p-SCHATTEN NORM INTEGRAL INEQUALITIES FOR THE WEIGHTED ČEBYŠEV'S FUNCTIONAL

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Abstract. An operator $A \in \mathcal{B}(H)$ is said to belong to the *von Neumann-Schatten class* $\mathcal{B}_p(H)$, $1 \leq p < \infty$ if the *p-Schatten norm* is finite

$$\|A\|_p := [\operatorname{tr}(|A|^p)]^{1/p} < \infty.$$

In this paper we show among others that, if $w : [a,b] \to [0,\infty)$ is integrable with $\int_a^b w(t)\,dt = 1$ and for $p,\ q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, $A : [a,b] \to B_p(H)$, $B : [a,b] \to B_q(H)$ are strongly differentiable, then

$$\begin{aligned}
&\left\| \int_a^b w(t) A(t) B(t)\,dt - \int_a^b w(t) A(t)\,dt \int_a^b w(t) B(t)\,dt \right\|_1 \\
&\leq \frac{1}{2} \left(\int_a^b \|B'(u)\|_q\,du \right) \\
&\times \int_a^b w(t) \left| \int_a^t \|A'(u)\|_p\,du - \int_a^b \left(\int_a^s \|A'(u)\|_p\,du \right) w(s)\,ds \right| dt \\
&\leq \frac{1}{2} \left(\int_a^b \|B'(u)\|_q\,du \right) \int_a^b w(s) \left[\left(\int_a^s \|A'(u)\|_p\,du \right)^2 ds \right. \\
&\left. - \left(\int_a^b w(s) \left(\int_a^s \|A'(u)\|_p\,du \right) ds \right)^2 \right]^{1/2} \\
&\leq \frac{1}{4} \left(\int_a^b \|A'(u)\|_p\,du \right) \left(\int_a^b \|B'(u)\|_q\,du \right),
\end{aligned}$$

provided that $\int_a^b \|A'(u)\|_p\,du$, $\int_a^b \|B'(u)\|_q\,du < \infty$.

Some examples of interest for the operator monotone functions are also given.

1. Introduction

Let $(\Omega, \mathcal{A}, \mu)$ be a measurable space consisting of a set Ω, a σ-algebra $\mathcal{A}$ of parts of Ω and a countably additive and positive measure μ on $\mathcal{A}$ with values in $\mathbb{R} \cup \{\infty\}$.

For a μ-measurable function $w : \Omega \to \mathbb{R}$, with $w(x) \geq 0$ for μ-a.e. $x \in \Omega$, consider the Lebesgue space $L_w(\Omega,\mu) := \{h : \Omega \to \mathbb{R},\ h$ is μ-measurable and $\int_\Omega w(x)|h(x)|\,d\mu(x) < \infty\}$. Assume $\int_\Omega w(x)\,d\mu(x) = 1$. In order to simplify the notation for the integrals, we do not write the variable, namely, instead of $\int_\Omega w(x)\,d\mu(x)$ we simply write $\int_\Omega w d\mu$.

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If $h,\ k:\Omega\to\mathbb{R}$ are μ-measurable functions and $h,\ k,\ hk\in L_w(\Omega,\mu)$, then we may consider the weighted Čebyšev functional in the following form

$$D_w(h,k):=\int_\Omega whkd\mu-\int_\Omega whd\mu\int_\Omega wkd\mu. \tag{1.1}$$

The following result is known in the literature as the Grüss inequality, see for instance [5]:

$$|D_w(h,k)|\le\frac{1}{4}(\Gamma-\gamma)(\Delta-\delta), \tag{1.2}$$

provided

$$-\infty<\gamma\le h\le\Gamma<\infty,\ \ -\infty<\delta\le k\le\Delta<\infty \tag{1.3}$$

for μ-a.e. on Ω. The constant $\frac{1}{4}$ is sharp in the sense that it cannot be replaced by a smaller constant.

In [5] Cerone and Dragomir proved among others the following *refinement of Grüss' inequality:*

Theorem 1. *For $h,\ k:\Omega\to\mathbb{R}$, μ-measurable functions and so that $-\infty<\gamma\le h\le\Gamma<\infty$, $-\infty<\delta\le k\le\Delta<\infty$ μ-a.e. on Ω,*

$$\begin{aligned}|D_w(h,k)|&\le\frac{1}{2}(\Delta-\delta)\int_\Omega w\left|h-\int_\Omega whd\mu\right|d\mu \\ &\le\frac{1}{2}(\Delta-\delta)\left[\int_\Omega wh^2d\mu-\left(\int_\Omega whd\mu\right)^2\right]^{\frac{1}{2}} \\ &\le\frac{1}{4}(\Delta-\delta)(\Gamma-\gamma),\end{aligned} \tag{1.4}$$

provided that $h,\ k,\ hk\in L_w(\Omega,\mu)$. The constants $\frac{1}{2}$ and $\frac{1}{4}$ are best possible.

