

**p -SCHATTEN NORM HÖLDER-TYPE INEQUALITIES FOR THE
WEIGHTED ČEBYŠEV'S FUNCTIONAL**

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ABSTRACT. An operator $A \in \mathcal{B}(H)$ is said to belong to the *von Neumann-Schatten class* $\mathcal{B}_p(H)$, $1 \leq p < \infty$ if the *p -Schatten norm* is finite

$$\|A\|_p := [\text{tr}(|A|^p)]^{1/p} < \infty.$$

In this paper we show among others that, Assume that $w : [a, b] \rightarrow [0, \infty)$ is integrable with $\int_a^b w(t) dt = 1$ and for $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, $A : [a, b] \rightarrow \mathcal{B}_p(H)$, $B : [a, b] \rightarrow \mathcal{B}_q(H)$ are strongly differentiable with $\int_a^b \|A'(u)\|_p^\alpha du$, $\int_a^b \|B'(u)\|_q^\beta du < \infty$, for $\alpha, \beta > 1$ with $\frac{1}{\alpha} + \frac{1}{\beta} = 1$, then

$$\begin{aligned} & \left\| \int_a^b w(t) A(t) B(t) dt - \int_a^b w(t) A(t) dt \int_a^b w(t) B(t) dt \right\|_1 \\ & \leq \frac{1}{2} \int_a^b w(t) \left| t - \int_a^b sw(s) ds \right| dt \\ & \times \left(\int_a^b \|A'(u)\|_p^\alpha du \right)^{1/\alpha} \left(\int_a^b \|B'(u)\|_q^\beta du \right)^{1/\beta} \\ & \leq \frac{1}{2} \left[\int_a^b s^2 w(s) ds - \left(\int_a^b sw(s) ds \right)^2 \right]^{\frac{1}{2}} \\ & \times \left(\int_a^b \|A'(u)\|_p^\alpha du \right)^{1/\alpha} \left(\int_a^b \|B'(u)\|_q^\beta du \right)^{1/\beta} \\ & \leq \frac{1}{4} (b-a) \left(\int_a^b \|A'(u)\|_p^\alpha du \right)^{1/\alpha} \left(\int_a^b \|B'(u)\|_q^\beta du \right)^{1/\beta}. \end{aligned}$$

Some examples of interest for the operator monotone functions are also given.

1. INTRODUCTION

Let $(\Omega, \mathcal{A}, \mu)$ be a measurable space consisting of a set Ω , a σ -algebra \mathcal{A} of parts of Ω and a countably additive and positive measure μ on \mathcal{A} with values in $\mathbb{R} \cup \{\infty\}$.

For a μ -measurable function $w : \Omega \rightarrow \mathbb{R}$, with $w(x) \geq 0$ for μ -a.e. $x \in \Omega$, consider the Lebesgue space $L_w(\Omega, \mu) := \{h : \Omega \rightarrow \mathbb{R}, h \text{ is } \mu\text{-measurable and } \int_\Omega w(x) |h(x)| d\mu(x) < \infty\}$. Assume $\int_\Omega w(x) d\mu(x) = 1$. In order to simplify the notation for the integrals, we do not write the variable, namely, instead of $\int_\Omega w(x) d\mu(x)$ we simply write $\int_\Omega w d\mu$.

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If $h, k : \Omega \rightarrow \mathbb{R}$ are μ -measurable functions and $h, k, hk \in L_w(\Omega, \mu)$, then we may consider the weighted Čebyšev functional in the following form

$$(1.1) \quad D_w(h, k) := \int_{\Omega} whkd\mu - \int_{\Omega} whd\mu \int_{\Omega} wk d\mu.$$

The following result is known in the literature as the Grüss inequality, see for instance [5]:

$$(1.2) \quad |D_w(h, k)| \leq \frac{1}{4}(\Gamma - \gamma)(\Delta - \delta),$$

provided

$$(1.3) \quad -\infty < \gamma \leq h \leq \Gamma < \infty, \quad -\infty < \delta \leq k \leq \Delta < \infty$$

for μ -a.e. on Ω . The constant $\frac{1}{4}$ is sharp in the sense that it cannot be replaced by a smaller constant.

In [5] Cerone and Dragomir proved among others the following *refinement of Grüss' inequality*:

Theorem 1. For $h, k : \Omega \rightarrow \mathbb{R}$, μ -measurable functions and so that $-\infty < \gamma \leq h \leq \Gamma < \infty$, $-\infty < \delta \leq k \leq \Delta < \infty$ μ -a.e. on Ω ,

$$(1.4) \quad \begin{aligned} |D_w(h, k)| &\leq \frac{1}{2}(\Delta - \delta) \int_{\Omega} w \left| h - \int_{\Omega} whd\mu \right| d\mu \\ &\leq \frac{1}{2}(\Delta - \delta) \left[\int_{\Omega} wh^2 d\mu - \left(\int_{\Omega} whd\mu \right)^2 \right]^{\frac{1}{2}} \\ &\leq \frac{1}{4}(\Delta - \delta)(\Gamma - \gamma), \end{aligned}$$

provided that $h, k, hk \in L_w(\Omega, \mu)$. The constants $\frac{1}{2}$ and $\frac{1}{4}$ are best possible.

Consider a probability density function w on $[a, b]$, i.e., $w \geq 0$ a.e. on $[a, b]$ with $\int_a^b w(t) dt = 1$, and the weighted Čebyšev functional for functions defined on a finite interval $[a, b]$,

$$D_w(h, k) := \int_a^b h(t)k(t)w(t)dt - \int_a^b h(t)w(t)dt \int_a^b k(t)w(t)dt.$$

From (1.4) we get

$$(1.5) \quad \begin{aligned} |D_w(h, k)| &\leq \frac{1}{2}(\Delta - \delta) \int_a^b w(t) \left| h(t) - \int_a^b h(s)w(s)ds \right| dt \\ &\leq \frac{1}{2}(\Delta - \delta) \left[\int_a^b w(s)h^2(s)ds - \left(\int_a^b h(s)w(s)ds \right)^2 \right]^{\frac{1}{2}} \\ &\leq \frac{1}{4}(\Delta - \delta)(\Gamma - \gamma), \end{aligned}$$

provided that $h, k : [a, b] \rightarrow \mathbb{R}$ are measurable functions and so that $-\infty < \gamma \leq h \leq \Gamma < \infty$, $-\infty < \delta \leq k \leq \Delta < \infty$ a.e. on $[a, b]$, and $h, k, hk \in L_w[a, b]$.

For classical and more recent upper bounds related to the Čebyšev functional see [4], [8]-[13] and [15].

In order to extend these results to p -Schatten norms we need the following preparations.

Let $(H; \langle \cdot, \cdot \rangle)$ be a complex Hilbert space and $\mathcal{B}(H)$ the Banach algebra of all bounded linear operators on H . If $\{e_i\}_{i \in I}$ an orthonormal basis of H , we say that $A \in \mathcal{B}(H)$ is of *trace class* if

$$(1.6) \quad \|A\|_1 := \sum_{i \in I} \langle |A| e_i, e_i \rangle < \infty.$$

The definition of $\|A\|_1$ does not depend on the choice of the orthonormal basis $\{e_i\}_{i \in I}$. We denote by $\mathcal{B}_1(H)$ the set of trace class operators in $\mathcal{B}(H)$.

We define the *trace* of a trace class operator $A \in \mathcal{B}_1(H)$ to be

$$(1.7) \quad \text{tr}(A) := \sum_{i \in I} \langle A e_i, e_i \rangle,$$

where $\{e_i\}_{i \in I}$ an orthonormal basis of H . Note that this coincides with the usual definition of the trace if H is finite-dimensional. We observe that the series (1.7) converges absolutely and it is independent from the choice of basis.

