B_q(H)$ are sequences of operators with $A_0=B_N=0$, then*

$$\begin{aligned}(2.20)\qquad \sum_{i=1}^{N-1}\|A_iB_i\|_1 &\leq \frac{1}{2}\left(\sum_{i=0}^{N-1}(N-i-1)(N-i)\|\Delta A_i\|_p^{\alpha}\right)^{1/\alpha}\\ &\quad\times\left(\sum_{i=0}^{N-1}(i+1)\,i\,\|\Delta B_i\|_q^{\beta}\right)^{1/\beta}\\ &\leq \frac{1}{2}N(N-1)\\ &\quad\times\begin{cases}\frac{1}{3}(N+1)\max_{i\in\{0,\dots,N-1\}}\|\Delta A_i\|_p\\ \times\max_{i\in\{0,\dots,N-1\}}\|\Delta B_i\|_q\,,\\ \\ \left(\sum_{i=0}^{N-1}\|\Delta A_i\|_p^{\alpha}\right)^{1/\alpha}\left(\sum_{i=0}^{N-1}\|\Delta B_i\|_q^{\beta}\right)^{1/\beta}.\end{cases}\end{aligned}$$

Also,

$$\begin{aligned}(2.21)\qquad &\sum_{i=1}^{N-1}\|A_iB_i\|_1\\ &\leq \frac{1}{2}\left(\sum_{i=0}^{N-1}(N-i-1)(N-i)\|\Delta A_i\|_p^{\alpha}\right)^{1/\alpha}\\ &\times\left(\sum_{i=0}^{N-1}(i+1)\,i\,\|\Delta B_i\|_q^{\beta}\right)^{1/\beta}\\ &\leq \frac{1}{2}\sum_{i=0}^{N-1}\left[\frac{1}{\alpha}(N-i-1)(N-i)\|\Delta A_i\|_p^{\alpha}+\frac{1}{\beta}(i+1)\,i\,\|\Delta B_i\|_q^{\beta}\right].\end{aligned}$$

Proof. Since $A_0=0$, and $B_N=0$, hence $A_i=\sum_{j=0}^{i-1}\Delta A_j$ and $B_i=-\sum_{j=i}^{N-1}\Delta B_j$. Then by Hölder's inequality

$$\begin{aligned}(2.22)\quad \sum_{i=1}^{N-1}\|A_iB_i\|_1 &\leq \sum_{i=1}^{N-1}\|A_i\|_p\|B_i\|_q=\sum_{i=1}^{N-1}\left\|\sum_{j=0}^{i-1}\Delta A_j\right\|_p\left\|\sum_{j=i}^{N-1}\Delta B_j\right\|_q\\ &\leq \sum_{i=1}^{N-1}i^{1/\beta}\left(\sum_{j=0}^{i-1}\|\Delta A_j\|_p^{\alpha}\right)^{1/\alpha}(N-i)^{1/\alpha}\left(\sum_{j=i}^{N-1}\|\Delta B_j\|_q^{\beta}\right)^{1/\beta}\end{aligned}$$

$$\leq\left(\sum_{i=1}^{N-1}\left[(N-i)^{1/\alpha}\left(\sum_{j=0}^{i-1}\|\Delta A_j\|_p^{\alpha}\right)^{1/\alpha}\right]^{\alpha}\right)^{1/\alpha}$$
$$\times\left(\sum_{i=1}^{N-1}\left[i^{1/\beta}\left(\sum_{j=i}^{N-1}\|\Delta B_j\|_q^{\beta}\right)^{1/\beta}\right]^{\beta}\right)^{1/\beta}$$
$$=\left(\sum_{i=1}^{N-1}(N-i)\left(\sum_{j=0}^{i-1}\|\Delta A_j\|_p^{\alpha}\right)\right)^{1/\alpha}\left(\sum_{i=1}^{N-1}i\left(\sum_{j=i}^{N-1}\|\Delta B_j\|_q^{\beta}\right)\right)^{1/\beta}=:\Xi.$$