Consider a probability density function w on $[a,b]$, i.e., $w\ge0$ a.e. on $[a,b]$ with $\int_a^b w(t)\,dt=1$, and the weighted Čebyšev functional for functions defined on a finite interval $[a,b]$,

$$D_w(h,k):=\int_a^b h(t)\,k(t)\,w(t)\,dt-\int_a^b h(t)\,w(t)\,dt\int_a^b k(t)\,w(t)\,dt.$$

From (1.4) we get

$$\begin{aligned}|D_w(h,k)|&\le\frac{1}{2}(\Delta-\delta)\int_a^b w(t)\left|h(t)-\int_a^b h(s)\,w(s)\,ds\right|dt \\ &\le\frac{1}{2}(\Delta-\delta)\left[\int_a^b w(s)\,h^2(s)\,ds-\left(\int_a^b h(s)\,w(s)\,ds\right)^2\right]^{\frac{1}{2}} \\ &\le\frac{1}{4}(\Delta-\delta)(\Gamma-\gamma),\end{aligned} \tag{1.5}$$

provided that $h,\ k:[a,b]\to\mathbb{R}$ are measurable functions and so that $-\infty<\gamma\le h\le\Gamma<\infty$, $-\infty<\delta\le k\le\Delta<\infty$ a.e. on $[a,b]$, and $h,\ k,\ hk\in L_w[a,b]$.

For classical and more recent upper bounds related to the Čebyšev functional see [4], [8]-[13] and [15].

In order to extend these results to p-Schatten norms we need the following preparations.

Let $(H; \langle .,. \rangle)$ be a complex Hilbert space and $\mathcal{B}(H)$ the Banach algebra of all bounded linear operators on H. If $\{e_i\}_{i\in I}$ an orthonormal basis of H, we say that $A \in \mathcal{B}(H)$ is of *trace class* if

$$\|A\|_1 := \sum_{i\in I} \langle |A| e_i, e_i \rangle < \infty. \tag{1.6}$$

The definition of $\|A\|_1$ does not depend on the choice of the orthonormal basis $\{e_i\}_{i\in I}$. We denote by $\mathcal{B}_1(H)$ the set of trace class operators in $\mathcal{B}(H)$.

We define the *trace* of a trace class operator $A \in \mathcal{B}_1(H)$ to be

$$\operatorname{tr}(A) := \sum_{i\in I} \langle Ae_i, e_i \rangle, \tag{1.7}$$

where $\{e_i\}_{i\in I}$ an orthonormal basis of H. Note that this coincides with the usual definition of the trace if H is finite-dimensional. We observe that the series (1.7) converges absolutely and it is independent from the choice of basis.

The following result collects some properties of the trace:

Theorem 2. *We have:*

(i) If $A \in \mathcal{B}_1(H)$ *then* $A^* \in \mathcal{B}_1(H)$ *and*

$$\operatorname{tr}(A^*) = \overline{\operatorname{tr}(A)}; \tag{1.8}$$

(ii) If $A \in \mathcal{B}_1(H)$ *and* $T \in \mathcal{B}(H)$, *then* $AT, TA \in \mathcal{B}_1(H)$ *and*

$$\operatorname{tr}(AT) = \operatorname{tr}(TA) \text{ and } |\operatorname{tr}(AT)| \le \|A\|_1 \|T\|; \tag{1.9}$$

(iii) $\operatorname{tr}(\cdot)$ *is a bounded linear functional on* $\mathcal{B}_1(H)$ *with* $\|\operatorname{tr}\| = 1$;

(iv) If $A, B \in \mathcal{B}_2(H)$ *then* $AB, BA \in \mathcal{B}_1(H)$ *and* $\operatorname{tr}(AB) = \operatorname{tr}(BA)$;

(v) $\mathcal{B}_{fin}(H)$ *is a dense subspace of* $\mathcal{B}_1(H)$.

An operator $A \in \mathcal{B}(H)$ is said to belong to the *von Neumann-Schatten class* $\mathcal{B}_p(H)$, $1 \le p < \infty$ if the *p-Schatten norm* is finite [17, p. 60-64]

$$\|A\|_p := [\operatorname{tr}(|A|^p)]^{1/p} < \infty.$$

For $1 < p < q < \infty$ we have that

$$\mathcal{B}_1(H) \subset \mathcal{B}_p(H) \subset \mathcal{B}_q(H) \subset \mathcal{B}(H) \tag{1.10}$$

and

$$\|A\|_1 \ge \|A\|_p \ge \|A\|_q \ge \|A\|. \tag{1.11}$$

For $p \ge 1$ the functional $\|\cdot\|_p$ is a *norm* on the $*$-ideal $\mathcal{B}_p(H)$ and $\left(\mathcal{B}_p(H), \|\cdot\|_p\right)$ is a Banach space.