The following result collects some properties of the trace:

Theorem 2. *We have:*

(i) *If $A \in \mathcal{B}_1(H)$ then $A^* \in \mathcal{B}_1(H)$ and*

$$(1.8) \quad \text{tr}(A^*) = \overline{\text{tr}(A)};$$

(ii) *If $A \in \mathcal{B}_1(H)$ and $T \in \mathcal{B}(H)$, then $AT, TA \in \mathcal{B}_1(H)$ and*

$$(1.9) \quad \text{tr}(AT) = \text{tr}(TA) \text{ and } |\text{tr}(AT)| \leq \|A\|_1 \|T\|;$$

(iii) *$\text{tr}(\cdot)$ is a bounded linear functional on $\mathcal{B}_1(H)$ with $\|\text{tr}\| = 1$;*

(iv) *If $A, B \in \mathcal{B}_2(H)$ then $AB, BA \in \mathcal{B}_1(H)$ and $\text{tr}(AB) = \text{tr}(BA)$;*

(v) *$\mathcal{B}_{fin}(H)$ is a dense subspace of $\mathcal{B}_1(H)$.*

An operator $A \in \mathcal{B}(H)$ is said to belong to the *von Neumann-Schatten class* $\mathcal{B}_p(H)$, $1 \leq p < \infty$ if the p -Schatten norm is finite [17, p. 60-64]

$$\|A\|_p := [\text{tr}(|A|^p)]^{1/p} < \infty.$$

For $1 < p < q < \infty$ we have that

$$(1.10) \quad \mathcal{B}_1(H) \subset \mathcal{B}_p(H) \subset \mathcal{B}_q(H) \subset \mathcal{B}(H)$$

and

$$(1.11) \quad \|A\|_1 \geq \|A\|_p \geq \|A\|_q \geq \|A\|.$$

For $p \geq 1$ the functional $\|\cdot\|_p$ is a *norm* on the $*$ -ideal $\mathcal{B}_p(H)$ and $(\mathcal{B}_p(H), \|\cdot\|_p)$ is a Banach space.

Also, see for instance [17, p. 60-64],

$$(1.12) \quad \|A\|_p = \|A^*\|_p, \quad A \in \mathcal{B}_p(H)$$

$$(1.13) \quad \|AB\|_p \leq \|A\|_p \|B\|_p, \quad A, B \in \mathcal{B}_p(H)$$

and

$$(1.14) \quad \|AB\|_p \leq \|A\|_p \|B\|, \quad \|BA\|_p \leq \|B\| \|A\|_p, \quad A \in \mathcal{B}_p(H), \quad B \in \mathcal{B}(H).$$

This implies that

$$(1.15) \quad \|CAB\|_p \leq \|C\| \|A\|_p \|B\|, \quad A \in \mathcal{B}_p(H), \quad B, C \in \mathcal{B}(H).$$

In terms of p -Schatten norm we have the Hölder inequality for $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$

$$(1.16) \quad (|\operatorname{tr}(AB)| \leq) \|AB\|_1 \leq \|A\|_p \|B\|_q, \quad A \in \mathcal{B}_p(H), \quad B \in \mathcal{B}_q(H).$$

For the theory of trace functionals and their applications the reader is referred to [16] and [17].

For some classical trace inequalities see [6], [7], and [14], which are continuations of the work of Bellman [1].

2. MAIN RESULTS

Consider a probability density function w on $[a, b]$, i.e., $w \geq 0$ a.e. on $[a, b]$ with $\int_a^b w(t) dt = 1$, and the weighted Čebyšev functional for operator-valued functions $A, B : [a, b] \rightarrow B(H)$ defined on a finite interval $[a, b]$,

$$D_w(A, B) := \int_a^b w(t) A(t) B(t) dt - \int_a^b w(t) A(t) dt \int_a^b w(t) B(t) dt.$$

The first main results is as follows:

Theorem 3. *Assume that $w : [a, b] \rightarrow [0, \infty)$ is integrable with $\int_a^b w(t) dt = 1$ and for $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, $A : [a, b] \rightarrow B_p(H)$, $B : [a, b] \rightarrow B_q(H)$ are strongly differentiable with $\left(\int_a^b \|A'(u)\|_p^\alpha du\right)^{1/\alpha} < \infty$ and $\left(\int_a^b \|B'(u)\|_q^\beta du\right)^{1/\beta} < \infty$ for $\alpha, \beta > 1$ with $\frac{1}{\alpha} + \frac{1}{\beta} = 1$. Then*

$$(2.1) \quad \begin{aligned} & \|D_w(A, B)\|_1 \\ & \leq \left[D_w \left(\ell, \int_a^\cdot \|A'(u)\|_p^\alpha du \right) \right]^{1/\alpha} \left[D_w \left(\ell, \int_a^\cdot \|B'(u)\|_q^\beta du \right) \right]^{1/\beta} \\ & \leq \frac{1}{2} \int_a^b w(t) \left| t - \int_a^b sw(s) ds \right| dt \\ & \quad \times \left(\int_a^b \|A'(u)\|_p^\alpha du \right)^{1/\alpha} \left(\int_a^b \|B'(u)\|_q^\beta du \right)^{1/\beta} \\ & \leq \frac{1}{2} \left[\int_a^b s^2 w(s) ds - \left(\int_a^b sw(s) ds \right)^2 \right]^{\frac{1}{2}} \\ & \quad \times \left(\int_a^b \|A'(u)\|_p^\alpha du \right)^{1/\alpha} \left(\int_a^b \|B'(u)\|_q^\beta du \right)^{1/\beta} \\ & \leq \frac{1}{4} (b-a) \left(\int_a^b \|A'(u)\|_p^\alpha du \right)^{1/\alpha} \left(\int_a^b \|B'(u)\|_q^\beta du \right)^{1/\beta}. \end{aligned}$$

In particular, if $\left(\int_a^b \|A'(u)\|_p^2 du\right)^{1/2} < \infty$ and $\left(\int_a^b \|B'(u)\|_q^2 du\right)^{1/2} < \infty$, then

$$\begin{aligned}
 (2.2) \quad & \|D_w(A, B)\|_1 \\
 & \leq \left[D_w\left(\ell, \int_a^\cdot \|A'(u)\|_p^2 du\right) \right]^{1/2} \left[D_w\left(\ell, \int_a^\cdot \|B'(u)\|_q^2 du\right) \right]^{1/2} \\
 & \leq \frac{1}{2} \int_a^b w(t) \left| t - \int_a^b sw(s) ds \right| dt \\
 & \quad \times \left(\int_a^b \|A'(u)\|_p^2 du \right)^{1/2} \left(\int_a^b \|B'(u)\|_q^2 du \right)^{1/2} \\
 & \leq \frac{1}{2} \left[\int_a^b s^2 w(s) ds - \left(\int_a^b sw(s) ds \right)^2 \right]^{\frac{1}{2}} \\
 & \quad \times \left(\int_a^b \|A'(u)\|_p^2 du \right)^{1/2} \left(\int_a^b \|B'(u)\|_q^2 du \right)^{1/2} \\
 & \leq \frac{1}{4} (b-a) \left(\int_a^b \|A'(u)\|_p^2 du \right)^{1/2} \left(\int_a^b \|B'(u)\|_q^2 du \right)^{1/2}.
 \end{aligned}$$

Proof. Observe that, by the use of integral's properties,

$$\begin{aligned}
 & \int_a^b \int_a^b w(t) w(s) [A(t) - A(s)] [B(t) - B(s)] dt ds \\
 & = \int_a^b \int_a^b w(t) w(s) (A(t)B(t) - A(s)B(t) - A(t)B(s) + A(s)B(s)) dt ds \\
 & = \int_a^b w(s) ds \int_a^b A(t)B(t) dt - \int_a^b w(s)A(s) ds \int_a^b w(t)B(t) dt \\
 & \quad - \int_a^b w(t)A(t) dt \int_a^b w(s)B(s) ds + \int_a^b w(t) dt \int_a^b A(s)B(s) ds = 2D_w(A, B),
 \end{aligned}$$

which give the weighted Korkine's identity for functions with operator values

$$D_w(A, B) = \frac{1}{2} \int_a^b \int_a^b w(t) w(s) [A(t) - A(s)] [B(t) - B(s)] dt ds.$$

For Korkine's classical identity for real-valued functions, see [13, p. 242].

