p-SCHATTEN NORM SUP-TYPE INEQUALITIES FOR THE WEIGHTED ČEBYŠEV'S FUNCTIONAL

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Abstract. An operator $A \in \mathcal{B}(H)$ is said to belong to the *von Neumann-Schatten class* $\mathcal{B}_p(H)$, $1 \leq p < \infty$ if the *p-Schatten norm* is finite

$$\|A\|_p := [\operatorname{tr}(|A|^p)]^{1/p} < \infty.$$

In this paper we show among others that,Assume that $w : [a,b] \to [0,\infty)$ is integrable with $\int_a^b w(t)\,dt = 1$ and for $p,\ q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, $A : [a,b] \to B_p(H)$, $B : [a,b] \to B_q(H)$ are strongly differentiable with $\sup_{u\in(a,b)} \|A'(u)\|_p < \infty$, then

$$\begin{aligned}
&\left\|\int_a^b w(t)\,A(t)\,B(t)\,dt - \int_a^b w(t)\,A(t)\,dt \int_a^b w(t)\,B(t)\,dt\right\|_1 \\
&\leq \frac{1}{2}\int_a^b w(t)\left|t - \int_a^b sw(s)\,ds\right| dt \sup_{u\in(a,b)} \|A'(u)\|_p \int_a^b \|B'(u)\|_q\,du \\
&\leq \frac{1}{2}\left[\int_a^b s^2 w(s)\,ds - \left(\int_a^b sw(s)\,ds\right)^2\right]^{\frac{1}{2}} \\
&\times \sup_{u\in(a,b)} \|A'(u)\|_p \int_a^b \|B'(u)\|_q\,du \\
&\leq \frac{1}{4}\sup_{u\in(a,b)} \|A'(u)\|_p \int_a^b \|B'(u)\|_q\,du.
\end{aligned}$$

Some examples of interest for the operator monotone functions are also given.

1. Introduction

Let $(\Omega, \mathcal{A}, \mu)$ be a measurable space consisting of a set Ω, a σ-algebra $\mathcal{A}$ of parts of Ω and a countably additive and positive measure μ on $\mathcal{A}$ with values in $\mathbb{R}\cup\{\infty\}$.

For a μ-measurable function $w : \Omega \to \mathbb{R}$, with $w(x) \geq 0$ for μ-a.e. $x \in \Omega$, consider the Lebesgue space $L_w(\Omega,\mu) := \{h : \Omega \to \mathbb{R},\ h$ is μ-measurable and $\int_\Omega w(x)\,|h(x)|\,d\mu(x) < \infty\}$. Assume $\int_\Omega w(x)\,d\mu(x) = 1$. In order to simplify the notation for the integrals, we do not write the variable, namely, instead of $\int_\Omega w(x)\,d\mu(x)$ we simply write $\int_\Omega wd\mu$.

If $h,\ k : \Omega \to \mathbb{R}$ are μ-measurable functions and $h,\ k,\ hk \in L_w(\Omega,\mu)$, then we may consider the weighted Čebyšev functional in the following form

$$D_w(h,k) := \int_\Omega whkd\mu - \int_\Omega whd\mu \int_\Omega wkd\mu. \tag{1.1}$$

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The following result is known in the literature as the Grüss inequality, see for instance [5]:

$$|D_w(h,k)| \le \frac{1}{4}(\Gamma-\gamma)(\Delta-\delta), \tag{1.2}$$

provided

$$-\infty<\gamma\le h\le\Gamma<\infty,\ \ -\infty<\delta\le k\le\Delta<\infty \tag{1.3}$$

for μ-a.e. on Ω. The constant $\frac{1}{4}$ is sharp in the sense that it cannot be replaced by a smaller constant.

In [5] Cerone and Dragomir proved among others the following *refinement of Grüss' inequality:*

Theorem 1. *For $h,\ k:\Omega\to\mathbb{R}$, μ-measurable functions and so that $-\infty<\gamma\le h\le\Gamma<\infty$, $-\infty<\delta\le k\le\Delta<\infty$ μ-a.e. on Ω,*

$$\begin{aligned}|D_w(h,k)| &\le \frac{1}{2}(\Delta-\delta)\int_\Omega w\left|h-\int_\Omega whd\mu\right|d\mu \\ &\le \frac{1}{2}(\Delta-\delta)\left[\int_\Omega wh^2d\mu-\left(\int_\Omega whd\mu\right)^2\right]^{\frac{1}{2}} \\ &\le \frac{1}{4}(\Delta-\delta)(\Gamma-\gamma),\end{aligned} \tag{1.4}$$

provided that $h,\ k,\ hk\in L_w(\Omega,\mu)$. The constants $\frac{1}{2}$ and $\frac{1}{4}$ are best possible.

Consider a probability density function w on $[a,b]$, i.e., $w\ge 0$ a.e. on $[a,b]$ with $\int_a^b w(t)\,dt=1$, and the weighted Čebyšev functional for functions defined on a finite interval $[a,b]$,

$$D_w(h,k):=\int_a^b h(t)\,k(t)\,w(t)\,dt-\int_a^b h(t)\,w(t)\,dt\int_a^b k(t)\,w(t)\,dt.$$

From (1.4) we get

$$\begin{aligned}|D_w(h,k)| &\le \frac{1}{2}(\Delta-\delta)\int_a^b w(t)\left|h(t)-\int_a^b h(s)\,w(s)\,ds\right|dt \\ &\le \frac{1}{2}(\Delta-\delta)\left[\int_a^b w(s)\,h^2(s)\,ds-\left(\int_a^b h(s)\,w(s)\,ds\right)^2\right]^{\frac{1}{2}} \\ &\le \frac{1}{4}(\Delta-\delta)(\Gamma-\gamma),\end{aligned} \tag{1.5}$$

provided that $h,\ k:[a,b]\to\mathbb{R}$ are measurable functions and so that $-\infty<\gamma\le h\le\Gamma<\infty$, $-\infty<\delta\le k\le\Delta<\infty$ a.e. on $[a,b]$, and $h,\ k,\ hk\in L_w[a,b]$.

For classical and more recent upper bounds related to the Čebyšev functional see [4], [8]-[13] and [15].