Also, see for instance [17, p. 60-64],

$$\|A\|_p = \|A^*\|_p, \ A \in \mathcal{B}_p(H) \tag{1.12}$$

$$\|AB\|_p \le \|A\|_p \|B\|_p, \ A, B \in \mathcal{B}_p(H) \tag{1.13}$$

and

$$\|AB\|_p \le \|A\|_p \|B\|, \ \|BA\|_p \le \|B\| \|A\|_p, \ A \in \mathcal{B}_p(H), \ B \in \mathcal{B}(H). \tag{1.14}$$

This implies that

$$\|CAB\|_p \leq \|C\| \|A\|_p \|B\| , \ A \in \mathcal{B}_p (H) , \ B, \ C \in \mathcal{B} (H) . \tag{1.15}$$

In terms of *p-Schatten norm* we have the *Hölder inequality* for $p, \ q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$

$$(|\operatorname{tr} (AB)| \leq) \|AB\|_1 \leq \|A\|_p \|B\|_q , \ A \in \mathcal{B}_p (H) , \ B \in \mathcal{B}_q (H) . \tag{1.16}$$

For the theory of trace functionals and their applications the reader is referred to [16] and [17].

For some classical trace inequalities see [6], [7], and [14], which are continuations of the work of Bellman [1].

2. Main Results

Consider a probability density function w on $[a, b]$, i.e., $w \geq 0$ a.e. on $[a, b]$ with $\int_a^b w (t) dt = 1$, and the weighted Čebyšev functional for operator-valued functions $A, \ B : [a, b] \to B (H)$ defined on a finite interval $[a, b]$,

$$D_w (A, B) := \int_a^b w (t) A (t) B (t) dt - \int_a^b w (t) A (t) dt \int_a^b w (t) B (t) dt.$$

The first main results is as follows:

Theorem 3. *Assume that* $w : [a, b] \to [0, \infty)$ *is integrable with* $\int_a^b w (t) dt = 1$ *and for* $p, \ q > 1$ *with* $\frac{1}{p} + \frac{1}{q} = 1, \ A : [a, b] \to B_p (H) , \ B : [a, b] \to B_q (H)$ *are strongly differentiable with* $\int_a^b \|A' (u)\|_p du, < \infty, \ \int_a^b \|B' (u)\|_q du < \infty,$ *then*

$$\begin{aligned} &\|D_w (A, B)\|_1 \\ &\leq D_w \left(\int_a^{\cdot} \|A' (u)\|_p du, \int_a^{\cdot} \|B' (u)\|_q du \right) \\ &\leq \frac{1}{2} \left(\int_a^b \|B' (u)\|_q du \right) \\ &\times \int_a^b w (t) \left| \int_a^t \|A' (u)\|_p du - \int_a^b \left(\int_a^s \|A' (u)\|_p du \right) w (s) ds \right| dt \\ &\leq \frac{1}{2} \left(\int_a^b \|B' (u)\|_q du \right) \int_a^b w (s) \left[\left(\int_a^s \|A' (u)\|_p du \right)^2 ds \right. \\ &\left. - \left(\int_a^b w (s) \left(\int_a^s \|A' (u)\|_p du \right) ds \right)^2 \right]^{1/2} \\ &\leq \frac{1}{4} \left(\int_a^b \|A' (u)\|_p du \right) \left(\int_a^b \|B' (u)\|_q du \right) . \end{aligned} \tag{2.1}$$

In particular, for $p=q=2$ we get

$$
\begin{aligned}
(2.2)\qquad & \left\| D_w(A,B)\right\|_1 \\
& \le D_w\left(\int_a^{\cdot}\left\| A'(u)\right\|_2 du, \int_a^{\cdot}\left\| B'(u)\right\|_2 du\right) \\
& \le \frac{1}{2}\left(\int_a^b \left\| B'(u)\right\|_2 du\right) \\
& \times \int_a^b w(t)\left| \int_a^t \left\| A'(u)\right\|_2 du - \int_a^b\left(\int_a^s \left\| A'(u)\right\|_2 du\right) w(s)\,ds\right| dt \\
& \le \frac{1}{2}\left(\int_a^b \left\| B'(u)\right\|_2 du\right)\left[\int_a^b w(s)\left(\int_a^s \left\| A'(u)\right\|_2 du\right)^2 ds\right. \\
& \left. - \left(\int_a^b w(s)\left(\int_a^s \left\| A'(u)\right\|_2 du\right) ds\right)^2\right]^{1/2} \\
& \le \frac{1}{4}\left(\int_a^b \left\| A'(u)\right\|_2 du\right)\left(\int_a^b \left\| B'(u)\right\|_2 du\right).
\end{aligned}
$$