If we take the 1-norm, use the integral's properties and the Hölder's inequality (1.16), then we get

$$\begin{aligned}
 (2.3) \quad & \|D_w(A, B)\|_1 \leq \frac{1}{2} \int_a^b \int_a^b w(t) w(s) \|[A(t) - A(s)] [B(t) - B(s)]\|_1 dt ds \\
 & \leq \frac{1}{2} \int_a^b \int_a^b w(t) w(s) \|A(t) - A(s)\|_p \|B(t) - B(s)\|_q dt ds.
 \end{aligned}$$

Observe also that for $s, t \in [a, b]$

$$A(t) - A(s) = \int_s^t A'(u) du, \quad B(t) - B(s) = \int_s^t B'(u) du.$$

Using Hölder's inequality for $\alpha, \beta > 1$ with $\frac{1}{\alpha} + \frac{1}{\beta} = 1$, then

$$\begin{aligned}
& \|A(t) - A(s)\|_p \|B(t) - B(s)\|_q \\
&= \left\| \int_s^t A'(u) du \right\|_p \left\| \int_s^t B'(u) du \right\|_q \\
&\leq \left| \int_s^t \|A'(u)\|_p^\alpha du \right| \left| \int_s^t \|B'(u)\|_q^\beta du \right| \\
&\leq |t-s|^{1/\beta} \left| \int_s^t \|A'(u)\|_p^\alpha du \right|^{1/\alpha} |t-s|^{1/\alpha} \left| \int_s^t \|B'(u)\|_q^\beta du \right|^{1/\beta} \\
&= |t-s| \left| \int_s^t \|A'(u)\|_p^\alpha du \right|^{1/\alpha} \left| \int_s^t \|B'(u)\|_q^\beta du \right|^{1/\beta}
\end{aligned}$$

for all $t, s \in [a, b]$.

By the weighted Hölder's inequality for double integral, we also have

$$\begin{aligned}
(2.4) \quad & \int_a^b \int_a^b w(s) w(t) \|A(t) - A(s)\|_p \|B(t) - B(s)\|_q dt ds \\
&\leq \int_a^b \int_a^b w(s) w(t) |t-s| \left| \int_s^t \|A'(u)\|_p^\alpha du \right|^{1/\alpha} \left| \int_s^t \|B'(u)\|_q^\beta du \right|^{1/\beta} dt ds \\
&\leq \left(\int_a^b \int_a^b w(s) w(t) |t-s| \left(\left| \int_s^t \|A'(u)\|_p^\alpha du \right|^{1/\alpha} \right)^\alpha dt ds \right)^{1/\alpha} \\
&\quad \times \left(\int_a^b \int_a^b w(s) w(t) |t-s| \left(\left| \int_s^t \|B'(u)\|_q^\beta du \right|^{1/\beta} \right)^\beta dt ds \right)^{1/\beta} \\
&= \left(\int_a^b \int_a^b w(s) w(t) |t-s| \left| \int_s^t \|A'(u)\|_p^\alpha du \right| dt ds \right)^{1/\alpha} \\
&\quad \times \left(\int_a^b \int_a^b w(s) w(t) |t-s| \left| \int_s^t \|B'(u)\|_q^\beta du \right| dt ds \right)^{1/\beta}.
\end{aligned}$$

Observe that

$$\begin{aligned}
& \int_a^b \int_a^b w(s) w(t) |t-s| \left| \int_s^t \|A'(u)\|_p^\alpha du \right| dt ds \\
&= \int_a^b \int_a^b w(s) w(t) (t-s) \left(\int_s^t \|A'(u)\|_p^\alpha du \right) dt ds \\
&= \int_a^b \int_a^b w(s) w(t) (t-s) \left(\int_a^t \|A'(u)\|_p^\alpha du - \int_a^s \|A'(u)\|_p^\alpha du \right) dt ds \\
&= 2D_w \left(\ell, \int_a^\cdot \|A'(u)\|_p^\alpha du \right)
\end{aligned}$$

and, similarly

$$\begin{aligned} & \int_a^b \int_a^b w(s) w(t) |t-s| \left| \int_s^t \|B'(u)\|_q^\beta du \right| dt ds \\ &= 2D_w \left(\ell, \int_a^\cdot \|B'(u)\|_q^\beta du \right). \end{aligned}$$

Therefore, by (2.3)

$$\begin{aligned} & \|D_w(A, B)\|_1 \\ & \leq \frac{1}{2} \int_a^b \int_a^b w(s) w(t) \|A(t) - A(s)\|_p \|B(t) - B(s)\|_q dt ds \\ & \leq \frac{1}{2} \left[2D_w \left(\ell, \int_a^\cdot \|A'(u)\|_p^\alpha du \right) \right]^{1/\alpha} \left[2D_w \left(\ell, \int_a^\cdot \|B'(u)\|_q^\beta du \right) \right]^{1/\beta} \\ & = \left[D_w \left(\ell, \int_a^\cdot \|A'(u)\|_p^\alpha du \right) \right]^{1/\alpha} \left[D_w \left(\ell, \int_a^\cdot \|B'(u)\|_q^\beta du \right) \right]^{1/\beta}, \end{aligned}$$

which proves the first inequality in (2.1).

If we use inequality (1.5) for $h = \ell$ and $k = \int_a^\cdot \|A'(u)\|_p^\alpha du$ and observe that $0 \leq \int_a^t \|A'(u)\|_p^\alpha du \leq \int_a^b \|A'(u)\|_p^\alpha du$ while $a \leq \ell(t) \leq b$ for $t \in [a, b]$, then

$$\begin{aligned} (2.5) \quad & D_w \left(\ell, \int_a^\cdot \|A'(u)\|_p^\alpha du \right) \\ & \leq \frac{1}{2} \left(\int_a^b \|A'(u)\|_p^\alpha du \right) \int_a^b w(t) \left| t - \int_a^b s w(s) ds \right| dt \\ & \leq \frac{1}{2} \left(\int_a^b \|A'(u)\|_p^\alpha du \right) \left[\int_a^b s^2 w(s) ds - \left(\int_a^b s w(s) ds \right)^2 \right]^{\frac{1}{2}} \\ & \leq \frac{1}{4} (b-a) \left(\int_a^b \|A'(u)\|_p^\alpha du \right), \end{aligned}$$

and, similarly

$$\begin{aligned} (2.6) \quad & D_w \left(\ell, \int_a^\cdot \|B'(u)\|_q^\beta du \right) \\ & \leq \frac{1}{2} \left(\int_a^b \|B'(u)\|_q^\beta du \right) \int_a^b w(t) \left| t - \int_a^b s w(s) ds \right| dt \\ & \leq \frac{1}{2} \left(\int_a^b \|B'(u)\|_q^\beta du \right) \left[\int_a^b s^2 w(s) ds - \left(\int_a^b s w(s) ds \right)^2 \right]^{\frac{1}{2}} \\ & \leq \frac{1}{4} (b-a) \left(\int_a^b \|B'(u)\|_q^\beta du \right). \end{aligned}$$

Therefore, by (2.5) and (2.6) we derive

$$\begin{aligned}
& \left[D_w \left(\ell, \int_a^{\cdot} \|A'(u)\|_p^\alpha du \right) \right]^{1/\alpha} \left[D_w \left(\ell, \int_a^{\cdot} \|B'(u)\|_q^\beta du \right) \right]^{1/\beta} \\
& \leq \left[\frac{1}{2} \left(\int_a^b \|A'(u)\|_p^\alpha du \right) \int_a^b w(t) \left| t - \int_a^b sw(s) ds \right| dt \right]^{1/\alpha} \\
& \quad \times \left[\frac{1}{2} \left(\int_a^b \|B'(u)\|_q^\beta du \right) \int_a^b w(t) \left| t - \int_a^b sw(s) ds \right| dt \right]^{1/\beta} \\
& \leq \left(\frac{1}{2} \left(\int_a^b \|A'(u)\|_p^\alpha du \right) \left[\int_a^b s^2 w(s) ds - \left(\int_a^b sw(s) ds \right)^2 \right]^{\frac{1}{2}} \right)^{1/\alpha} \\
& \quad \times \left(\frac{1}{2} \left(\int_a^b \|B'(u)\|_q^\beta du \right) \left[\int_a^b s^2 w(s) ds - \left(\int_a^b sw(s) ds \right)^2 \right]^{\frac{1}{2}} \right)^{1/\beta} \\
& \leq \left[\frac{1}{4} (b-a) \left(\int_a^b \|A'(u)\|_p^\alpha du \right) \right]^{1/\alpha} \left[\frac{1}{4} (b-a) \left(\int_a^b \|B'(u)\|_q^\beta du \right) \right]^{1/\beta} \\
& = \frac{1}{4} (b-a) \left(\int_a^b \|A'(u)\|_p^\alpha du \right)^{1/\alpha} \left(\int_a^b \|B'(u)\|_q^\beta du \right)^{1/\beta}
\end{aligned}$$