In order to extend these results to p-Schatten norms we need the following preparations.

Let $(H;\langle .,.\rangle)$ be a complex Hilbert space and $\mathcal{B}(H)$ the Banach algebra of all bounded linear operators on H. If $\{e_i\}_{i\in I}$ an orthonormal basis of H, we say that

$A\in\mathcal{B}(H)$ is of *trace class* if

$$\|A\|_1:=\sum_{i\in I}\langle |A|e_i,e_i\rangle<\infty. \tag{1.6}$$

The definition of $\|A\|_1$ does not depend on the choice of the orthonormal basis $\{e_i\}_{i\in I}$. We denote by $\mathcal{B}_1(H)$ the set of trace class operators in $\mathcal{B}(H)$.

We define the *trace* of a trace class operator $A\in\mathcal{B}_1(H)$ to be

$$\operatorname{tr}(A):=\sum_{i\in I}\langle Ae_i,e_i\rangle, \tag{1.7}$$

where $\{e_i\}_{i\in I}$ an orthonormal basis of H. Note that this coincides with the usual definition of the trace if H is finite-dimensional. We observe that the series (1.7) converges absolutely and it is independent from the choice of basis.

The following result collects some properties of the trace:

Theorem 2. *We have:*

(i) If $A\in\mathcal{B}_1(H)$ *then* $A^*\in\mathcal{B}_1(H)$ *and*

$$\operatorname{tr}(A^*)=\overline{\operatorname{tr}(A)}; \tag{1.8}$$

(ii) If $A\in\mathcal{B}_1(H)$ *and* $T\in\mathcal{B}(H)$, *then* $AT,\ TA\in\mathcal{B}_1(H)$ *and*

$$\operatorname{tr}(AT)=\operatorname{tr}(TA)\ \text{ and }\ |\operatorname{tr}(AT)|\le\|A\|_1\|T\|; \tag{1.9}$$

(iii) $\operatorname{tr}(\cdot)$ *is a bounded linear functional on* $\mathcal{B}_1(H)$ *with* $\|\operatorname{tr}\|=1$;

(iv) If $A,\ B\in\mathcal{B}_2(H)$ *then* $AB,\ BA\in\mathcal{B}_1(H)$ *and* $\operatorname{tr}(AB)=\operatorname{tr}(BA)$;

(v) $\mathcal{B}_{fin}(H)$ *is a dense subspace of* $\mathcal{B}_1(H)$.

An operator $A\in\mathcal{B}(H)$ is said to belong to the *von Neumann-Schatten class* $\mathcal{B}_p(H)$, $1\le p<\infty$ if the *p-Schatten norm* is finite [17, p. 60-64]

$$\|A\|_p:=\left[\operatorname{tr}(|A|^p)\right]^{1/p}<\infty.$$

For $1<p<q<\infty$ we have that

$$\mathcal{B}_1(H)\subset\mathcal{B}_p(H)\subset\mathcal{B}_q(H)\subset\mathcal{B}(H) \tag{1.10}$$

and

$$\|A\|_1\ge\|A\|_p\ge\|A\|_q\ge\|A\|. \tag{1.11}$$

For $p\ge1$ the functional $\|\cdot\|_p$ is a *norm* on the $*$-ideal $\mathcal{B}_p(H)$ and $\left(\mathcal{B}_p(H),\|\cdot\|_p\right)$ is a Banach space.

Also, see for instance [17, p. 60-64],

$$\|A\|_p=\|A^*\|_p,\ A\in\mathcal{B}_p(H) \tag{1.12}$$

$$\|AB\|_p\le\|A\|_p\|B\|_p,\ A,B\in\mathcal{B}_p(H) \tag{1.13}$$

and

$$\|AB\|_p\le\|A\|_p\|B\|,\ \|BA\|_p\le\|B\|\|A\|_p,\ A\in\mathcal{B}_p(H),\ B\in\mathcal{B}(H). \tag{1.14}$$

This implies that

$$\|CAB\|_p\le\|C\|\|A\|_p\|B\|,\ A\in\mathcal{B}_p(H),\ B,\ C\in\mathcal{B}(H). \tag{1.15}$$

In terms of *p-Schatten norm* we have the *Hölder inequality* for $p,\ q>1$ with $\frac{1}{p}+\frac{1}{q}=1$

$$(|\operatorname{tr}(AB)|\leq)\ \|AB\|_1\leq\|A\|_p\,\|B\|_q\,,\ A\in\mathcal{B}_p(H)\,,\ B\in\mathcal{B}_q(H)\,. \tag{1.16}$$

For the theory of trace functionals and their applications the reader is referred to [16] and [17].

For some classical trace inequalities see [6], [7], and [14], which are continuations of the work of Bellman [1].

2. Main Results

Consider a probability density function w on $[a,b]$, i.e., $w\geq 0$ a.e. on $[a,b]$ with $\int_a^b w(t)\,dt=1$, and the weighted Čebyšev functional for operator-valued functions $A,\ B:[a,b]\to B(H)$ defined on a finite interval $[a,b]$,

$$D_w(A,B):=\int_a^b w(t)\,A(t)\,B(t)\,dt-\int_a^b w(t)\,A(t)\,dt\int_a^b w(t)\,B(t)\,dt.$$

The first main results is as follows:

Theorem 3. *Assume that $w:[a,b]\to[0,\infty)$ is integrable with $\int_a^b w(t)\,dt=1$ and for $p,\ q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, $A:[a,b]\to B_p(H)$, $B:[a,b]\to B_q(H)$ are strongly differentiable with $\sup_{u\in(a,b)}\|A'(u)\|_p<\infty$, then*

$$\begin{aligned}
&\|D_w(A,B)\|_1 \\
&\leq\sup_{u\in(a,b)}\|A'(u)\|_p\,D_w\left(\ell,\int_a^{\cdot}\|B'(u)\|_q\,du\right)\\
&\leq\frac{1}{2}\int_a^b w(t)\left|t-\int_a^b sw(s)\,ds\right|dt\sup_{u\in(a,b)}\|A'(u)\|_p\int_a^b\|B'(u)\|_q\,du\\
&\leq\frac{1}{2}\left[\int_a^b s^2w(s)\,ds-\left(\int_a^b sw(s)\,ds\right)^2\right]^{\frac{1}{2}}\sup_{u\in(a,b)}\|A'(u)\|_p\int_a^b\|B'(u)\|_q\,du\\
&\leq\frac{1}{4}\sup_{u\in(a,b)}\|A'(u)\|_p\int_a^b\|B'(u)\|_q\,du,
\end{aligned} \tag{2.1}$$

where $\ell(t)=t,\ t\in[a,b]$.