We use the identity

$$\sum_{i=1}^{N}a_i\Delta b_i=a_Nb_{N+1}-a_1b_1-\sum_{i=1}^{N-1}b_{i+1}\Delta a_i. \tag{2.23}$$

Take $a_i=\sum_{j=0}^{i-1}\|\Delta A_j\|_p^{\alpha}$ and $b_i=-\frac{1}{2}(N-i)(N-i+1)$, then

$$\Delta b_i=b_{i+1}-b_i=-\frac{1}{2}(N-i-1)(N-i)+\frac{1}{2}(N-i)(N-i+1)$$
$$=\frac{1}{2}(N-i)[N-i+1-(N-i-1)]=(N-i),$$

$$\Delta a_i=a_{i+1}-a_i=\sum_{j=0}^{i}\|\Delta A_j\|_p^{\alpha}-\sum_{j=0}^{i-1}\|\Delta A_j\|_p^{\alpha}=\|\Delta A_i\|_p^{\alpha}$$

and by (2.23)

$$\sum_{i=1}^{N-1}(N-i)\left(\sum_{j=0}^{i-1}\|\Delta A_j\|_p^{\alpha}\right) \tag{2.24}$$
$$=\frac{1}{2}N(N-1)\|\Delta A_0\|_p^{\alpha}+\frac{1}{2}\sum_{i=1}^{N-1}(N-i-1)(N-i)\|\Delta A_i\|_p^{\alpha}$$
$$=\frac{1}{2}\sum_{i=0}^{N-1}(N-i-1)(N-i)\|\Delta A_i\|_p^{\alpha}.$$

Take $a_i=\sum_{j=i}^{N-1}\|\Delta B_j\|_q^{\beta}$ and $b_i=\frac{1}{2}i(i-1)$, then

$$\Delta b_i=b_{i+1}-b_i=\frac{1}{2}(i+1)i-\frac{1}{2}i(i-1)=\frac{1}{2}i(i+1-i+1)=i,$$

$$\Delta a_i=a_{i+1}-a_i=\sum_{j=i+1}^{N-1}\|\Delta B_j\|_q^{\beta}-\sum_{j=i}^{N-1}\|\Delta B_j\|_q^{\beta}=-\|\Delta B_i\|_q^{\beta}$$

and by the identity

$$\sum_{i=1}^{N-1}a_i\Delta b_i=a_{N-1}b_N-a_1b_1-\sum_{i=1}^{N-2}b_{i+1}\Delta a_i.$$

we get

$$\sum_{i=1}^{N-1} i \left(\sum_{j=i}^{N-1} \|\Delta B_j\|_q^\beta \right) = \frac{1}{2} N (N-1) \|\Delta B_{N-1}\|_q^\beta + \frac{1}{2} \sum_{i=1}^{N-2} (i+1) i \|\Delta B_i\|_q^\beta = \frac{1}{2} \sum_{i=1}^{N-1} (i+1) i \|\Delta B_i\|_q^\beta = \frac{1}{2} \sum_{i=0}^{N-1} (i+1) i \|\Delta B_i\|_q^\beta . \tag{2.25}$$

Therefore

$$\Xi = \left(\frac{1}{2} \sum_{i=0}^{N-1} (N-i-1)(N-i) \|\Delta A_i\|_p^\alpha \right)^{1/\alpha} \left(\frac{1}{2} \sum_{i=0}^{N-1} (i+1) i \|\Delta B_i\|_q^\beta \right)^{1/\beta}$$
$$= \frac{1}{2} \left(\sum_{i=0}^{N-1} (N-i-1)(N-i) \|\Delta A_i\|_p^\alpha \right)^{1/\alpha} \left(\sum_{i=0}^{N-1} (i+1) i \|\Delta B_i\|_q^\beta \right)^{1/\beta},$$

which proves the first inequality in (2.20)