which proves the last part of (2.1). \square

Remark 1. If $A, B : [a, b] \rightarrow B_2(H)$ are strongly differentiable with $\left(\int_a^b \|A'(u)\|_2^\alpha du \right)^{1/\alpha} < \infty$ and $\left(\int_a^b \|B'(u)\|_2^\beta du \right)^{1/\beta} < \infty$ for $\alpha, \beta > 1$ with $\frac{1}{\alpha} + \frac{1}{\beta} = 1$. Then

$$\begin{aligned}
(2.7) \quad & \|D_w(A, B)\|_1 \\
& \leq \left[D_w \left(\ell, \int_a^{\cdot} \|A'(u)\|_2^\alpha du \right) \right]^{1/\alpha} \left[D_w \left(\ell, \int_a^{\cdot} \|B'(u)\|_2^\beta du \right) \right]^{1/\beta} \\
& \leq \frac{1}{2} \int_a^b w(t) \left| t - \int_a^b sw(s) ds \right| dt \\
& \quad \times \left(\int_a^b \|A'(u)\|_2^\alpha du \right)^{1/\alpha} \left(\int_a^b \|B'(u)\|_2^\beta du \right)^{1/\beta}
\end{aligned}$$

$$\begin{aligned}
 &\leq \frac{1}{2} \left[\int_a^b s^2 w(s) ds - \left(\int_a^b s w(s) ds \right)^2 \right]^{\frac{1}{2}} \\
 &\times \left(\int_a^b \|A'(u)\|_2^\alpha du \right)^{1/\alpha} \left(\int_a^b \|B'(u)\|_2^\beta du \right)^{1/\beta} \\
 &\leq \frac{1}{4} (b-a) \left(\int_a^b \|A'(u)\|_2^\alpha du \right)^{1/\alpha} \left(\int_a^b \|B'(u)\|_2^\beta du \right)^{1/\beta}.
 \end{aligned}$$

In particular, if $\left(\int_a^b \|A'(u)\|_2^2 du \right)^{1/2} < \infty$ and $\left(\int_a^b \|B'(u)\|_2^2 du \right)^{1/2} < \infty$, then

$$\begin{aligned}
 (2.8) \quad &\|D_w(A, B)\|_1 \\
 &\leq \left[D_w \left(\ell, \int_a^b \|A'(u)\|_2^2 du \right) \right]^{1/2} \left[D_w \left(\ell, \int_a^b \|B'(u)\|_2^2 du \right) \right]^{1/2} \\
 &\leq \frac{1}{2} \int_a^b w(t) \left| t - \int_a^b s w(s) ds \right| dt \\
 &\times \left(\int_a^b \|A'(u)\|_2^2 du \right)^{1/2} \left(\int_a^b \|B'(u)\|_2^2 du \right)^{1/2} \\
 &\leq \frac{1}{2} \left[\int_a^b s^2 w(s) ds - \left(\int_a^b s w(s) ds \right)^2 \right]^{\frac{1}{2}} \\
 &\times \left(\int_a^b \|A'(u)\|_2^2 du \right)^{1/2} \left(\int_a^b \|B'(u)\|_2^2 du \right)^{1/2} \\
 &\leq \frac{1}{4} (b-a) \left(\int_a^b \|A'(u)\|_2^2 du \right)^{1/2} \left(\int_a^b \|B'(u)\|_2^2 du \right)^{1/2}.
 \end{aligned}$$

Remark 2. We consider the uniform distribution $w_0(t) = \frac{1}{b-a}$ for which we have

$$D_{w_0}(A, B) := \frac{1}{b-a} \int_a^b A(t) B(t) dt - \frac{1}{b-a} \int_a^b A(t) dt \frac{1}{b-a} \int_a^b B(t) dt.$$

Since

$$\begin{aligned}
 \int_a^b w_0(t) \left| t - \int_a^b s w_0(s) ds \right| dt &= \frac{1}{b-a} \int_a^b \left| t - \frac{1}{b-a} \int_a^b s ds \right| dt \\
 &= \frac{1}{b-a} \int_a^b \left| t - \frac{a+b}{2} \right| dt = \frac{1}{4} (b-a),
 \end{aligned}$$

hence by (2.1) we get

$$\begin{aligned}
(2.9) \quad & \|D_{w_0}(A, B)\|_1 \\
& \leq \left[D_{w_0} \left(\ell, \int_a^\cdot \|A'(u)\|_p^\alpha du \right) \right]^{1/\alpha} \left[D_{w_0} \left(\ell, \int_a^\cdot \|B'(u)\|_q^\beta du \right) \right]^{1/\beta} \\
& \leq \frac{1}{8} (b-a) \left(\int_a^b \|A'(u)\|_p^\alpha du \right)^{1/\alpha} \left(\int_a^b \|B'(u)\|_q^\beta du \right)^{1/\beta}.
\end{aligned}$$

In particular, if $\left(\int_a^b \|A'(u)\|_p^2 du \right)^{1/2} < \infty$ and $\left(\int_a^b \|B'(u)\|_q^2 du \right)^{1/2} < \infty$, then

$$\begin{aligned}
(2.10) \quad & \|D_{w_0}(A, B)\|_1 \\
& \leq \left[D_{w_0} \left(\ell, \int_a^\cdot \|A'(u)\|_p^2 du \right) \right]^{1/2} \left[D_w \left(\ell, \int_a^\cdot \|B'(u)\|_q^2 du \right) \right]^{1/2} \\
& \leq \frac{1}{8} (b-a) \left(\int_a^b \|A'(u)\|_p^2 du \right)^{1/2} \left(\int_a^b \|B'(u)\|_q^2 du \right)^{1/2}.
\end{aligned}$$

For $p = q = 2$ we also have

$$\begin{aligned}
(2.11) \quad & \|D_{w_0}(A, B)\|_1 \\
& \leq \left[D_{w_0} \left(\ell, \int_a^\cdot \|A'(u)\|_2^2 du \right) \right]^{1/2} \left[D_w \left(\ell, \int_a^\cdot \|B'(u)\|_2^2 du \right) \right]^{1/2} \\
& \leq \frac{1}{8} (b-a) \left(\int_a^b \|A'(u)\|_2^2 du \right)^{1/2} \left(\int_a^b \|B'(u)\|_2^2 du \right)^{1/2}.
\end{aligned}$$

We also have:

Theorem 4. Assume that $w : [a, b] \rightarrow [0, \infty)$ is integrable with $\int_a^b w(t) dt = 1$ and for $p \geq 1$, $A, B : [a, b] \rightarrow B_p(H)$ are strongly differentiable with $\left(\int_a^b \|A'(u)\|_p^\alpha du \right)^{1/\alpha} <$

∞ and $\left(\int_a^b \|B'(u)\|_p^\beta du\right)^{1/\beta} < \infty$ for $\alpha, \beta > 1$ with $\frac{1}{\alpha} + \frac{1}{\beta} = 1$. Then

$$\begin{aligned}
 (2.12) \quad & \|D_w(A, B)\|_1 \\
 & \leq \left[D_w \left(\ell, \int_a^\cdot \|A'(u)\|_p^\alpha du \right) \right]^{1/\alpha} \left[D_w \left(\ell, \int_a^\cdot \|B'(u)\|_p^\beta du \right) \right]^{1/\beta} \\
 & \leq \frac{1}{2} \int_a^b w(t) \left| t - \int_a^b sw(s) ds \right| dt \\
 & \times \left(\int_a^b \|A'(u)\|_p^\alpha du \right)^{1/\alpha} \left(\int_a^b \|B'(u)\|_p^\beta du \right)^{1/\beta} \\
 & \leq \frac{1}{2} \left[\int_a^b s^2 w(s) ds - \left(\int_a^b sw(s) ds \right)^2 \right]^{\frac{1}{2}} \\
 & \times \left(\int_a^b \|A'(u)\|_p^\alpha du \right)^{1/\alpha} \left(\int_a^b \|B'(u)\|_p^\beta du \right)^{1/\beta} \\
 & \leq \frac{1}{4} (b-a) \left(\int_a^b \|A'(u)\|_p^\alpha du \right)^{1/\alpha} \left(\int_a^b \|B'(u)\|_p^\beta du \right)^{1/\beta}.
 \end{aligned}$$