Proof. Observe that, by the use of integral's properties,

$$
\begin{aligned}
& \int_a^b\int_a^b w(t)\,w(s)\,[A(t)-A(s)]\,[B(t)-B(s)]\,dtds \\
& = \int_a^b\int_a^b w(t)\,w(s)\,(A(t)B(t)-A(s)B(t)-A(t)B(s)+A(s)B(s))\,dtds \\
& = \int_a^b w(s)\,ds\int_a^b A(t)B(t)\,dt - \int_a^b w(s)A(s)\,ds\int_a^b w(t)B(t)\,dt \\
& - \int_a^b w(t)A(t)\,dt\int_a^b w(s)B(s)\,ds + \int_a^b w(t)\,dt\int_a^b A(s)B(s)\,ds = 2D_w(A,B),
\end{aligned}
$$

which give the weighted Korkine's identity for functions with operator values

$$
D_w(A,B) = \frac{1}{2}\int_a^b\int_a^b w(t)\,w(s)\,[A(t)-A(s)]\,[B(t)-B(s)]\,dtds.
$$

For Korkine's classical identity for real-valued functions, see [13, p. 242].

If we take the 1-norm, use the integral's properties and the Hölder's inequality (1.16), then we get

$$
\begin{aligned}
(2.3)\quad \left\| D_w(A,B)\right\|_1 & \le \frac{1}{2}\int_a^b\int_a^b w(t)\,w(s)\left\| [A(t)-A(s)]\,[B(t)-B(s)]\right\|_1 dtds \\
& \le \frac{1}{2}\int_a^b\int_a^b w(t)\,w(s)\left\| A(t)-A(s)\right\|_p\left\| B(t)-B(s)\right\|_q dtds.
\end{aligned}
$$

Observe that for $s,\ t\in[a,b]$

$$
A(t)-A(s) = \int_s^t A'(u)\,du,\ \ B(t)-B(s) = \int_s^t B'(u)\,du,
$$

which implies that

$$\begin{aligned}\|A(t)-A(s)\|_p\|B(t)-B(s)\|_q &= \left\|\int_s^t A'(u)\,du\right\|_p\left\|\int_s^t B'(u)\,du\right\|_q\\ &\le \left|\int_s^t \|A'(u)\|_p\,du\right|\left|\int_s^t \|B'(u)\|_q\,du\right|\\ &= \left(\int_s^t \|A'(u)\|_p\,du\right)\left(\int_s^t \|B'(u)\|_q\,du\right)\end{aligned}$$

for all $s,\,t\in[a,b]$.

By (2.3) we get

$$\begin{aligned}&\|D_w(A,B)\|_1\\ &\le \frac{1}{2}\int_a^b\int_a^b w(t)\,w(s)\left(\int_s^t \|A'(u)\|_p\,du\right)\left(\int_s^t \|B'(u)\|_q\,du\right)dtds.\end{aligned}\tag{2.4}$$

Since

$$\begin{aligned}&\left(\int_s^t \|A'(u)\|_p\,du\right)\left(\int_s^t \|B'(u)\|_q\,du\right)\\ =\;&\left(\int_a^t \|A'(u)\|_p\,du-\int_a^s \|A'(u)\|_p\,du\right)\left(\int_a^t \|B'(u)\|_q\,du-\int_a^s \|B'(u)\|_q\,du\right),\end{aligned}$$

hence by Korkine's identity for real valued functions $\int_a^t \|A'(u)\|_p\,du$ and $\int_a^t \|B'(u)\|_q\,du$, we have

$$\begin{aligned}&\frac{1}{2}\int_a^b\int_a^b w(t)\,w(s)\left(\int_a^t \|A'(u)\|_p\,du-\int_a^s \|A'(u)\|_p\,du\right)\\ &\times\left(\int_a^t \|B'(u)\|_q\,du-\int_a^s \|B'(u)\|_q\,du\right)dtds\\ &= D_w\left(\int_a^{\cdot} \|A'(u)\|_p\,du,\int_a^{\cdot} \|B'(u)\|_q\,du\right).\end{aligned}\tag{2.5}$$

By utilising (2.4) and (2.5), we deduce the first inequality in (2.1).