In particular, for $p=q=2$ *we get*

$$\begin{aligned}(2.2)\quad & \left\|D_w(A,B)\right\|_1 \\ & \le \sup_{u\in(a,b)}\left\|A'(u)\right\|_2 D_w\left(\ell,\int_a^{\cdot}\left\|B'(u)\right\|_2 du\right) \\ & \le \frac{1}{2}\int_a^b w(t)\left|t-\int_a^b sw(s)\,ds\right| dt \sup_{u\in(a,b)}\left\|A'(u)\right\|_2\int_a^b\left\|B'(u)\right\|_2 du \\ & \le \frac{1}{2}\left[\int_a^b s^2 w(s)\,ds-\left(\int_a^b sw(s)\,ds\right)^2\right]^{\frac{1}{2}}\sup_{u\in(a,b)}\left\|A'(u)\right\|_2\int_a^b\left\|B'(u)\right\|_2 du \\ & \le \frac{1}{4}\sup_{u\in(a,b)}\left\|A'(u)\right\|_2\int_a^b\left\|B'(u)\right\|_2 du.\end{aligned}$$

Proof. Observe that, by the use of integral's properties,

$$\begin{aligned}&\int_a^b\int_a^b w(t)\,w(s)\left[A(t)-A(s)\right]\left[B(t)-B(s)\right]dtds \\ &=\int_a^b\int_a^b w(t)\,w(s)\left(A(t)\,B(t)-A(s)\,B(t)-A(t)\,B(s)+A(s)\,B(s)\right)dtds \\ &=\int_a^b w(s)\,ds\int_a^b A(t)\,B(t)\,dt-\int_a^b w(s)\,A(s)\,ds\int_a^b w(t)\,B(t)\,dt \\ &-\int_a^b w(t)\,A(t)\,dt\int_a^b w(s)\,B(s)\,ds+\int_a^b w(t)\,dt\int_a^b A(s)\,B(s)\,ds=2D_w(A,B),\end{aligned}$$

which give the weighted Korkine's identity for functions with operator values

$$D_w(A,B)=\frac{1}{2}\int_a^b\int_a^b w(t)\,w(s)\left[A(t)-A(s)\right]\left[B(t)-B(s)\right]dtds.$$

For Korkine's classical identity for real-valued functions, see [13, p. 242].

If we take the 1-norm, use the integral's properties and the Hölder's inequality (1.16), then we get

$$\begin{aligned}(2.3)\quad \left\|D_w(A,B)\right\|_1 &\le \frac{1}{2}\int_a^b\int_a^b w(t)\,w(s)\left\|\left[A(t)-A(s)\right]\left[B(t)-B(s)\right]\right\|_1 dtds \\ &\le \frac{1}{2}\int_a^b\int_a^b w(t)\,w(s)\left\|A(t)-A(s)\right\|_p\left\|B(t)-B(s)\right\|_q dtds.\end{aligned}$$

Observe that for $s,\,t\in[a,b]$

$$A(t)-A(s)=\int_s^t A'(u)\,du,\;\;B(t)-B(s)=\int_s^t B'(u)\,du,$$

which implies that

$$\begin{aligned}\|A(t)-A(s)\|_p\|B(t)-B(s)\|_q &= \left\|\int_s^t A'(u)\,du\right\|_p\left\|\int_s^t B'(u)\,du\right\|_q \\ &\le \left|\int_s^t \|A'(u)\|_p\,du\right|\left|\int_s^t \|B'(u)\|_q\,du\right| \\ &\le \sup_{u\in(a,b)}\|A'(u)\|_p\,|t-s|\left|\int_s^t \|B'(u)\|_q\,du\right| \\ &= \sup_{u\in(a,b)}\|A'(u)\|_p\,(t-s)\int_s^t \|B'(u)\|_q\,du,\end{aligned}$$

for all $s,\ t\in[a,b]$.

By (2.3) we get

$$\begin{aligned}&\|D_w(A,B)\|_1 \\ &\le \sup_{u\in(a,b)}\|A'(u)\|_p\,\frac{1}{2}\int_a^b\int_a^b w(t)\,w(s)\,(t-s)\left(\int_s^t\|B'(u)\|_q\,du\right)dtds.\end{aligned}\tag{2.4}$$

Since

$$(t-s)\left(\int_s^t\|B'(u)\|_q\,du\right)=(t-s)\left(\int_a^t\|B'(u)\|_q\,du-\int_a^s\|B'(u)\|_q\,du\right),$$

hence by Korkine's identity for real valued functions $A(t)=\ell(t)$ and $\int_a^t\|B'(u)\|_q\,du$, we have

$$\begin{aligned}&\frac{1}{2}\int_a^b\int_a^b w(t)\,w(s)\,(t-s)\left(\int_a^t\|B'(u)\|_q\,du-\int_a^s\|B'(u)\|_q\,du\right)dtds \\ &= D_w\left(\ell,\int_a^{\cdot}\|B'(u)\|_q\,du\right).\end{aligned}\tag{2.5}$$

By utilising (2.4) and (2.5), we deduce the first inequality in (2.1).

