p-SCHATTEN NORM INEQUALITIES OF LUPAŞ TYPE FOR ČEBYŠEV'S FUNCTIONAL WITH COMPLEX WEIGHTS

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Abstract. An operator $A\in\mathcal{B}(H)$ is said to belong to the *von Neumann-Schatten class* $\mathcal{B}_p(H)$, $1\leq p<\infty$ if the *p-Schatten norm* is finite

$$\|A\|_p := [\operatorname{tr}(|A|^p)]^{1/p} < \infty.$$

For $p,\ q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, assume that $\{B_i\}_{i=0}^n\subset B_p(H)$, $\{C_i\}_{i=0}^n\subset B_q(H)$ are sequences of operators and $\alpha_i\in\mathbb{C}\ \ (i=1,...,n)$. Then we have the inequality:

$$\left\|\sum_{i=1}^n \alpha_i \sum_{i=1}^n \alpha_i B_i C_i - \sum_{i=1}^n \alpha_i B_i \sum_{i=1}^n \alpha_i C_i\right\|_1 \leq \max_{1\leq j\leq n-1}\|\Delta B_j\|_p \max_{1\leq j\leq n-1}\|\Delta C_j\|_q \left[\sum_{i=1}^n|\alpha_i|\sum_{i=1}^n i^2|\alpha_i| - \left(\sum_{i=1}^n i|\alpha_i|\right)^2\right],$$

where $\Delta B_i := B_{i+1}-B_i\ \ (i=1,...,n-1)$ are the usual forward difference. Applications for operator functions defined by power series with complex coefficients with applications for the inverse and exponential functions are also provided.

1. Introduction

In 1935, G. Grüss [15] proved the following integral inequality which gives an approximation of the integral mean of the product in terms of the product of the integral means integrals as follows:

$$\left|\frac{1}{b-a}\int_a^b f(x)g(x)\,dx - \frac{1}{b-a}\int_a^b f(x)\,dx\cdot\frac{1}{b-a}\int_a^b g(x)\,dx\right| \leq \frac{1}{4}(\Phi-\varphi)(\Gamma-\gamma) \tag{1.1}$$

where $f,g:[a,b]\to\mathbb{R}$ are integrable on $[a,b]$ and satisfying the assumption

$$\varphi\leq f(x)\leq\Phi, \gamma\leq g(x)\leq\Gamma$$

for each $x\in[a,b]$ where $\varphi,\Phi,\gamma,\Gamma$ are given real constants.

Moreover the constant $\frac{1}{4}$ is sharp in the sense that it can not be replaced by a smaller one.

For a simple proof of (1.1) as well as for some other integral inequalities of Grüss' type see the Chapter X of the recent book [16] by Mitrinović, Pečarić and Fink.

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In 1950, M. Biernacki, H. Pidek and C. Ryll-Nardzewski [3] established the following discrete version of Grüss' inequality, see also [16, Ch. X]:

Theorem 1. *Let $a=(a_1,\dots,a_n)$, $b=(b_1,\dots,b_n)$ be two n-tuples of real numbers such that $r\le a_i\le R$ and $s\le b_i\le S$ for $i=1,\dots,n$. Then one has the inequality:*

$$\left|\frac{1}{n}\sum_{i=1}^{n}a_ib_i-\frac{1}{n}\sum_{i=1}^{n}a_i\cdot\frac{1}{n}\sum_{i=1}^{n}b_i\right| \le\frac{1}{n}\left[\frac{n}{2}\right]\left(1-\frac{1}{n}\left[\frac{n}{2}\right]\right)(R-r)(S-s) \tag{1.2}$$

when $[x]$ is the integer part of x, $x\in\mathbb{R}$.

In 1981, A. Lupaş [16, Ch. X] proved some similar results for the first difference of a as follows :

Theorem 2. *Let a,b two monotonic n-tuples in the same sense and p a positive n-tuple. Then*

$$\begin{aligned}&\min_{1\le i\le n-1}|a_{i+1}-a_i|\min_{1\le i\le n-1}|b_{i+1}-b_i|\left[\frac{1}{P_n}\sum_{i=1}^{n}i^2p_i-\left(\frac{1}{P_n}\sum_{i=1}^{n}ip_i\right)^2\right]\\&\le\frac{1}{P_n}\sum_{i=1}^{n}p_ia_ib_i-\frac{1}{P_n}\sum_{i=1}^{n}p_ia_i\cdot\frac{1}{P_n}\sum_{i=1}^{n}p_ib_i\\&\le\max_{1\le i\le n-1}|a_{i+1}-a_i|\max_{1\le i\le n-1}|b_{i+1}-b_i|\left[\frac{1}{P_n}\sum_{i=1}^{n}i^2p_i-\left(\frac{1}{P_n}\sum_{i=1}^{n}ip_i\right)^2\right].\end{aligned} \tag{1.3}$$

If there exists the numbers $\bar{a},\bar{a}_1,r,r_1,(rr_1>0)$ such that $a_k=\bar{a}+kr$ and $b_k=\bar{a}_1+kr_1$, then in (1.3) the equality holds.

In order to extend these results for p-Schatten norms we need the following preparations.