Observe that

$$0\le\int_a^t \|B'(u)\|_q\,du\le\int_a^b \|B'(u)\|_q\,du$$

and

$$0\le\int_a^t \|A'(u)\|_p\,du\le\int_a^b \|A'(u)\|_p\,du$$

for all $t \in [a,b]$, then by (1.5) for the functions $\int_a^t \|A'(u)\|_p du$ and $\int_a^t \|B'(u)\|_q du$, $t \in [a,b]$, we get

$$
\begin{aligned}
(2.6)\qquad & (0 \leq) D_w \left(\int_a^{\cdot} \|A'(u)\|_p du, \int_a^{\cdot} \|B'(u)\|_q du \right) \\
& \leq \frac{1}{2} \left(\int_a^b \|B'(u)\|_q du \right) \\
& \times \int_a^b w(t) \left| \int_a^t \|A'(u)\|_p du - \int_a^b \left(\int_a^s \|A'(u)\|_p du \right) w(s) ds \right| dt \\
& \leq \frac{1}{2} \left(\int_a^b \|B'(u)\|_q du \right) \left[\int_a^b w(s) \left(\int_a^s \|A'(u)\|_p du \right)^2 ds \right. \\
& \left. - \left(\int_a^b w(s) \left(\int_a^s \|A'(u)\|_p du \right) ds \right)^2 \right]^{1/2} \\
& \leq \frac{1}{4} \left(\int_a^b \|A'(u)\|_p du \right) \left(\int_a^b \|B'(u)\|_q du \right),
\end{aligned}
$$

which proves the last part of (2.1). □

Remark 1. *We consider the uniform distribution $w_0(t) = \frac{1}{b-a}$ for which we have*

$$
D_{w_0}(A,B) := \frac{1}{b-a} \int_a^b A(t) B(t) dt - \frac{1}{b-a} \int_a^b A(t) dt \frac{1}{b-a} \int_a^b B(t) dt.
$$

then by (2.1) we get

$$
\begin{aligned}
(2.7)\qquad \|D_{w_0}(A,B)\|_1 & \leq D_{w_0} \left(\int_a^{\cdot} \|A'(u)\|_p du, \int_a^{\cdot} \|B'(u)\|_q du \right) \\
& \leq \frac{1}{4} \int_a^b \|A'(u)\|_p du \int_a^b \|B'(u)\|_q du
\end{aligned}
$$

for $p,\ q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1,$ and $A : [a,b] \to B_p(H)$, $B : [a,b] \to B_q(H)$ strongly differentiable with $\int_a^b \|A'(u)\|_p du$, $\int_a^b \|B'(u)\|_q du < \infty$.

In particular, for $p = q = 2$ we derive

$$
\begin{aligned}
(2.8)\qquad \|D_{w_0}(A,B)\|_1 & \leq D_{w_0} \left(\int_a^{\cdot} \|A'(u)\|_2 du, \int_a^{\cdot} \|B'(u)\|_2 du \right) \\
& \leq \frac{1}{4} \int_a^b \|A'(u)\|_2 du \int_a^b \|B'(u)\|_2 du.
\end{aligned}
$$

Theorem 4. *Assume that $w : [a,b] \to [0,\infty)$ is integrable with $\int_a^b w(t) dt = 1$ and for $p \geq 1,\ A, B : [a,b] \to B_p(H)$, are strongly differentiable with $\int_a^b \|A'(u)\|_p du$,*

$< \infty$, $\int_a^b \|B'(u)\|_p du < \infty$, *then*

$$\begin{aligned} &\|D_w(A,B)\|_p \\ &\le D_w\left(\int_a^{\cdot} \|A'(u)\|_p du, \int_a^{\cdot} \|B'(u)\|_p du\right) \\ &\le \frac{1}{2}\left(\int_a^b \|B'(u)\|_p du\right) \\ &\times \int_a^b w(t)\left|\int_a^t \|A'(u)\|_p du - \int_a^b \left(\int_a^s \|A'(u)\|_p du\right) w(s)\,ds\right| dt \\ &\le \frac{1}{2}\left(\int_a^b \|B'(u)\|_p du\right)\left[\int_a^b w(s)\left(\int_a^s \|A'(u)\|_p du\right)^2 ds\right. \\ &\left.-\left(\int_a^b w(s)\left(\int_a^s \|A'(u)\|_p du\right) ds\right)^2\right]^{1/2} \\ &\le \frac{1}{4}\left(\int_a^b \|A'(u)\|_p du\right)\left(\int_a^b \|B'(u)\|_p du\right). \end{aligned} \tag{2.9}$$

In particular,

$$\begin{aligned} &\|D_w(A,B)\|_2 \\ &\le D_w\left(\int_a^{\cdot} \|A'(u)\|_2 du, \int_a^{\cdot} \|B'(u)\|_2 du\right) \\ &\le \frac{1}{2}\left(\int_a^b \|B'(u)\|_2 du\right) \\ &\times \int_a^b w(t)\left|\int_a^t \|A'(u)\|_2 du - \int_a^b \left(\int_a^s \|A'(u)\|_2 du\right) w(s)\,ds\right| dt \\ &\le \frac{1}{2}\left(\int_a^b \|B'(u)\|_2 du\right)\left[\int_a^b w(s)\left(\int_a^s \|A'(u)\|_2 du\right)^2 ds\right. \\ &\left.-\left(\int_a^b w(s)\left(\int_a^s \|A'(u)\|_2 du\right) ds\right)^2\right]^{1/2} \\ &\le \frac{1}{4}\left(\int_a^b \|A'(u)\|_2 du\right)\left(\int_a^b \|B'(u)\|_2 du\right). \end{aligned} \tag{2.10}$$