Now, observe that

$$\sum_{i=0}^{N-1} (N-i-1)(N-i) \|\Delta A_i\|_p^\alpha \le \max_{i\in\{0,\ldots,N-1\}} \|\Delta A_i\|_p^\alpha \sum_{i=0}^{N-1} (N-i-1)(N-i) = \max_{i\in\{0,\ldots,N-1\}} \|\Delta A_i\|_p^\alpha \frac{1}{3} (N-1) N (N+1)$$

and

$$\sum_{i=0}^{N-1} (i+1) i \|\Delta B_i\|_q^\beta \le \max_{i\in\{0,\ldots,N-1\}} \|\Delta B_i\|_q^\beta \sum_{i=0}^{N-1} (i+1) i = \max_{i\in\{0,\ldots,N-1\}} \|\Delta B_i\|_q^\beta \frac{1}{3} (N-1) N (N+1),$$

which proves the first branch in (2.20).

Observe also that

$$\sum_{i=0}^{N-1} (N-i-1)(N-i) \|\Delta A_i\|_p^\alpha \le \max_{i\in\{0,\ldots,N-1\}} [(N-i-1)(N-i)] \sum_{i=0}^{N-1} \|\Delta A_i\|_p^\alpha = N (N-1) \sum_{i=0}^{N-1} \|\Delta A_i\|_p^\alpha$$

and

$$\sum_{i=0}^{N-1} (i+1) i \|\Delta B_i\|_q^\beta \le \max_{i\in\{0,\ldots,N-1\}} [i(i+1)] \sum_{i=0}^{N-1} \|\Delta B_i\|_q^\beta \le N (N-1) \sum_{i=0}^{N-1} \|\Delta B_i\|_q^\beta ,$$

which proves the second branch in (2.20). □

Corollary 3. *For $p,\ q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, assume that $\{A_i\}_{i=0}^{N}\subset B_p(H)\cap B_q(H)$ with $A_0=A_N=0$, then*

$$\begin{aligned}(2.26)\quad \sum_{i=1}^{N-1}\|A_i\|_2^2 &\le \frac{1}{2}\left(\sum_{i=0}^{N-1}(N-i-1)(N-i)\|\Delta A_i\|_p^{\alpha}\right)^{1/\alpha} \\ &\quad\times\left(\sum_{i=0}^{N-1}(i+1)\,i\,\|\Delta A_i\|_q^{\beta}\right)^{1/\beta} \\ &\le \frac{1}{2}N(N-1) \\ &\quad\times\begin{cases}\frac{1}{3}(N+1) \\ \times\max_{i\in\{0,\dots,N-1\}}\|\Delta A_i\|_p\max_{i\in\{0,\dots,N-1\}}\|\Delta A_i\|_q \\ \\ \left(\sum_{i=0}^{N-1}\|\Delta A_i\|_p^{\alpha}\right)^{1/\alpha}\left(\sum_{i=0}^{N-1}\|\Delta A_i\|_p^{\beta}\right)^{1/\beta}.\end{cases}\end{aligned}$$

Also,

$$\begin{aligned}(2.27)\quad \sum_{i=1}^{N-1}\|A_i\|_2^2 &\le \frac{1}{2}\left(\sum_{i=0}^{N-1}(N-i-1)(N-i)\|\Delta A_i\|_p^{\alpha}\right)^{1/\alpha} \\ &\quad\times\left(\sum_{i=0}^{N-1}(i+1)\,i\,\|\Delta A_i\|_q^{\beta}\right)^{1/\beta} \\ &\le \frac{1}{2}\sum_{i=0}^{N-1}\left[\frac{1}{\alpha}(N-i-1)(N-i)\|\Delta A_i\|_p^{\alpha}+\frac{1}{\beta}(i+1)\,i\,\|\Delta A_i\|_q^{\beta}\right].\end{aligned}$$