In particular, if $\left(\int_a^b \|A'(u)\|_p^2 du\right)^{1/2} < \infty$ and $\left(\int_a^b \|B'(u)\|_p^2 du\right)^{1/2} < \infty$, then

$$\begin{aligned}
 (2.13) \quad & \|D_w(A, B)\|_1 \\
 & \leq \left[D_w \left(\ell, \int_a^\cdot \|A'(u)\|_p^2 du \right) \right]^{1/2} \left[D_w \left(\ell, \int_a^\cdot \|B'(u)\|_p^2 du \right) \right]^{1/2} \\
 & \leq \frac{1}{2} \int_a^b w(t) \left| t - \int_a^b sw(s) ds \right| dt \\
 & \times \left(\int_a^b \|A'(u)\|_p^2 du \right)^{1/2} \left(\int_a^b \|B'(u)\|_p^2 du \right)^{1/2} \\
 & \leq \frac{1}{2} \left[\int_a^b s^2 w(s) ds - \left(\int_a^b sw(s) ds \right)^2 \right]^{\frac{1}{2}} \\
 & \times \left(\int_a^b \|A'(u)\|_p^2 du \right)^{1/2} \left(\int_a^b \|B'(u)\|_p^2 du \right)^{1/2} \\
 & \leq \frac{1}{4} (b-a) \left(\int_a^b \|A'(u)\|_p^2 du \right)^{1/2} \left(\int_a^b \|B'(u)\|_p^2 du \right)^{1/2}.
 \end{aligned}$$

Proof. If we take the p -norm, use the integral's properties and the inequality (1.13), then we get

$$\begin{aligned}
(2.14) \quad & \|D_w(A, B)\|_p \\
& \leq \frac{1}{2} \int_a^b \int_a^b w(t) w(s) \|[A(t) - A(s)][B(t) - B(s)]\|_p dt ds \\
& \leq \frac{1}{2} \int_a^b \int_a^b w(t) w(s) \|A(t) - A(s)\|_p \|B(t) - B(s)\|_p dt ds.
\end{aligned}$$

Since by Hölder's inequality

$$\begin{aligned}
& \|A(t) - A(s)\|_p \|B(t) - B(s)\|_p \\
& = \left\| \int_s^t A'(u) du \right\|_p \left\| \int_s^t B'(u) du \right\|_p \\
& \leq \left| \int_s^t \|A'(u)\|_p du \right| \left| \int_s^t \|B'(u)\|_p du \right| \\
& \leq |t - s|^{1/\beta} \left| \int_s^t \|A'(u)\|_p^\alpha du \right|^{1/\alpha} |t - s|^{1/\alpha} \left| \int_s^t \|B'(u)\|_p^\beta du \right|^{1/\beta} \\
& = |t - s| \left| \int_s^t \|A'(u)\|_p^\alpha du \right|^{1/\alpha} \left| \int_s^t \|B'(u)\|_p^\beta du \right|^{1/\beta}
\end{aligned}$$

hence

$$\begin{aligned}
& \|D_w(A, B)\|_1 \\
& \leq \frac{1}{2} \int_a^b \int_a^b w(s) w(t) \|A(t) - A(s)\|_p \|B(t) - B(s)\|_p dt ds \\
& \leq \frac{1}{2} \left[2D_w \left(\ell, \int_a^\cdot \|A'(u)\|_p^\alpha du \right) \right]^{1/\alpha} \left[2D_w \left(\ell, \int_a^\cdot \|B'(u)\|_p^\beta du \right) \right]^{1/\beta} \\
& = \left[D_w \left(\ell, \int_a^\cdot \|A'(u)\|_p^\alpha du \right) \right]^{1/\alpha} \left[D_w \left(\ell, \int_a^\cdot \|B'(u)\|_p^\beta du \right) \right]^{1/\beta}.
\end{aligned}$$

Further, by utilising a similar argument to the one in the proof of Theorem ?? we derive (2.12). \square

Remark 3. *With the assumptions of Theorem 3 we get*

$$\begin{aligned}
(2.15) \quad & \|D_{w_0}(A, B)\|_p \\
& \leq \left[D_{w_0} \left(\ell, \int_a^\cdot \|A'(u)\|_p^\alpha du \right) \right]^{1/\alpha} \left[D_{w_0} \left(\ell, \int_a^\cdot \|B'(u)\|_p^\beta du \right) \right]^{1/\beta} \\
& \leq \frac{1}{8} (b - a) \left(\int_a^b \|A'(u)\|_p^\alpha du \right)^{1/\alpha} \left(\int_a^b \|B'(u)\|_p^\beta du \right)^{1/\beta}.
\end{aligned}$$

In particular, if $\left(\int_a^b \|A'(u)\|_p^2 du\right)^{1/2} < \infty$ and $\left(\int_a^b \|B'(u)\|_p^2 du\right)^{1/2} < \infty$, then

$$\begin{aligned}
 (2.16) \quad & \|D_{w_0}(A, B)\|_p \\
 & \leq \left[D_{w_0} \left(\ell, \int_a^\cdot \|A'(u)\|_p^2 du \right) \right]^{1/2} \left[D_w \left(\ell, \int_a^\cdot \|B'(u)\|_p^2 du \right) \right]^{1/2} \\
 & \leq \frac{1}{8} (b-a) \left(\int_a^b \|A'(u)\|_p^2 du \right)^{1/2} \left(\int_a^b \|B'(u)\|_p^2 du \right)^{1/2}.
 \end{aligned}$$

3. THE CASE OF INFINITE INTERVALS

Similar results may be stated for the probability distributions that are supported on the whole axis $\mathbb{R} = (-\infty, \infty)$. Namely, if $I = (-\infty, \infty)$ and $w(s) > 0$ for $s \in \mathbb{R}$ with $\int_{-\infty}^{\infty} w(s) ds = 1$, i.e. w is a probability density function on $(-\infty, \infty)$, then for $A, B : (-\infty, \infty) \rightarrow B(H)$ we can consider the functional

$$D_{w, \mathbb{R}}(A, B) := \int_{-\infty}^{\infty} w(t) A(t) B(t) dt - \int_{-\infty}^{\infty} w(t) A(t) dt \int_{-\infty}^{\infty} w(t) B(t) dt.$$

From (2.1) we then have

$$\begin{aligned}
 (3.1) \quad & \|D_w(A, B)\|_1 \\
 & \leq D_w \left(\ell, \int_{-\infty}^\cdot \|A'(u)\|_p du \right) D_w \left(\ell, \int_{-\infty}^\cdot \|B'(u)\|_q du \right) \\
 & \leq \frac{1}{2} \int_{-\infty}^{\infty} w(t) \left| t - \int_{-\infty}^{\infty} sw(s) ds \right| dt \int_{-\infty}^{\infty} \|A'(u)\|_p \int_{-\infty}^{\infty} \|B'(u)\|_q du \\
 & \leq \frac{1}{2} \left[\int_{-\infty}^{\infty} s^2 w(s) ds - \left(\int_{-\infty}^{\infty} sw(s) ds \right)^2 \right]^{\frac{1}{2}} \\
 & \quad \times \int_{-\infty}^{\infty} \|A'(u)\|_p \int_{-\infty}^{\infty} \|B'(u)\|_q du \\
 & \leq \frac{1}{4} \int_{-\infty}^{\infty} \|A'(u)\|_p \int_{-\infty}^{\infty} \|B'(u)\|_q du,
 \end{aligned}$$

with the assumptions that for $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, $A : (-\infty, \infty) \rightarrow B_p(H)$, $B : (-\infty, \infty) \rightarrow B_q(H)$ are strongly differentiable, $\int_{-\infty}^{\infty} \|A'(u)\|_p < \infty$ and $\int_{-\infty}^{\infty} \|B'(u)\|_q du < \infty$.