Proof. If we take the p-norm, use the integral's properties and the inequality (1.13), then we get

$$\begin{aligned} \|D_w(A,B)\|_p &\le \frac{1}{2}\int_a^b \int_a^b w(t)\,w(s)\,\|[A(t)-A(s)]\,[B(t)-B(s)]\|_p\,dtds \\ &\le \frac{1}{2}\int_a^b \int_a^b w(t)\,w(s)\,\|A(t)-A(s)\|_p\,\|B(t)-B(s)\|_p\,dtds. \end{aligned} \tag{2.11}$$

Since

$$\begin{aligned}\|A(t)-A(s)\|_p\|B(t)-B(s)\|_p &= \left\|\int_s^t A'(u)\,du\right\|_p\left\|\int_s^t B'(u)\,du\right\|_p\\ &\le \left|\int_s^t \|A'(u)\|_p\,du\right|\left|\int_s^t \|B'(u)\|_p\,du\right|\\ &= \left(\int_s^t \|A'(u)\|_p\,du\right)\left(\int_s^t \|B'(u)\|_q\,du\right)\end{aligned}$$

hence by (2.11) we get

$$\|D_w(A,B)\|_p \le \frac{1}{2}\int_a^b\int_a^b w(t)\,w(s)\left(\int_s^t \|A'(u)\|_p\,du\right)\left(\int_s^t \|B'(u)\|_q\,du\right)dtds..$$

Further, by utilising a similar argument to the one in the proof of Theorem 3 we derive (2.9). □

Remark 2. *Let $A,\ B : [a,b] \to B_p(H)\,,\ p \ge 1,$ be strongly differentiable with $\int_a^b \|A'(u)\|_p\,du,\ \int_a^b \|B'(u)\|_p\,du < \infty$, then*

$$\begin{aligned}\|D_{w_0}(A,B)\|_p &\le D_{w_0}\left(\int_a^{\cdot} \|A'(u)\|_p\,du, \int_a^{\cdot} \|B'(u)\|_p\,du\right)\\ &\le \frac{1}{4}\int_a^b \|A'(u)\|_p\,du\int_a^b \|B'(u)\|_p\,du.\end{aligned} \tag{2.12}$$

3. Inequalities for Operator Monotone Functions

A real valued continuous function h on $[0,\infty)$ is said to be operator monotone if $h(A) \ge h(B)$ holds for any $A \ge B \ge 0$.

We have the following representation of operator monotone functions, see for instance [2, p. 144-145]:

Theorem 5. *A function $h : [0,\infty) \to \mathbb{R}$ is operator monotone in $[0,\infty)$ if and only if it has the representation*

$$h(t) = h(0) + bt + \int_0^\infty \frac{t\lambda}{t+\lambda}\,d\mu(\lambda)\,, \tag{3.1}$$

where $b \ge 0$ and a positive measure μ on $[0,\infty)$ such that

$$\int_0^\infty \frac{\lambda}{1+\lambda}\,d\mu(\lambda) < \infty. \tag{3.2}$$

We have the following representation result:

Lemma 1. *Let $h : [0,\infty) \to \mathbb{R}$ be operator monotone in $[0,\infty)$. Assume that $U \ge 0$, then for all selfadjoint operators V we have*

$$Dh(U)(V) = bV + \int_0^\infty \lambda^2\left[(\lambda+U)^{-1}V(\lambda+U)^{-1}\right]d\mu(\lambda)\,. \tag{3.3}$$

Proof. From (3.1) we get

$$h(t) = h(0) + bt + \int_0^\infty \left(\lambda - \frac{\lambda^2}{t+\lambda}\right)d\mu(\lambda)\,.$$

Assume that $U \geq 0$, then for all selfadjoint operator V we have, by the representation of h and for t in a small open interval around 0, that

$$\begin{aligned}
&h(U+tV)-h(U)\\
&=btV+\int_0^\infty \left(\lambda-\lambda^2(U+tV+\lambda)^{-1}\right)d\mu(\lambda)-\int_0^\infty \left(\lambda-\lambda^2(U+\lambda)^{-1}\right)d\mu(\lambda)\\
&=btV+\int_0^\infty \lambda^2\left[(\lambda+U)^{-1}-(\lambda+U+tV)^{-1}\right]d\mu(\lambda)\\
&=btV+\int_0^\infty \lambda^2\left[(\lambda+U)^{-1}(\lambda+U+tV-\lambda-U)(\lambda+U+tV)^{-1}\right]d\mu(\lambda)\\
&=btV+t\int_0^\infty \lambda^2\left[(\lambda+U)^{-1}V(\lambda+U+tV)^{-1}\right]d\mu(\lambda).
\end{aligned}$$