Remark 3. *If $\{A_i\}_{i=0}^{N}\subset B_2(H)$ with $A_0=A_N=0$, then for $\alpha,\ \beta>1$ with $\frac{1}{\alpha}+\frac{1}{\beta}=1$,*

$$\begin{aligned}(2.28)\quad \sum_{i=1}^{N-1}\|A_i\|_2^2 &\le \frac{1}{2}\left(\sum_{i=0}^{N-1}(N-i-1)(N-i)\|\Delta A_i\|_2^{\alpha}\right)^{1/\alpha} \\ &\quad\times\left(\sum_{i=0}^{N-1}(i+1)\,i\,\|\Delta A_i\|_2^{\beta}\right)^{1/\beta} \\ &\le \frac{1}{2}N(N-1) \\ &\quad\times\begin{cases}\frac{1}{3}(N+1)\max_{i\in\{0,\dots,N-1\}}\|\Delta A_i\|_p^{2} \\ \\ \left(\sum_{i=0}^{N-1}\|\Delta A_i\|_p^{\alpha}\right)^{1/\alpha}\left(\sum_{i=0}^{N-1}\|\Delta A_i\|_2^{\beta}\right)^{1/\beta}.\end{cases}\end{aligned}$$

In particular, for $\alpha = \beta = 2$ we obtain

$$\sum_{i=1}^{N-1} \|A_i\|_2^2 \leq \frac{1}{2}\left(\sum_{i=0}^{N-1}(N-i-1)(N-i)\|\Delta A_i\|_2^2\right)^{1/2}\left(\sum_{i=0}^{N-1}(i+1)\,i\,\|\Delta A_i\|_2^2\right)^{1/2}. \tag{2.29}$$

3. p-Schatten Norm Inequalities

We also have the following result:

Theorem 8. *For $p \geq 1$, assume that $\{A_i\}_{i=0}^N, \{B_i\}_{i=0}^N \subset B_p(H)$ are sequences of operators with $A_0 = B_0 = 0$, then*

$$\begin{aligned} \sum_{i=1}^{N}\|A_iB_i\|_p &\leq \frac{1}{2}\left(\sum_{i=0}^{N-1}(N-i)(N+i+1)\|\Delta A_i\|_p^\alpha\right)^{1/\alpha} \\ &\times\left(\sum_{i=0}^{N-1}(N-i)(N+i+1)\|\Delta B_i\|_p^\beta\right)^{1/\beta} \\ &\leq \frac{1}{2}N(N+1) \\ &\times\begin{cases} \frac{1}{3}(2N+1) \\ \times \max_{i\in\{0,\dots,N-1\}}\|\Delta A_i\|_p \max_{i\in\{0,\dots,N-1\}}\|\Delta B_i\|_p, \\ \\ \left(\sum_{i=0}^{N-1}\|\Delta A_i\|_p^\alpha\right)^{1/\alpha}\left(\sum_{i=0}^{N-1}\|\Delta B_i\|_p^\beta\right)^{1/\beta}, \end{cases} \end{aligned} \tag{3.1}$$

for α, $\beta > 1$ with $\frac{1}{\alpha} + \frac{1}{\beta} = 1$.

Also,

$$\begin{aligned} \sum_{i=1}^{N}\|A_iB_i\|_p &\leq \frac{1}{2}\left(\sum_{i=0}^{N-1}(N-i)(N+i+1)\|\Delta A_i\|_p^\alpha\right)^{1/\alpha} \\ &\times\left(\sum_{i=0}^{N-1}(N-i)(N+i+1)\|\Delta B_i\|_p^\beta\right)^{1/\beta} \\ &\leq \frac{1}{2}\sum_{i=0}^{N-1}(N-i)(N+i+1)\left(\frac{1}{\alpha}\|\Delta A_i\|_p^\alpha + \frac{1}{\beta}\|\Delta B_i\|_p^\beta\right). \end{aligned} \tag{3.2}$$