For $p = q = 2$ we derive

$$(3.2) \quad \|D_w(A, B)\|_1 \leq D_w \left(\ell, \int_{-\infty}^\cdot \|A'(u)\|_2 du \right) D_w \left(\ell, \int_{-\infty}^\cdot \|B'(u)\|_2 du \right)$$

$$\begin{aligned}
&\leq \frac{1}{2} \int_{-\infty}^{\infty} w(t) \left| t - \int_{-\infty}^{\infty} sw(s) ds \right| dt \int_{-\infty}^{\infty} \|A'(u)\|_2 du \int_{-\infty}^{\infty} \|B'(u)\|_2 du \\
&\leq \frac{1}{2} \left[\int_{-\infty}^{\infty} s^2 w(s) ds - \left(\int_{-\infty}^{\infty} sw(s) ds \right)^2 \right]^{\frac{1}{2}} \\
&\quad \times \int_{-\infty}^{\infty} \|A'(u)\|_2 du \int_{-\infty}^{\infty} \|B'(u)\|_2 du \\
&\leq \frac{1}{4} \int_{-\infty}^{\infty} \|A'(u)\|_2 du \int_{-\infty}^{\infty} \|B'(u)\|_2 du,
\end{aligned}$$

with the assumptions that $A, B : (-\infty, \infty) \rightarrow B_2(H)$ are strongly differentiable, $\int_{-\infty}^{\infty} \|A'(u)\|_2 du < \infty$ and $\int_{-\infty}^{\infty} \|B'(u)\|_2 du < \infty$.

The probability density of the *normal distribution* on $(-\infty, \infty)$ is

$$w_{\mu, \sigma^2}(x) := \frac{1}{\sqrt{2\pi\sigma}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right), \quad x \in \mathbb{R},$$

where μ is the *mean* or *expectation* of the distribution (and also its *median* and *mode*), σ is the *standard deviation*, and σ^2 is the *variance*.

The cumulative distribution function is

$$W_{\mu, \sigma^2}(x) = \frac{1}{2} + \frac{1}{2} \operatorname{erf}\left(\frac{x-\mu}{\sigma\sqrt{2}}\right),$$

where the *error function* erf is defined by

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x \exp(-t^2) dt.$$

Consider the functional

$$\begin{aligned}
&D_{N, \sigma, \mu}(A, B) \\
&:= \frac{1}{\sqrt{2\pi\sigma}} \int_{-\infty}^{\infty} \exp\left(-\frac{(t-\mu)^2}{2\sigma^2}\right) A(t) B(t) dt \\
&\quad - \frac{1}{\sqrt{2\pi\sigma}} \int_{-\infty}^{\infty} \exp\left(-\frac{(t-\mu)^2}{2\sigma^2}\right) A(t) dt \frac{1}{\sqrt{2\pi\sigma}} \int_{-\infty}^{\infty} \exp\left(-\frac{(t-\mu)^2}{2\sigma^2}\right) B(t) dt
\end{aligned}$$

with the parameters μ and σ as above.

Then from (3.1) we get

$$\begin{aligned}
(3.3) \quad &\|D_{N, \sigma, \mu}(A, B)\|_1 \\
&\leq D_{N, \sigma, \mu}\left(\ell, \int_{-\infty}^{\infty} \|A'(u)\|_p du\right) D_{N, \sigma, \mu}\left(\ell, \int_{-\infty}^{\infty} \|B'(u)\|_q du\right) \\
&\leq \frac{1}{2\sqrt{2\pi\sigma}} \int_{-\infty}^{\infty} \exp\left(-\frac{(t-\mu)^2}{2\sigma^2}\right) |t-\mu| dt \\
&\quad \times \int_{-\infty}^{\infty} \|A'(u)\|_p \int_{-\infty}^{\infty} \|B'(u)\|_q du \\
&\leq \frac{1}{2}\sigma \int_{-\infty}^{\infty} \|A'(u)\|_p \int_{-\infty}^{\infty} \|B'(u)\|_q du
\end{aligned}$$

with the assumptions that for $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, $A : (-\infty, \infty) \rightarrow B_p(H)$, $B : (-\infty, \infty) \rightarrow B_q(H)$ are strongly differentiable, $\sup_{u \in (-\infty, \infty)} \|A'(u)\|_p < \infty$ and $\int_{-\infty}^{\infty} \|B'(u)\|_q du < \infty$.

In particular, for $p = q = 2$ we get

$$\begin{aligned}
 (3.4) \quad & \|D_{N, \sigma, \mu}(A, B)\|_1 \\
 & \leq D_{N, \sigma, \mu} \left(\ell, \int_{-\infty}^{\cdot} \|A'(u)\|_2 du \right) D_{N, \sigma, \mu} \left(\ell, \int_{-\infty}^{\cdot} \|B'(u)\|_2 du \right) \\
 & \leq \frac{1}{2\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} \exp\left(-\frac{(t-\mu)^2}{2\sigma^2}\right) |t-\mu| dt \\
 & \quad \times \int_{-\infty}^{\infty} \|A'(u)\|_2 du \int_{-\infty}^{\infty} \|B'(u)\|_2 du \\
 & \leq \frac{1}{2}\sigma \int_{-\infty}^{\infty} \|A'(u)\|_2 du \int_{-\infty}^{\infty} \|B'(u)\|_2 du
 \end{aligned}$$

with the assumptions that $A, B : (-\infty, \infty) \rightarrow B_2(H)$ are strongly differentiable, $\sup_{u \in (-\infty, \infty)} \|A'(u)\|_2 < \infty$ and $\int_{-\infty}^{\infty} \|B'(u)\|_2 du < \infty$.

4. INEQUALITIES FOR OPERATOR MONOTONE FUNCTIONS

A real valued continuous function h on $[0, \infty)$ is said to be operator monotone if $h(A) \geq h(B)$ holds for any $A \geq B \geq 0$.

We have the following representation of operator monotone functions, see for instance [2, p. 144-145]:

Theorem 5. *A function $h : [0, \infty) \rightarrow \mathbb{R}$ is operator monotone in $[0, \infty)$ if and only if it has the representation*

$$(4.1) \quad h(t) = h(0) + bt + \int_0^{\infty} \frac{t\lambda}{t+\lambda} d\mu(\lambda),$$

where $b \geq 0$ and a positive measure μ on $[0, \infty)$ such that

$$(4.2) \quad \int_0^{\infty} \frac{\lambda}{1+\lambda} d\mu(\lambda) < \infty.$$

We have the following representation result:

Lemma 1. *Let $h : [0, \infty) \rightarrow \mathbb{R}$ be operator monotone in $[0, \infty)$. Assume that $U \geq 0$, then for all selfadjoint operators V we have*

$$(4.3) \quad Dh(U)(V) = bV + \int_0^{\infty} \lambda^2 [(\lambda + U)^{-1} V (\lambda + U)^{-1}] d\mu(\lambda).$$

Proof. From (4.1) we get

$$h(t) = h(0) + bt + \int_0^{\infty} \left(\lambda - \frac{\lambda^2}{t+\lambda} \right) d\mu(\lambda).$$

Assume that $U \geq 0$, then for all selfadjoint operator V we have, by the representation of h and for t in a small open interval around 0, that

$$\begin{aligned}
& h(U + tV) - h(U) \\
&= btV + \int_0^\infty \left(\lambda - \lambda^2 (U + tV + \lambda)^{-1} \right) d\mu(\lambda) - \int_0^\infty \left(\lambda - \lambda^2 (U + \lambda)^{-1} \right) d\mu(\lambda) \\
&= btV + \int_0^\infty \lambda^2 \left[(\lambda + U)^{-1} - (\lambda + U + tV)^{-1} \right] d\mu(\lambda) \\
&= btV + \int_0^\infty \lambda^2 \left[(\lambda + U)^{-1} (\lambda + U + tV - \lambda - U) (\lambda + U + tV)^{-1} \right] d\mu(\lambda) \\
&= btV + t \int_0^\infty \lambda^2 \left[(\lambda + U)^{-1} V (\lambda + U + tV)^{-1} \right] d\mu(\lambda).
\end{aligned}$$