Observe that

$$0\le\int_a^t\|B'(u)\|_q\,du\le\int_a^b\|B'(u)\|_q\,du$$

for all $t\in[a,b]$, then by (1.5) for the functions $h(t)=\ell(t)$ and $k(t)=\int_a^t|B'(u)|\,du$, $t\in[a,b]$, we get

$$\begin{aligned}&\left|D_w\left(\ell,\int_a^{\cdot}\|B'(u)\|_q\,du\right)\right| \\ &\le\frac{1}{2}\left(\int_a^b\|B'(u)\|_q\,du\right)\int_a^b w(t)\left|t-\int_a^b sw(s)\,ds\right|dt \\ &\le\frac{1}{2}\left(\int_a^b\|B'(u)\|_q\,du\right)\left[\int_a^b s^2w(s)\,ds-\left(\int_a^b sw(s)\,ds\right)^2\right]^{\frac{1}{2}} \\ &\le\frac{1}{4}(b-a)\left(\int_a^b\|B'(u)\|_q\,du\right),\end{aligned}\tag{2.6}$$

which proves the last part of (2.1). □

Remark 1. *We consider the uniform distribution* $w_0(t) = \frac{1}{b-a}$ *for which we have*

$$D_{w_0}(A,B) := \frac{1}{b-a}\int_a^b A(t)B(t)\,dt - \frac{1}{b-a}\int_a^b A(t)\,dt \frac{1}{b-a}\int_a^b B(t)\,dt.$$

Since

$$\int_a^b w_0(t)\left|t - \int_a^b s w_0(s)\,ds\right| dt = \frac{1}{b-a}\int_a^b \left|t - \frac{1}{b-a}\int_a^b s ds\right| dt$$
$$= \frac{1}{b-a}\int_a^b \left|t - \frac{a+b}{2}\right| dt = \frac{1}{4}(b-a),$$

hence by (2.1) we get

$$\left\|D_{w_0}(A,B)\right\|_1 \le \sup_{u\in(a,b)} \left\|A'(u)\right\|_p D_{w_0}\left(\ell, \int_a^{\cdot} \left\|B'(u)\right\|_q du\right) \le \frac{1}{8}(b-a)\sup_{u\in(a,b)} \left\|A'(u)\right\|_p \int_a^b \left\|B'(u)\right\|_q du \tag{2.7}$$

for $p,\ q > 1$ *with* $\frac{1}{p}+\frac{1}{q} = 1,$ *and* $A : [a,b] \to B_p(H),\ B : [a,b] \to B_q(H)$ *strongly differentiable with* $\sup_{u\in(a,b)} \left\|A'(u)\right\|_p < \infty$.

In particular, for $p = q = 2$ *we derive*

$$\left\|D_{w_0}(A,B)\right\|_1 \le \sup_{u\in(a,b)} \left\|A'(u)\right\|_2 D_{w_0}\left(\ell, \int_a^{\cdot} \left\|B'(u)\right\|_2 du\right) \le \frac{1}{8}(b-a)\sup_{u\in(a,b)} \left\|A'(u)\right\|_2 \int_a^b \left\|B'(u)\right\|_2 du. \tag{2.8}$$

Theorem 4. *Assume that* $w : [a,b] \to [0,\infty)$ *is integrable with* $\int_a^b w(t)\,dt = 1$ *and for* $p \ge 1,\ A,\ B : [a,b] \to B_p(H)$ *are strongly differentiable with* $\sup_{u\in(a,b)} \left\|A'(u)\right\|_p < \infty$, *then*

$$\begin{aligned} &\left\|D_w(A,B)\right\|_p \\ &\le \sup_{u\in(a,b)} \left\|A'(u)\right\|_p D_w\left(\ell, \int_a^{\cdot} \left\|B'(u)\right\|_p du\right) \\ &\le \frac{1}{2}\int_a^b w(t)\left|t - \int_a^b s w(s)\,ds\right| dt \sup_{u\in(a,b)} \left\|A'(u)\right\|_p \int_a^b \left\|B'(u)\right\|_p du \\ &\le \frac{1}{2}\left[\int_a^b s^2 w(s)\,ds - \left(\int_a^b s w(s)\,ds\right)^2\right]^{\frac{1}{2}} \sup_{u\in(a,b)} \left\|A'(u)\right\|_p \int_a^b \left\|B'(u)\right\|_p du \\ &\le \frac{1}{4}\sup_{u\in(a,b)} \left\|A'(u)\right\|_p \int_a^b \left\|B'(u)\right\|_p du. \end{aligned} \tag{2.9}$$

Proof. If we take the p-norm, use the integral's properties and the inequality (1.13), then we get

$$\begin{aligned} (2.10)\quad \left\| D_w(A,B) \right\|_p &\leq \frac{1}{2}\int_a^b \int_a^b w(t)\,w(s) \left\| [A(t)-A(s)]\,[B(t)-B(s)] \right\|_p dtds \\ &\leq \frac{1}{2}\int_a^b \int_a^b w(t)\,w(s) \left\| A(t)-A(s) \right\|_p \left\| B(t)-B(s) \right\|_p dtds. \end{aligned}$$

Since

$$\begin{aligned} \left\| A(t)-A(s) \right\|_p \left\| B(t)-B(s) \right\|_p &= \left\| \int_s^t A'(u)\,du \right\|_p \left\| \int_s^t B'(u)\,du \right\|_p \\ &\leq \left| \int_s^t \left\| A'(u) \right\|_p du \right| \left| \int_s^t \left\| B'(u) \right\|_p du \right| \\ &\leq \sup_{u\in(a,b)} \left\| A'(u) \right\|_p |t-s| \left| \int_s^t \left\| B'(u) \right\|_p du \right| \\ &= \sup_{u\in(a,b)} \left\| A'(u) \right\|_p (t-s) \int_s^t \left\| B'(u) \right\|_p du, \end{aligned}$$

hence by (2.10) we get

$$\begin{aligned} &\left\| D_w(A,B) \right\|_p \\ &\leq \sup_{u\in(a,b)} \left\| A'(u) \right\|_p \frac{1}{2}\int_a^b \int_a^b w(t)\,w(s)\,(t-s) \left(\int_s^t \left\| B'(u) \right\|_p du \right) dtds. \end{aligned}$$

Further, by utilising a similar argument to the one in the proof of Theorem 3 we derive (2.9). □