Dividing by $t \neq 0$, we get

$$\frac{h(U+tV)-h(U)}{t}=bV+\int_0^\infty \lambda^2\left[(\lambda+U)^{-1}V(\lambda+U+tV)^{-1}\right]d\mu(\lambda)$$

and by taking the limit over $t \to 0$, we get

$$Dh(U)(V)=bV+\int_0^\infty \lambda^2\left[(\lambda+U)^{-1}V(\lambda+U)^{-1}\right]d\mu(\lambda)$$

for all selfadjoint operator V we have (3.3). □

Lemma 2. *Let $h:[0,\infty)\to\mathbb{R}$ be operator monotone in $[0,\infty)$. Assume that $U \geq u > 0$, then for all selfadjoint operators $V \in B_p(H)$, $p \geq 1$ we have*

$$\|Dh(U)(V)\|_p \leq h'(u)\|V\|_p. \tag{3.4}$$

Proof. From (3.3) and using (1.15) we get

$$\begin{aligned}
\|Dh(U)(V)-bV\|_p &\leq \int_0^\infty \lambda^2\left\|(\lambda+U)^{-1}V(\lambda+U)^{-1}\right\|_p d\mu(\lambda) \qquad (3.5)\\
&\leq \|V\|_p\int_0^\infty \lambda^2\left\|(\lambda+U)^{-1}\right\|^2 d\mu(\lambda).
\end{aligned}$$

Observe that $\lambda+U \geq \lambda+u > 0$ for $\lambda \in [0,\infty)$. Then $0<(\lambda+U)^{-1}\leq(\lambda+u)^{-1}$, which implies that $\left\|(\lambda+U)^{-1}\right\| \leq (\lambda+u)^{-1}$, namely $\left\|(\lambda+U)^{-1}\right\|^2 \leq (\lambda+u)^{-2}$ for $\lambda \in [0,\infty)$.

Therefore by (3.5) we get

$$\|Dh(U)(V)-bV\|_p \leq \|V\|_p\int_0^\infty \lambda^2(\lambda+u)^{-2}d\mu(\lambda). \tag{3.6}$$

If we take the derivative over t in (3.1) then we have

$$h'(t)=b+\int_0^\infty \frac{\lambda(t+\lambda)-\lambda t}{(t+\lambda)^2}d\mu(\lambda)=b+\int_0^\infty \frac{\lambda^2}{(t+\lambda)^2}d\mu(\lambda) \tag{3.7}$$

for $t>0$.

From (3.7) we get

$$\int_0^\infty \lambda^2(\lambda+u)^{-2}d\mu(\lambda)=h'(u)-b,$$

and by (3.6) we derive

$$\|Dh(U)(V)-bV\|_p \le \|V\|_p h'(u) - b\|V\|_p .$$

Finally, by the triangle inequality and by the fact that $b \ge 0$, we obtain that

$$\|Dh(U)(V)\|_p - b\|V\|_p \le \|Dh(U)(V)-bV\|_p ,$$

which proves the desired result (3.4). □

For a continuous function h on $(0,\infty)$ and $C,\ E>0$ we consider the auxiliary function $h_{C,E}:[0,1]\to\mathbb{R}$ defined by

$$h_{C,E}(t) := h((1-t)C+tE),\ t\in[0,1].$$

We have the following representations of the derivatives:

Lemma 3. *Assume that the operator function generated by h is Fréchet differentiable in each $C\ge 0$, then for $E\ge 0$ we have that $h_{C,E}$ is differentiable on $[0,1]$ and*

$$h'_{C,E}(t) = D(h)((1-t)C+tE)(E-C) \tag{3.8}$$

for $t\in[0,1]$, where in 0 and 1 the derivatives are the right and left derivatives.

Proof. We prove only for the interior points $t\in(0,1)$. Let h be in a small interval around 0 such that $t+h\in(0,1)$. Then for $h\neq 0$,

$$\begin{aligned}&\frac{h_{C,E}(t+h)-h(t)}{h}\\&=\frac{h((1-(t+h))C+(t+h)E)-h((1-t)C+tE)}{h}\\&=\frac{h((1-t)C+tE+h(E-C))-h((1-t)C+tE)}{h}\end{aligned}$$

and by taking the limit over $h\to 0$, we get

$$\begin{aligned}h'_{C,E}(t)&=\lim_{h\to 0}\frac{h_{C,E}(t+h)-h(t)}{h}\\&=\lim_{h\to 0}\left[\frac{h((1-t)C+tE+h(E-C))-h((1-t)C+tE)}{h}\right]\\&=D(h)((1-t)C+tE)(E-C),\end{aligned}$$

which proves (3.8). □

Corollary 1. *Let $h:[0,\infty)\to\mathbb{R}$ be operator monotone in $[0,\infty)$. Then for all $C\ge c>0,\ E\ge d>0$ with $C,E\in B_p(H),\ p\ge 1$ we have*

$$\begin{aligned}\left\|h'_{C,E}(t)\right\|_p &= \|D(h)((1-t)C+tE)(E-C)\|_p\\&\le h'((1-t)c+td)\|E-C\|_p\end{aligned} \tag{3.9}$$

for all $t\in[0,1]$.