Dividing by $t \neq 0$, we get

$$\frac{h(U + tV) - h(U)}{t} = bV + \int_0^\infty \lambda^2 \left[(\lambda + U)^{-1} V (\lambda + U + tV)^{-1} \right] d\mu(\lambda)$$

and by taking the limit over $t \rightarrow 0$, we get

$$Dh(U)(V) = bV + \int_0^\infty \lambda^2 \left[(\lambda + U)^{-1} V (\lambda + U)^{-1} \right] d\mu(\lambda)$$

for all selfadjoint operator V we have (4.3). \square

Lemma 2. *Let $h : [0, \infty) \rightarrow \mathbb{R}$ be operator monotone in $[0, \infty)$. Assume that $U \geq u > 0$, then for all selfadjoint operators $V \in B_p(H)$, $p \geq 1$ we have*

$$(4.4) \quad \|Dh(U)(V)\|_p \leq h'(u) \|V\|_p.$$

Proof. From (4.3) and using (1.15) we get

$$\begin{aligned}
(4.5) \quad \|Dh(U)(V) - bV\|_p &\leq \int_0^\infty \lambda^2 \left\| (\lambda + U)^{-1} V (\lambda + U)^{-1} \right\|_p d\mu(\lambda) \\
&\leq \|V\|_p \int_0^\infty \lambda^2 \left\| (\lambda + U)^{-1} \right\|_p^2 d\mu(\lambda).
\end{aligned}$$

Observe that $\lambda + U \geq \lambda + u > 0$ for $\lambda \in [0, \infty)$. Then $0 < (\lambda + U)^{-1} \leq (\lambda + u)^{-1}$, which implies that $\left\| (\lambda + U)^{-1} \right\|_p \leq (\lambda + u)^{-1}$, namely $\left\| (\lambda + U)^{-1} \right\|_p^2 \leq (\lambda + u)^{-2}$ for $\lambda \in [0, \infty)$.

Therefore by (4.5) we get

$$(4.6) \quad \|Dh(U)(V) - bV\|_p \leq \|V\|_p \int_0^\infty \lambda^2 (\lambda + u)^{-2} d\mu(\lambda).$$

If we take the derivative over t in (4.1) then we have

$$(4.7) \quad h'(t) = b + \int_0^\infty \frac{\lambda(t + \lambda) - \lambda t}{(t + \lambda)^2} d\mu(\lambda) = b + \int_0^\infty \frac{\lambda^2}{(t + \lambda)^2} d\mu(\lambda)$$

for $t > 0$.

From (4.7) we get

$$\int_0^\infty \lambda^2 (\lambda + u)^{-2} d\mu(\lambda) = h'(u) - b,$$

and by (4.6) we derive

$$\|Dh(U)(V) - bV\|_p \leq \|V\|_p \|h'(u) - b\|_p.$$

Finally, by the triangle inequality and by the fact that $b \geq 0$, we obtain that

$$\|Dh(U)(V)\|_p - b\|V\|_p \leq \|Dh(U)(V) - bV\|_p,$$

which proves the desired result (4.4). \square

For a continuous function h on $(0, \infty)$ and $C, E > 0$ we consider the auxiliary function $h_{C,E} : [0, 1] \rightarrow \mathbb{R}$ defined by

$$h_{C,E}(t) := h((1-t)C + tE), \quad t \in [0, 1].$$

We have the following representations of the derivatives:

Lemma 3. *Assume that the operator function generated by h is Fréchet differentiable in each $C \geq 0$, then for $E \geq 0$ we have that $h_{C,E}$ is differentiable on $[0, 1]$ and*

$$(4.8) \quad h'_{C,E}(t) = D(h)((1-t)C + tE)(E - C)$$

for $t \in [0, 1]$, where in 0 and 1 the derivatives are the right and left derivatives.

Proof. We prove only for the interior points $t \in (0, 1)$. Let h be in a small interval around 0 such that $t + h \in (0, 1)$. Then for $h \neq 0$,

$$\begin{aligned} & \frac{h_{C,E}(t+h) - h_{C,E}(t)}{h} \\ &= \frac{h((1-(t+h))C + (t+h)E) - h((1-t)C + tE)}{h} \\ &= \frac{h((1-t)C + tE + h(E - C)) - h((1-t)C + tE)}{h} \end{aligned}$$

and by taking the limit over $h \rightarrow 0$, we get

$$\begin{aligned} h'_{C,E}(t) &= \lim_{h \rightarrow 0} \frac{h_{C,E}(t+h) - h_{C,E}(t)}{h} \\ &= \lim_{h \rightarrow 0} \left[\frac{h((1-t)C + tE + h(E - C)) - h((1-t)C + tE)}{h} \right] \\ &= D(h)((1-t)C + tE)(E - C), \end{aligned}$$

which proves (4.8). \square

Corollary 1. *Let $h : [0, \infty) \rightarrow \mathbb{R}$ be operator monotone in $[0, \infty)$. Then for all $C \geq c > 0$, $E \geq d > 0$ with $C, E \in B_p(H)$, $p \geq 1$ we have*

$$(4.9) \quad \begin{aligned} \|h'_{C,E}(t)\|_p &= \|D(h)((1-t)C + tE)(E - C)\|_p \\ &\leq h'((1-t)c + td) \|E - C\|_p \end{aligned}$$

for all $t \in [0, 1]$.

The proof follows by Lemma 2 and Lemma 3.

Consider a probability density function w on $[a, b]$, i.e., $w \geq 0$ a.e. on $[a, b]$ with $\int_a^b w(t) dt = 1$. For the operator monotone functions $h, k : [0, \infty) \rightarrow \mathbb{R}$ and two operators $C, E \geq 0$ we consider the *Čebyšev functional*

$$D_w(h, k, C, E) := \int_0^1 w(t) h((1-t)C + tE) k((1-t)C + tE) dt \\ - \int_0^1 w(t) h((1-t)C + tE) dt \int_0^1 w(t) k((1-t)C + tE) dt.$$

For $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ assume that $C \geq c > 0, E \geq d > 0$ with $C, E \in B_p(H) \cap B_q(H)$. If we use Theorem 3 and Corollary 1 we have for $\alpha, \beta > 1$ with $\frac{1}{\alpha} + \frac{1}{\beta} = 1$ that

$$(4.10) \quad \|D_w(h, k, C, E)\|_1 \\ \leq \frac{1}{2} \|E - C\|_p \|E - C\|_q \int_a^b w(t) \left| t - \int_a^b sw(s) ds \right| dt \\ \times \left(\int_a^b [h'((1-t)c + td)]^\alpha dt \right)^{1/\alpha} \left(\int_a^b [k'((1-t)c + td)]^\beta du \right)^{1/\beta} \\ \leq \frac{1}{2} \|E - C\|_p \|E - C\|_q \left[\int_a^b s^2 w(s) ds - \left(\int_a^b sw(s) ds \right)^2 \right]^{\frac{1}{2}} \\ \times \left(\int_a^b [h'((1-t)c + td)]^\alpha dt \right)^{1/\alpha} \left(\int_a^b [k'((1-t)c + td)]^\beta du \right)^{1/\beta} \\ \leq \frac{1}{4} \|E - C\|_p \|E - C\|_q \\ \times \left(\int_a^b [h'((1-t)c + td)]^\alpha dt \right)^{1/\alpha} \left(\int_a^b [k'((1-t)c + td)]^\beta du \right)^{1/\beta}.$$

In particular, for $C, E \in B_2(H)$ and $C \geq c > 0, E \geq d > 0$ we have

$$(4.11) \quad \|D_w(h, k, C, E)\|_1 \\ \leq \frac{1}{2} \|E - C\|_2^2 \int_a^b w(t) \left| t - \int_a^b sw(s) ds \right| dt \\ \times \left(\int_a^b [h'((1-t)c + td)]^\alpha dt \right)^{1/\alpha} \left(\int_a^b [k'((1-t)c + td)]^\beta du \right)^{1/\beta}$$

$$\begin{aligned}
 &\leq \frac{1}{2} \|E - C\|_2^2 \left[\int_a^b s^2 w(s) ds - \left(\int_a^b s w(s) ds \right)^2 \right]^{\frac{1}{2}} \\
 &\times \left(\int_a^b [h'((1-t)c + td)]^\alpha dt \right)^{1/\alpha} \left(\int_a^b [k'((1-t)c + td)]^\beta du \right)^{1/\beta} \\
 &\leq \frac{1}{4} \|E - C\|_2^2 \\
 &\times \left(\int_a^b [h'((1-t)c + td)]^\alpha dt \right)^{1/\alpha} \left(\int_a^b [k'((1-t)c + td)]^\beta du \right)^{1/\beta}.
 \end{aligned}$$