Remark 2. *Let A, $B : [a,b] \to B_p(H)$, $p \geq 1$, be strongly differentiable with $\sup_{u\in(a,b)} \left\| A'(u) \right\|_p < \infty$, then*

$$\begin{aligned} (2.11)\quad \left\| D_{w_0}(A,B) \right\|_p &\leq \sup_{u\in(a,b)} \left\| A'(u) \right\|_p D_{w_0}\left(\ell, \int_a^{\cdot} \left\| B'(u) \right\|_p du \right) \\ &\leq \frac{1}{8}(b-a) \left\| A'(u) \right\|_p \int_a^b \left\| B'(u) \right\|_p du. \end{aligned}$$

3. The Case of Infinite Intervals

Similar results may be stated for the probability distributions that are supported on the whole axis $\mathbb{R} = (-\infty,\infty)$. Namely, if $I = (-\infty,\infty)$ and $w(s) > 0$ for $s \in \mathbb{R}$ with $\int_{-\infty}^{\infty} w(s)\,ds = 1$, i.e. w is a probability density function on $(-\infty,\infty)$, then for A, $B : (-\infty,\infty) \to B(H)$ we can consider the functional

$$D_{w,\mathbb{R}}(A,B) := \int_{-\infty}^{\infty} w(t)\,A(t)\,B(t)\,dt - \int_{-\infty}^{\infty} w(t)\,A(t)\,dt \int_{-\infty}^{\infty} w(t)\,B(t)\,dt.$$

From (2.1) we then have

$$\begin{aligned}
(3.1)\quad & \left\| D_w(A,B)\right\|_1 \\
&\le \sup_{u\in(-\infty,\infty)} \left\| A'(u)\right\|_p D_w\left(\ell, \int_{-\infty}^{\cdot} \left\| B'(u)\right\|_q du\right) \\
&\le \frac{1}{2}\int_{-\infty}^{\infty} w(t)\left| t-\int_{-\infty}^{\infty} sw(s)\,ds\right| dt \sup_{u\in(-\infty,\infty)} \left\| A'(u)\right\|_p \int_{-\infty}^{\infty} \left\| B'(u)\right\|_q du \\
&\le \frac{1}{2}\left[\int_{-\infty}^{\infty} s^2 w(s)\,ds-\left(\int_{-\infty}^{\infty} sw(s)\,ds\right)^2\right]^{\frac{1}{2}} \\
&\times \sup_{u\in(-\infty,\infty)} \left\| A'(u)\right\|_p \int_{-\infty}^{\infty} \left\| B'(u)\right\|_q du \\
&\le \frac{1}{4}\sup_{u\in(-\infty,\infty)} \left\| A'(u)\right\|_p \int_{-\infty}^{\infty} \left\| B'(u)\right\|_q du,
\end{aligned}$$

with the assumptions that for $p,\ q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, $A:(-\infty,\infty)\to B_p(H)$, $B:(-\infty,\infty)\to B_q(H)$ are strongly differentiable, $\sup_{u\in(-\infty,\infty)}\left\| A'(u)\right\|_p<\infty$ and $\int_{-\infty}^{\infty}\left\| B'(u)\right\|_q du<\infty$.

For $p=q=2$ we derive

$$\begin{aligned}
(3.2)\quad & \left\| D_w(A,B)\right\|_1 \le \sup_{u\in(-\infty,\infty)} \left\| A'(u)\right\|_2 D_w\left(\ell, \int_{-\infty}^{\cdot} \left\| B'(u)\right\|_2 du\right) \\
&\le \frac{1}{2}\int_{-\infty}^{\infty} w(t)\left| t-\int_{-\infty}^{\infty} sw(s)\,ds\right| dt \sup_{u\in(-\infty,\infty)} \left\| A'(u)\right\|_2 \int_{-\infty}^{\infty} \left\| B'(u)\right\|_2 du \\
&\le \frac{1}{2}\left[\int_{-\infty}^{\infty} s^2 w(s)\,ds-\left(\int_{-\infty}^{\infty} sw(s)\,ds\right)^2\right]^{\frac{1}{2}} \\
&\times \sup_{u\in(-\infty,\infty)} \left\| A'(u)\right\|_2 \int_{-\infty}^{\infty} \left\| B'(u)\right\|_2 du \\
&\le \frac{1}{4}\sup_{u\in(-\infty,\infty)} \left\| A'(u)\right\|_2 \int_{-\infty}^{\infty} \left\| B'(u)\right\|_2 du,
\end{aligned}$$

with the assumptions that $A,\ B:(-\infty,\infty)\to B_2(H)$ are strongly differentiable, $\sup_{u\in(-\infty,\infty)}\left\| A'(u)\right\|_2<\infty$ and $\int_{-\infty}^{\infty}\left\| B'(u)\right\|_2 du<\infty$.

The probability density of the *normal distribution* on $(-\infty,\infty)$ is

$$w_{\mu,\sigma^2}(x):=\frac{1}{\sqrt{2\pi}\sigma}\exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right),\ x\in\mathbb{R},$$

where μ is the *mean* or *expectation* of the distribution (and also its *median* and *mode*), σ is the *standard deviation*, and σ^2 is the *variance*.

The cumulative distribution function is

$$W_{\mu,\sigma^2}(x)=\frac{1}{2}+\frac{1}{2}\operatorname{erf}\left(\frac{x-\mu}{\sigma\sqrt{2}}\right),$$

where the *error function* erf is defined by

$$\operatorname{erf}(x)=\frac{2}{\sqrt{\pi}}\int_0^x \exp\left(-t^2\right)dt.$$

Consider the functional

$$\begin{aligned}&D_{N,\sigma,\mu}(A,B)\\&:=\frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty}\exp\left(-\frac{(t-\mu)^2}{2\sigma^2}\right)A(t)B(t)\,dt\\&-\frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty}\exp\left(-\frac{(t-\mu)^2}{2\sigma^2}\right)A(t)\,dt\frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty}\exp\left(-\frac{(t-\mu)^2}{2\sigma^2}\right)B(t)\,dt\end{aligned}$$

with the parameters μ and σ as above.