The proof follows by Lemma 2 and Lemma 3.

Consider a probability density function w on $[a,b]$, i.e., $w \geq 0$ a.e. on $[a,b]$ with $\int_a^b w(t)\,dt = 1$. For the operator monotone functions h, $k : [0,\infty) \to \mathbb{R}$ and two operators C, $E \geq 0$ we consider the *Čebyšev functional*

$$D_w(h,k,C,E) := \int_0^1 w(t)\,h((1-t)C+tE)\,k((1-t)C+tE)\,dt - \int_0^1 w(t)\,h((1-t)C+tE)\,dt \int_0^1 w(t)\,k((1-t)C+tE)\,dt.$$

For p, $q > 1$ with $\frac{1}{p}+\frac{1}{q} = 1$ assume that $C \geq c > 0$, $E \geq d > 0$ with C, $E \in B_p(H) \cap B_q(H)$. If we use Theorem 3 and Corollary 1 we have

$$\|D_w(h,k,C,E)\|_1 \leq \frac{1}{4}\|E-C\|_p \|E-C\|_q \times \begin{cases} \frac{h(d)-h(c)}{d-c} & \text{if } d \neq c \\ h'(c) & \text{if } d = c \end{cases} \times \begin{cases} \frac{k(d)-k(c)}{d-c} & \text{if } d \neq c \\ k'(c) & \text{if } d = c \end{cases} \tag{3.10}$$

and, in particular,

$$\|D_w(h,k,C,E)\|_1 \leq \frac{1}{4}\|E-C\|_2^2 \times \begin{cases} \frac{h(d)-h(c)}{d-c} & \text{if } d \neq c \\ h'(c) & \text{if } d = c \end{cases} \times \begin{cases} \frac{k(d)-k(c)}{d-c} & \text{if } d \neq c \\ k'(c) & \text{if } d = c. \end{cases} \tag{3.11}$$

For $C \geq c > 0$, $E \geq d > 0$ with C, $E \in B_p(H)$, $p \geq 1$, then by Theorem 4 we have

$$\|D_w(h,k,C,E)\|_p \leq \frac{1}{4}\|E-C\|_p^2 \times \begin{cases} \frac{h(d)-h(c)}{d-c} & \text{if } d \neq c \\ h'(c) & \text{if } d = c \end{cases} \times \begin{cases} \frac{k(d)-k(c)}{d-c} & \text{if } d \neq c \\ k'(c) & \text{if } d = c \end{cases} \tag{3.12}$$

If we consider the operator monotone functions $h(t) = t^m$, $k(t) = t^n$ with m, $n \in (0,1)$ and put

$$D_w(m,n,C,E) := \int_0^1 w(t)\,((1-t)C+tE)^{m+n}\,dt - \int_0^1 w(t)\,((1-t)C+tE)^m\,dt \int_0^1 w(t)\,((1-t)C+tE)^n\,dt,$$

then by (3.10) we derive

$$\|D_w(h,k,C,E)\|_1 \leq \frac{1}{4}\|E-C\|_p \|E-C\|_q \times \begin{cases} \frac{d^m-c^m}{d-c} & \text{if } d \neq c \\ \frac{m}{c^{1-m}} & \text{if } d = c \end{cases} \times \begin{cases} \frac{d^n-c^n}{d-c} & \text{if } d \neq c \\ \frac{n}{c^{1-n}} & \text{if } d = c \end{cases} \tag{3.13}$$

provided that $C \geq c > 0$, $E \geq d > 0$ with $C,\, E \in B_p(H) \cap B_q(H)$.

For $C \geq c > 0$, $E \geq d > 0$ with $C,\, E \in B_p(H)$, $p \geq 1$, then we have

$$\|D_w(h,k,C,E)\|_p \leq \frac{1}{4}\|E-C\|_p^2 \times \begin{cases} \frac{d^m-c^m}{d-c} \text{ if } d \neq c \\ \frac{m}{c^{1-m}} \text{ if } d = c \end{cases} \times \begin{cases} \frac{d^n-c^n}{d-c} \text{ if } d \neq c \\ \frac{n}{c^{1-n}} \text{ if } d = c. \end{cases} \tag{3.14}$$

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