Moreover, for $\alpha = \beta = 2$ we get

$$\begin{aligned}
 (4.12) \quad &\|D_w(h, k, C, E)\|_1 \\
 &\leq \frac{1}{2} \|E - C\|_2^2 \int_a^b w(t) \left| t - \int_a^b s w(s) ds \right| dt \\
 &\times \left(\int_a^b [h'((1-t)c + td)]^2 dt \right)^{1/2} \left(\int_a^b [k'((1-t)c + td)]^2 du \right)^{1/2} \\
 &\leq \frac{1}{2} \|E - C\|_2^2 \left[\int_a^b s^2 w(s) ds - \left(\int_a^b s w(s) ds \right)^2 \right]^{\frac{1}{2}} \\
 &\times \left(\int_a^b [h'((1-t)c + td)]^2 dt \right)^{1/2} \left(\int_a^b [k'((1-t)c + td)]^2 du \right)^{1/2} \\
 &\leq \frac{1}{4} \|E - C\|_2^2 \\
 &\times \left(\int_a^b [h'((1-t)c + td)]^2 dt \right)^{1/2} \left(\int_a^b [k'((1-t)c + td)]^2 du \right)^{1/2}.
 \end{aligned}$$

If we consider the uniform distribution $w_0(t) = \frac{1}{b-a}$, $t \in [a, b]$ then

$$\begin{aligned}
 &D_{w_0}(h, k, C, E) \\
 &= \frac{1}{b-a} \int_0^1 h((1-t)C + tE) k((1-t)C + tE) dt \\
 &\quad - \frac{1}{b-a} \int_0^1 h((1-t)C + tE) dt \frac{1}{b-a} \int_0^1 k((1-t)C + tE) dt.
 \end{aligned}$$

and by (4.10) we derive

$$\begin{aligned}
 (4.13) \quad &\|D_w(h, k, C, E)\|_1 \\
 &\leq \frac{1}{8} (b-a) \|E - C\|_p \|E - C\|_q \\
 &\times \left(\int_a^b [h'((1-t)c + td)]^\alpha dt \right)^{1/\alpha} \left(\int_a^b [k'((1-t)c + td)]^\beta du \right)^{1/\beta},
 \end{aligned}$$

from (4.11) we get

$$(4.14) \quad \begin{aligned} & \|D_w(h, k, C, E)\|_1 \\ & \leq \frac{1}{8} \|E - C\|_2^2 \\ & \quad \times \left(\int_a^b [h'((1-t)c + td)]^\alpha dt \right)^{1/\alpha} \left(\int_a^b [k'((1-t)c + td)]^\beta du \right)^{1/\beta} \end{aligned}$$

while from (4.12) we obtain

$$(4.15) \quad \begin{aligned} & \|D_w(h, k, C, E)\|_1 \\ & \leq \frac{1}{8} \|E - C\|_2^2 \\ & \quad \times \left(\int_a^b [h'((1-t)c + td)]^2 dt \right)^{1/2} \left(\int_a^b [k'((1-t)c + td)]^2 du \right)^{1/2}. \end{aligned}$$

If we consider the operator monotone functions $h(t) = t^m$, $k(t) = t^n$ with $m, n \in (0, 1)$ and put

$$\begin{aligned} D_w(m, n, C, E) & := \int_0^1 w(t) ((1-t)C + tE)^{m+n} dt \\ & \quad - \int_0^1 w(t) ((1-t)C + tE)^m dt \int_0^1 w(t) ((1-t)C + tE)^n dt. \end{aligned}$$

From (4.10) we get

$$(4.16) \quad \begin{aligned} & \|D_w(m, n, C, E)\|_1 \\ & \leq \frac{1}{2} \|E - C\|_p \|E - C\|_q \int_a^b w(t) \left| t - \int_a^b sw(s) ds \right| dt \\ & \quad \times mn \left(\frac{d^{(m-1)\alpha+1} - a^{(m-1)\alpha+1}}{[(m-1)\alpha+1](d-a)} \right)^{1/\alpha} \left(\int_a^b \frac{d^{(n-1)\beta+1} - a^{(n-1)\beta+1}}{[(n-1)\beta+1](d-a)} \right)^{1/\beta} \end{aligned}$$

provided that $(m-1)\alpha+1, (n-1)\beta+1 \neq 0$.

The case of uniform distributions is as follows

$$(4.17) \quad \begin{aligned} & \|D_{w_0}(m, n, C, E)\|_1 \\ & \leq \frac{1}{8} (b-a) \|E - C\|_p \|E - C\|_q \\ & \quad \times mn \left(\frac{d^{(m-1)\alpha+1} - a^{(m-1)\alpha+1}}{[(m-1)\alpha+1](d-a)} \right)^{1/\alpha} \left(\int_a^b \frac{d^{(n-1)\beta+1} - a^{(n-1)\beta+1}}{[(n-1)\beta+1](d-a)} \right)^{1/\beta}, \end{aligned}$$

where

$$\begin{aligned} & D_{w_0}(m, n, C, E) \\ & = \frac{1}{b-a} \int_0^1 ((1-t)C + tE)^{m+n} dt \\ & \quad - \frac{1}{b-a} \int_0^1 ((1-t)C + tE)^m dt \frac{1}{b-a} \int_0^1 ((1-t)C + tE)^n dt. \end{aligned}$$

REFERENCES

- [1] R. Bellman, Some inequalities for positive definite matrices, in: E.F. Beckenbach (Ed.), *General Inequalities 2*, Proceedings of the 2nd International Conference on General Inequalities, Birkhäuser, Basel, 1980, pp. 89–90.
- [2] R. Bhatia, *Matrix Analysis*. Graduate Texts in Mathematics, 169. Springer-Verlag, New York, 1997. xii+347 pp. ISB
- [3] N. S. Barnett, C. Buşe, P. Cerone and S. S. Dragomir, Ostrowski’s inequality for vector-valued functions and applications, *Computers and Mathematics with Applications* **44** (2002), 559–572.
- [4] P. Cerone and S. S. Dragomir, New bounds for the Čebyšev functional, *Appl. Math. Lett.*, **18** (2005), 603–611.
- [5] P. Cerone and S. S. Dragomir, A refinement of the Grüss inequality and applications, *Tamkang J. Math.* **38**(2007), No. 1, 37–49. Preprint available at *RGMA Res. Rep. Coll.*, **5**(2) (2002), Art. 14. [ONLINE <http://rgmia.vu.edu.au/v5n2.htm>
- [6] D. Chang, A matrix trace inequality for products of Hermitian matrices, *J. Math. Anal. Appl.* **237** (1999) 721–725.
- [7] I. D. Coop, On matrix trace inequalities and related topics for products of Hermitian matrix, *J. Math. Anal. Appl.* **188** (1994) 999–1001.
- [8] P. L. Chebyshev, Sur les expressions approximatives des intégrals définis par les autres prises entre les même limites, *Proc. Math. Soc. Charkov*, **2** (1882), 93–98.
- [9] X.-L. Cheng and J. Sun, Note on the perturbed trapezoid inequality, *J. Inequal. Pure Appl. Math.*, **3**(2) (2002), Art. 29. [ONLINE <http://jipam.vu.edu.au/article.php?sid=181>
- [10] S. S. Dragomir, New Grüss’ type inequalities for functions of bounded variation and applications. *Appl. Math. Lett.* **25** (2012), no. 10, 1475–1479.
- [11] G. Grüss, Über das maximum des absoluten Betrages von $\frac{1}{b-a} \int_a^b f(x)g(x)dx - \frac{1}{(b-a)^2} \int_a^b f(x)dx \cdot \int_a^b g(x)dx$, *Math. Z.*, **39** (1934), 215–226.
- [12] A. Lupaş, The best constant in an integral inequality, *Mathematica (Cluj)*, **15** (38) (1973), No. 2, 219–222.
- [13] D. S. Mitrinović, J. E. Pečarić and A. M. Fink, *Classical and New Inequalities in Analysis*, Kluwer Academic Publishers, Dordrecht, 1993.
- [14] H. Neudecker, A matrix trace inequality, *J. Math. Anal. Appl.* **166** (1992) 302–303.
- [15] A. M. Ostrowski, On an integral inequality, *Aequationes Math.*, **4** (1970), 358–373.
- [16] B. Simon, *Trace Ideals and Their Applications*, Cambridge University Press, Cambridge, 1979.
- [17] V. A. Zagrebov, *Gibbs Semigroups*, Operator Theory: Advances and Applications, Volume 273, Birkhäuser, 2019

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