Then from (3.1) we get

$$\begin{aligned}\left\|D_{N,\sigma,\mu}(A,B)\right\|_1&\le\sup_{u\in(-\infty,\infty)}\left\|A'(u)\right\|_p D_{N,\sigma,\mu}\left(\ell,\int_{-\infty}^{\cdot}\left\|B'(u)\right\|_q du\right)\\&\le\frac{1}{2\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty}\exp\left(-\frac{(t-\mu)^2}{2\sigma^2}\right)|t-\mu|\,dt\\&\times\sup_{u\in(-\infty,\infty)}\left\|A'(u)\right\|_p\int_{-\infty}^{\infty}\left\|B'(u)\right\|_q du\\&\le\frac{1}{2}\sigma\sup_{u\in(-\infty,\infty)}\left\|A'(u)\right\|_p\int_{-\infty}^{\infty}\left\|B'(u)\right\|_q du\end{aligned}\tag{3.3}$$

with the assumptions that for $p,\ q>1$ with $\frac{1}{p}+\frac{1}{q}=1,\ A:(-\infty,\infty)\to B_p(H)$, $B:(-\infty,\infty)\to B_q(H)$ are strongly differentiable, $\sup_{u\in(-\infty,\infty)}\left\|A'(u)\right\|_p<\infty$ and $\int_{-\infty}^{\infty}\left\|B'(u)\right\|_q du<\infty$.

In particular, for $p=q=2$ we get

$$\begin{aligned}\left\|D_{N,\sigma,\mu}(A,B)\right\|_1&\le\sup_{u\in(-\infty,\infty)}\left\|A'(u)\right\|_2 D_{N,\sigma,\mu}\left(\ell,\int_{-\infty}^{\cdot}\left\|B'(u)\right\|_2 du\right)\\&\le\frac{1}{2\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty}\exp\left(-\frac{(t-\mu)^2}{2\sigma^2}\right)|t-\mu|\,dt\\&\times\sup_{u\in(-\infty,\infty)}\left\|A'(u)\right\|_2\int_{-\infty}^{\infty}\left\|B'(u)\right\|_2 du\\&\le\frac{1}{2}\sigma\sup_{u\in(-\infty,\infty)}\left\|A'(u)\right\|_2\int_{-\infty}^{\infty}\left\|B'(u)\right\|_2 du\end{aligned}\tag{3.4}$$

with the assumptions that $A,\ B:(-\infty,\infty)\to B_2(H)$ are strongly differentiable, $\sup_{u\in(-\infty,\infty)}\left\|A'(u)\right\|_2<\infty$ and $\int_{-\infty}^{\infty}\left\|B'(u)\right\|_2 du<\infty$.

4. Inequalities for Operator Monotone Functions

A real valued continuous function h on $[0,\infty)$ is said to be operator monotone if $h(A)\ge h(B)$ holds for any $A\ge B\ge 0$.

We have the following representation of operator monotone functions, see for instance [2, p. 144-145]:

Theorem 5. *A function $h:[0,\infty)\to\mathbb{R}$ is operator monotone in $[0,\infty)$ if and only if it has the representation*

$$h(t)=h(0)+bt+\int_0^\infty \frac{t\lambda}{t+\lambda}d\mu(\lambda), \tag{4.1}$$

where $b\geq 0$ and a positive measure μ on $[0,\infty)$ such that

$$\int_0^\infty \frac{\lambda}{1+\lambda}d\mu(\lambda)<\infty. \tag{4.2}$$

We have the following representation result:

Lemma 1. *Let $h:[0,\infty)\to\mathbb{R}$ be operator monotone in $[0,\infty)$. Assume that $U\geq 0$, then for all selfadjoint operators V we have*

$$Dh(U)(V)=bV+\int_0^\infty \lambda^2\left[(\lambda+U)^{-1}V(\lambda+U)^{-1}\right]d\mu(\lambda). \tag{4.3}$$

Proof. From (4.1) we get

$$h(t)=h(0)+bt+\int_0^\infty\left(\lambda-\frac{\lambda^2}{t+\lambda}\right)d\mu(\lambda).$$

Assume that $U\geq 0$, then for all selfadjoint operator V we have, by the representation of h and for t in a small open interval around 0, that

$$\begin{aligned}
&h(U+tV)-h(U)\\
&=btV+\int_0^\infty\left(\lambda-\lambda^2(U+tV+\lambda)^{-1}\right)d\mu(\lambda)-\int_0^\infty\left(\lambda-\lambda^2(U+\lambda)^{-1}\right)d\mu(\lambda)\\
&=btV+\int_0^\infty\lambda^2\left[(\lambda+U)^{-1}-(\lambda+U+tV)^{-1}\right]d\mu(\lambda)\\
&=btV+\int_0^\infty\lambda^2\left[(\lambda+U)^{-1}(\lambda+U+tV-\lambda-U)(\lambda+U+tV)^{-1}\right]d\mu(\lambda)\\
&=btV+t\int_0^\infty\lambda^2\left[(\lambda+U)^{-1}V(\lambda+U+tV)^{-1}\right]d\mu(\lambda).
\end{aligned}$$

Dividing by $t\neq 0$, we get

$$\frac{h(U+tV)-h(U)}{t}=bV+\int_0^\infty\lambda^2\left[(\lambda+U)^{-1}V(\lambda+U+tV)^{-1}\right]d\mu(\lambda)$$

and by taking the limit over $t\to 0$, we get

$$Dh(U)(V)=bV+\int_0^\infty\lambda^2\left[(\lambda+U)^{-1}V(\lambda+U)^{-1}\right]d\mu(\lambda)$$

for all selfadjoint operator V we have (4.3). □

Lemma 2. *Let $h:[0,\infty)\to\mathbb{R}$ be operator monotone in $[0,\infty)$. Assume that $U\geq u>0$, then for all selfadjoint operators $V\in B_p(H)$, $p\geq 1$ we have*

$$\|Dh(U)(V)\|_p\leq h'(u)\|V\|_p. \tag{4.4}$$

Proof. From (4.3) and using (1.15) we get

$$\|Dh(U)(V)-bV\|_p \le \int_0^\infty \lambda^2 \left\|(\lambda+U)^{-1} V (\lambda+U)^{-1}\right\|_p d\mu(\lambda) \le \|V\|_p \int_0^\infty \lambda^2 \left\|(\lambda+U)^{-1}\right\|^2 d\mu(\lambda). \tag{4.5}$$

Observe that $\lambda+U \ge \lambda+u>0$ for $\lambda\in[0,\infty)$. Then $0<(\lambda+U)^{-1}\le(\lambda+u)^{-1}$, which implies that $\left\|(\lambda+U)^{-1}\right\| \le (\lambda+u)^{-1}$, namely $\left\|(\lambda+U)^{-1}\right\|^2 \le (\lambda+u)^{-2}$ for $\lambda\in[0,\infty)$.