Proof. Since $A_0 = B_0 = 0$, hence $A_i = \sum_{j=0}^{i-1} \Delta A_j$ and $B_i = \sum_{j=0}^{i-1} \Delta B_j$ for $i = 1, ..., N$. Then by (1.14)

$$\begin{aligned} \sum_{i=1}^{N} \|A_i B_i\|_p &\leq \sum_{i=1}^{N} \|A_i\|_p \|B_i\|_p \\ &= \sum_{i=1}^{N} \left\| \sum_{j=0}^{i-1} \Delta A_j \right\|_p \left\| \sum_{j=0}^{i-1} \Delta B_j \right\|_p \\ &= \sum_{i=1}^{N} i \frac{1}{i^{1/\beta}} \left\| \sum_{j=0}^{i-1} \Delta A_j \right\|_p \frac{1}{i^{1/\alpha}} \left\| \sum_{j=0}^{i-1} \Delta B_j \right\|_p . \end{aligned}$$

By utilising a similar argument with the one from Theorem 5 we deduce the desired results. □

Corollary 4. *With the assumptions of Theorem 8*

$$\begin{aligned} \sum_{i=1}^{N} \|A_i B_i\|_p &\leq \frac{1}{2} \left(\sum_{i=0}^{N-1} (N-i)(N+i+1) \|\Delta A_i\|_p^2 \right)^{1/2} \\ &\quad \times \left(\sum_{i=0}^{N-1} (N-i)(N+i+1) \|\Delta B_i\|_p^2 \right)^{1/2} \\ &\leq \frac{1}{2} N(N+1) \\ &\quad \times \begin{cases} \frac{1}{3}(2N+1) \\ \times \max_{i\in\{0,\dots,N-1\}} \|\Delta A_i\|_p \max_{i\in\{0,\dots,N-1\}} \|\Delta B_i\|_p , \\ \\ \left(\sum_{i=0}^{N-1} \|\Delta A_i\|_p^2 \right)^{1/2} \left(\sum_{i=0}^{N-1} \|\Delta B_i\|_p^2 \right)^{1/2}, \end{cases} \end{aligned} \tag{3.3}$$

for α, $\beta > 1$ with $\frac{1}{\alpha} + \frac{1}{\beta} = 1$.

Theorem 9. *Also,*

$$\begin{aligned} \sum_{i=1}^{N} \|A_i B_i\|_p &\leq \frac{1}{2} \left(\sum_{i=0}^{N-1} (N-i)(N+i+1) \|\Delta A_i\|_p^2 \right)^{1/2} \\ &\quad \times \left(\sum_{i=0}^{N-1} (N-i)(N+i+1) \|\Delta B_i\|_p^2 \right)^{1/2} \\ &\leq \frac{1}{4} \sum_{i=0}^{N-1} (N-i)(N+i+1) \left(\|\Delta A_i\|_p^2 + \|\Delta B_i\|_p^2 \right). \end{aligned} \tag{3.4}$$

Remark 4. *If* $\{A_i\}_{i=0}^{N} \subset B_p(H)$, $p \geq 1$ *with* $A_0 = 0$, *then for* $\alpha, \beta > 1$ *with* $\frac{1}{\alpha} + \frac{1}{\beta} = 1$

$$\sum_{i=1}^{N} \|A_i\|_{2p}^2 \leq \frac{1}{2} \left(\sum_{i=0}^{N-1} (N-i)(N+i+1) \|\Delta A_i\|_p^\alpha \right)^{1/\alpha} \times \left(\sum_{i=0}^{N-1} (N-i)(N+i+1) \|\Delta A_i\|_p^\beta \right)^{1/\beta}. \tag{3.5}$$

In particular, we have

$$\sum_{i=1}^{N} \|A_i\|_{2p}^2 \leq \frac{1}{2} \sum_{i=0}^{N-1} (N-i)(N+i+1) \|\Delta A_i\|_p^2. \tag{3.6}$$

Also, we have:

Theorem 10. *For* $p \geq 1$, *assume that* $\{A_i\}_{i=0}^{N}, \{B_i\}_{i=0}^{N} \subset B_p(H)$ *are sequences of operators with* $A_N = B_N = 0$, *then*

$$\begin{aligned} \sum_{i=0}^{N-1} \|A_i B_i\|_p &\leq \frac{1}{2} \left(\sum_{i=0}^{N-1} (i+1)(2N-i) \|\Delta A_i\|_p^\alpha \right)^{1/\alpha} \times \left(\sum_{i=0}^{N-1} (i+1)(2N-i) \|\Delta B_i\|_p^\beta \right)^{1/\beta} \\ &\leq \frac{1}{2} N(N+1) \times \begin{cases} \frac{1}{3}(2N+1) \max_{i\in\{0,\dots,N-1\}} \|\Delta A_i\|_p \times \max_{i\in\{0,\dots,N-1\}} \|\Delta B_i\|_p, \\ \\ \left(\sum_{i=0}^{N-1} \|\Delta A_i\|_p^\alpha \right)^{1/\alpha} \left(\sum_{i=0}^{N-1} \|\Delta B_i\|_p^\beta \right)^{1/\beta}, \end{cases} \end{aligned} \tag{3.7}$$

for $\alpha, \beta > 1$ *with* $\frac{1}{\alpha} + \frac{1}{\beta} = 1$.

Moreover, we have

$$\begin{aligned} \sum_{i=0}^{N-1} \|A_i B_i\|_p &\leq \frac{1}{2} \left(\sum_{i=0}^{N-1} (i+1)(2N-i) \|\Delta A_i\|_p^\alpha \right)^{1/\alpha} \times \left(\sum_{i=0}^{N-1} (i+1)(2N-i) \|\Delta B_i\|_p^\beta \right)^{1/\beta} \\ &\leq \frac{1}{2} \sum_{i=0}^{N-1} (i+1)(2N-i) \left(\frac{1}{\alpha} \|\Delta A_i\|_p^\alpha + \frac{1}{\beta} \|\Delta B_i\|_p^\beta \right). \end{aligned} \tag{3.8}$$

Finally, we also have:

Theorem 11. *For $p \geq 1$, assume that $\{A_i\}_{i=0}^N$, $\{B_i\}_{i=0}^N \subset B_p(H)$ are sequences of operators with $A_0 = B_N = 0$, then*

$$\sum_{i=1}^{N-1} \|A_i B_i\|_p \leq \frac{1}{2} \left(\sum_{i=0}^{N-1} (N-i-1)(N-i) \|\Delta A_i\|_p^\alpha \right)^{1/\alpha} \times \left(\sum_{i=0}^{N-1} (i+1) i \|\Delta B_i\|_p^\beta \right)^{1/\beta} \leq \frac{1}{2} N(N-1) \times \begin{cases} \frac{1}{3}(N+1) \max_{i\in\{0,\dots,N-1\}} \|\Delta A_i\|_p \\ \times \max_{i\in\{0,\dots,N-1\}} \|\Delta B_i\|_p, \\ \\ \left(\sum_{i=0}^{N-1} \|\Delta A_i\|_p^\alpha\right)^{1/\alpha} \left(\sum_{i=0}^{N-1} \|\Delta B_i\|_p^\beta\right)^{1/\beta}. \end{cases} \tag{3.9}$$

Also,

$$\sum_{i=1}^{N-1} \|A_i B_i\|_p \leq \frac{1}{2} \left(\sum_{i=0}^{N-1} (N-i-1)(N-i) \|\Delta A_i\|_p^\alpha \right)^{1/\alpha} \times \left(\sum_{i=0}^{N-1} (i+1) i \|\Delta B_i\|_p^\beta \right)^{1/\beta} \leq \frac{1}{2} \sum_{i=0}^{N-1} \left[\frac{1}{\alpha} (N-i-1)(N-i) \|\Delta A_i\|_p^\alpha + \frac{1}{\beta} (i+1) i \|\Delta B_i\|_p^\beta \right]. \tag{3.10}$$

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