Therefore by (4.5) we get

$$\|Dh(U)(V)-bV\|_p \le \|V\|_p \int_0^\infty \lambda^2 (\lambda+u)^{-2} d\mu(\lambda). \tag{4.6}$$

If we take the derivative over t in (4.1) then we have

$$h'(t) = b+\int_0^\infty \frac{\lambda(t+\lambda)-\lambda t}{(t+\lambda)^2} d\mu(\lambda) = b+\int_0^\infty \frac{\lambda^2}{(t+\lambda)^2} d\mu(\lambda) \tag{4.7}$$

for $t>0$.

From (4.7) we get

$$\int_0^\infty \lambda^2(\lambda+u)^{-2} d\mu(\lambda) = h'(u)-b,$$

and by (4.6) we derive

$$\|Dh(U)(V)-bV\|_p \le \|V\|_p h'(u) - b\|V\|_p.$$

Finally, by the triangle inequality and by the fact that $b\ge 0$, we obtain that

$$\|Dh(U)(V)\|_p - b\|V\|_p \le \|Dh(U)(V)-bV\|_p,$$

which proves the desired result (4.4). □

For a continuous function h on $(0,\infty)$ and $C,\ E>0$ we consider the auxiliary function $h_{C,E}:[0,1]\to\mathbb{R}$ defined by

$$h_{C,E}(t) := h((1-t)C+tE),\ t\in[0,1].$$

We have the following representations of the derivatives:

Lemma 3. *Assume that the operator function generated by h is Fréchet differentiable in each $C\ge 0$, then for $E\ge 0$ we have that $h_{C,E}$ is differentiable on $[0,1]$ and*

$$h'_{C,E}(t) = D(h)((1-t)C+tE)(E-C) \tag{4.8}$$

for $t\in[0,1]$, where in 0 and 1 the derivatives are the right and left derivatives.

Proof. We prove only for the interior points $t\in(0,1)$. Let h be in a small interval around 0 such that $t+h\in(0,1)$. Then for $h\ne 0$,

$$\frac{h_{C,E}(t+h)-h(t)}{h} = \frac{h((1-(t+h))C+(t+h)E)-h((1-t)C+tE)}{h} = \frac{h((1-t)C+tE+h(E-C))-h((1-t)C+tE)}{h}$$

and by taking the limit over $h \to 0$, we get

$$\begin{aligned} h'_{C,E}(t) &= \lim_{h\to 0} \frac{h_{C,E}(t+h) - h(t)}{h} \\ &= \lim_{h\to 0} \left[\frac{h((1-t)C + tE + h(E-C)) - h((1-t)C+tE)}{h} \right] \\ &= D(h)((1-t)C+tE)(E-C), \end{aligned}$$

which proves (4.8). □

Corollary 1. *Let $h:[0,\infty)\to\mathbb{R}$ be operator monotone in $[0,\infty)$. Then for all $C\geq c>0$, $E\geq d>0$ with $C,E\in B_p(H)$, $p\geq 1$ we have*

$$\begin{aligned} \left\| h'_{C,E}(t) \right\|_p &= \| D(h)((1-t)C+tE)(E-C)\|_p \\ &\leq h'((1-t)c+td)\|E-C\|_p \end{aligned} \tag{4.9}$$

for all $t\in[0,1]$.

The proof follows by Lemma 2 and Lemma 3.

Consider a probability density function w on $[a,b]$, i.e., $w\geq 0$ a.e. on $[a,b]$ with $\int_a^b w(t)\,dt = 1$. For the operator monotone functions h, $k:[0,\infty)\to\mathbb{R}$ and two operators C, $E\geq 0$ we consider the *Čebyšev functional*

$$\begin{aligned} D_w(h,k,C,E) := &\int_0^1 w(t)\,h((1-t)C+tE)\,k((1-t)C+tE)\,dt \\ &- \int_0^1 w(t)\,h((1-t)C+tE)\,dt \int_0^1 w(t)\,k((1-t)C+tE)\,dt. \end{aligned}$$

For p, $q>1$ with $\frac{1}{p}+\frac{1}{q}=1$ assume that $C\geq c>0$, $E\geq d>0$ with C, $E\in B_p(H)\cap B_q(H)$. If we use Theorem 3 and Corollary 1 we have

$$\begin{aligned} &\|D_w(h,k,C,E)\|_1 \\ &\leq \frac{1}{2}\|E-C\|_p\|E-C\|_q \int_a^b w(t)\left| t - \int_a^b s w(s)\,ds \right| dt \\ &\times \sup_{t\in(a,b)} h'((1-t)c+td) \int_a^b k'((1-t)c+td)\,dt \\ &\leq \frac{1}{2}\|E-C\|_p\|E-C\|_q \left[\int_a^b s^2 w(s)\,ds - \left(\int_a^b s w(s)\,ds \right)^2 \right]^{\frac{1}{2}} \\ &\times \sup_{t\in(a,b)} h'((1-t)c+td) \int_a^b k'((1-t)c+td)\,dt \\ &\leq \frac{1}{4}\|E-C\|_p\|E-C\|_q \sup_{t\in(a,b)} h'((1-t)c+td) \int_a^b k'((1-t)c+td)\,dt, \end{aligned} \tag{4.10}$$

and, in particular,

$$\begin{aligned}
(4.11)\qquad & \left\| D_w\left(h,k,C,E\right)\right\|_1 \\
& \leq \frac{1}{2}\left\| E-C\right\|_2^2 \int_a^b w\left(t\right)\left| t-\int_a^b sw\left(s\right)ds\right| dt \\
& \times \sup_{t\in(a,b)} h'\left(\left(1-t\right)c+td\right)\int_a^b k'\left(\left(1-t\right)c+td\right)dt \\
& \leq \frac{1}{2}\left\| E-C\right\|_2^2 \left[\int_a^b s^2 w\left(s\right)ds-\left(\int_a^b sw\left(s\right)ds\right)^2\right]^{\frac{1}{2}} \\
& \times \sup_{t\in(a,b)} h'\left(\left(1-t\right)c+td\right)\int_a^b k'\left(\left(1-t\right)c+td\right)dt \\
& \leq \frac{1}{4}\left\| E-C\right\|_2^2 \sup_{t\in(a,b)} h'\left(\left(1-t\right)c+td\right)\int_a^b k'\left(\left(1-t\right)c+td\right)dt.
\end{aligned}$$

For $C\geq c>0,\ E\geq d>0$ with $C,\ E\in B_p\left(H\right),\ p\geq 1$, then by Theorem 4 we have

$$\begin{aligned}
(4.12)\qquad & \left\| D_w\left(h,k,C,E\right)\right\|_p \\
& \leq \frac{1}{2}\left\| E-C\right\|_p^2 \int_a^b w\left(t\right)\left| t-\int_a^b sw\left(s\right)ds\right| dt \\
& \times \sup_{u\in(a,b)} h'\left(\left(1-t\right)c+td\right)\int_a^b k'\left(\left(1-t\right)c+td\right)dt \\
& \leq \frac{1}{2}\left\| E-C\right\|_p^2 \left[\int_a^b s^2 w\left(s\right)ds-\left(\int_a^b sw\left(s\right)ds\right)^2\right]^{\frac{1}{2}} \\
& \times \sup_{u\in(a,b)} h'\left(\left(1-t\right)c+td\right)\int_a^b k'\left(\left(1-t\right)c+td\right)dt \\
& \leq \frac{1}{4}\left\| E-C\right\|_p^2 \sup_{u\in(a,b)} h'\left(\left(1-t\right)c+td\right)\int_a^b k'\left(\left(1-t\right)c+td\right)dt.
\end{aligned}$$

We consider the uniform distribution $w_0\left(t\right)=\frac{1}{b-a}$ for which we have

$$\begin{aligned}
& D\left(h,k,C,E\right) \\
& := D_{w_0}\left(h,k,C,E\right)=\frac{1}{b-a}\int_0^1 h\left(\left(1-t\right)C+tE\right)k\left(\left(1-t\right)C+tE\right)dt \\
& -\frac{1}{b-a}\int_0^1 h\left(\left(1-t\right)C+tE\right)dt\frac{1}{b-a}\int_0^1 k\left(\left(1-t\right)C+tE\right)dt.
\end{aligned}$$

By (4.10) we get

$$\left\| D\left(h,k,C,E\right)\right\|_1 \leq \frac{1}{8}\left(b-a\right)\left\| E-C\right\|_p \left\| E-C\right\|_q \times \sup_{t\in(a,b)} h'\left(\left(1-t\right)c+td\right) \int_a^b k'\left(\left(1-t\right)c+td\right)dt \tag{4.13}$$

for $C\geq c>0$, $E\geq d>0$ with C, $E\in B_p\left(H\right)\cap B_q\left(H\right)$, and in particular

$$\left\| D\left(h,k,C,E\right)\right\|_1 \leq \frac{1}{8}\left(b-a\right)\left\| E-C\right\|_p \left\| E-C\right\|_q \times \sup_{t\in(a,b)} h'\left(\left(1-t\right)c+td\right) \int_a^b k'\left(\left(1-t\right)c+td\right)dt. \tag{4.14}$$

From (4.12) we obtain

$$\left\| D\left(h,k,C,E\right)\right\|_p \leq \frac{1}{8}\left(b-a\right)\left\| E-C\right\|_p^2 \times \sup_{t\in(a,b)} h'\left(\left(1-t\right)c+td\right) \int_a^b k'\left(\left(1-t\right)c+td\right)dt \tag{4.15}$$

for $C\geq c>0$, $E\geq d>0$ with C, $E\in B_p\left(H\right)$, $p\geq 1$.

If we consider the operator monotone functions $h\left(t\right)=t^m$, $k\left(t\right)=t^n$ with m, $n\in\left(0,1\right)$ and put

$$D_w\left(m,n,C,E\right):=\int_0^1 w\left(t\right)\left(\left(1-t\right)C+tE\right)^{m+n}dt - \int_0^1 w\left(t\right)\left(\left(1-t\right)C+tE\right)^{m}dt \int_0^1 w\left(t\right)\left(\left(1-t\right)C+tE\right)^{n}dt,$$

then by (4.10) we derive

$$\begin{aligned} &\left\| D_w\left(m,n,C,E\right)\right\|_1 \\ &\leq \frac{1}{2}\left\| E-C\right\|_p \left\| E-C\right\|_q \int_a^b w\left(t\right)\left| t-\int_a^b sw\left(s\right)ds\right| dt \\ &\times \frac{m}{\min\left\{c^{1-m},d^{1-m}\right\}}\times\begin{cases} \frac{d^n-c^n}{d-c} & \text{if } d\neq c \\ \frac{n}{c^{1-n}} & \text{if } d=c \end{cases} \\ &\leq \frac{1}{2}\left\| E-C\right\|_p \left\| E-C\right\|_q \left[\int_a^b s^2w\left(s\right)ds-\left(\int_a^b sw\left(s\right)ds\right)^2\right]^{\frac{1}{2}} \\ &\times \frac{m}{\min\left\{c^{1-m},d^{1-m}\right\}}\times\begin{cases} \frac{d^n-c^n}{d-c} & \text{if } d\neq c \\ \frac{n}{c^{1-n}} & \text{if } d=c \end{cases} \\ &\leq \frac{1}{4}\left\| E-C\right\|_p \left\| E-C\right\|_q \times \frac{m}{\min\left\{c,d\right\}}\times\begin{cases} \frac{d^n-c^n}{d-c} & \text{if } d\neq c \\ \frac{n}{c^{1-n}} & \text{if } d=c, \end{cases} \end{aligned} \tag{4.16}$$

provided that $C\geq c>0$, $E\geq d>0$ with C, $E\in B_p\left(H\right)\cap B_q\left(H\right)$.

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