Let $(H;\langle .,.\rangle)$ be a complex Hilbert space and $\mathcal{B}(H)$ the Banach algebra of all bounded linear operators on H. If $\{e_i\}_{i\in I}$ an orthonormal basis of H, we say that $A\in\mathcal{B}(H)$ is of *trace class* if

$$\|A\|_1:=\sum_{i\in I}\langle|A|e_i,e_i\rangle<\infty. \tag{1.4}$$

The definition of $\|A\|_1$ does not depend on the choice of the orthonormal basis $\{e_i\}_{i\in I}$. We denote by $\mathcal{B}_1(H)$ the set of trace class operators in $\mathcal{B}(H)$.

We define the *trace* of a trace class operator $A\in\mathcal{B}_1(H)$ to be

$$\operatorname{tr}(A):=\sum_{i\in I}\langle Ae_i,e_i\rangle, \tag{1.5}$$

where $\{e_i\}_{i\in I}$ an orthonormal basis of H. Note that this coincides with the usual definition of the trace if H is finite-dimensional. We observe that the series (1.5) converges absolutely and it is independent from the choice of basis.

The following result collects some properties of the trace:

Theorem 3. *We have:*

(i) If $A \in \mathcal{B}_1(H)$ *then* $A^* \in \mathcal{B}_1(H)$ *and*

$$\operatorname{tr}(A^*) = \overline{\operatorname{tr}(A)}; \tag{1.6}$$

(ii) If $A \in \mathcal{B}_1(H)$ *and* $T \in \mathcal{B}(H)$, *then* AT, $TA \in \mathcal{B}_1(H)$ *and*

$$\operatorname{tr}(AT) = \operatorname{tr}(TA) \text{ and } |\operatorname{tr}(AT)| \leq \|A\|_1 \|T\|; \tag{1.7}$$

(iii) $\operatorname{tr}(\cdot)$ *is a bounded linear functional on* $\mathcal{B}_1(H)$ *with* $\|\operatorname{tr}\| = 1$;
(iv) If A, $B \in \mathcal{B}_2(H)$ *then* AB, $BA \in \mathcal{B}_1(H)$ *and* $\operatorname{tr}(AB) = \operatorname{tr}(BA)$;
(v) $\mathcal{B}_{fin}(H)$, *the space of operators of finite rank, is a dense subspace of* $\mathcal{B}_1(H)$.

An operator $A \in \mathcal{B}(H)$ is said to belong to the *von Neumann-Schatten class* $\mathcal{B}_p(H)$, $1 \leq p < \infty$ if the *p-Schatten norm* is finite [20, p. 60-64]

$$\|A\|_p := [\operatorname{tr}(|A|^p)]^{1/p} < \infty.$$

For $1 < p < q < \infty$ we have that

$$\mathcal{B}_1(H) \subset \mathcal{B}_p(H) \subset \mathcal{B}_q(H) \subset \mathcal{B}(H) \tag{1.8}$$

and

$$\|A\|_1 \geq \|A\|_p \geq \|A\|_q \geq \|A\|. \tag{1.9}$$

For $p \geq 1$ the functional $\|\cdot\|_p$ is a *norm* on the $*$-ideal $\mathcal{B}_p(H)$ and $\left(\mathcal{B}_p(H), \|\cdot\|_p\right)$ is a Banach space.

Also, see for instance [20, p. 60-64],

$$\|A\|_p = \|A^*\|_p, \ A \in \mathcal{B}_p(H) \tag{1.10}$$

$$\|AB\|_p \leq \|A\|_p \|B\|_p, \ A, B \in \mathcal{B}_p(H) \tag{1.11}$$

and

$$\|AB\|_p \leq \|A\|_p \|B\|, \ \|BA\|_p \leq \|B\| \|A\|_p, \ A \in \mathcal{B}_p(H), \ B \in \mathcal{B}(H). \tag{1.12}$$

This implies that

$$\|CAB\|_p \leq \|C\| \|A\|_p \|B\|, \ A \in \mathcal{B}_p(H), \ B, \ C \in \mathcal{B}(H). \tag{1.13}$$

In terms of *p-Schatten norm* we have the *Hölder inequality* for p, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$

$$(|\operatorname{tr}(AB)| \leq) \|AB\|_1 \leq \|A\|_p \|B\|_q, \ A \in \mathcal{B}_p(H), \ B \in \mathcal{B}_q(H). \tag{1.14}$$

For the theory of trace functionals and their applications the reader is referred to [19] and [20].

For some classical trace inequalities see [7], [8], and [18], which are continuations of the work of Bellman [1].

2. Some Grüss-Lupaş' Type Inequalities for p-Schatten Norm

The following inequality of Grüss'-Lupaş type in Banach algebras holds:

Theorem 4. *For $p,\ q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, assume that $\{B_i\}_{i=0}^{n}\subset B_p(H)$, $\{C_i\}_{i=0}^{n}\subset B_q(H)$ are sequences of operators and $\alpha_i\in\mathbb{C}$ $(i=1,...,n)$. Then we have the inequality:*

$$\left\|\sum_{i=1}^{n}\alpha_i\sum_{i=1}^{n}\alpha_iB_iC_i-\sum_{i=1}^{n}\alpha_iB_i\sum_{i=1}^{n}\alpha_iC_i\right\|_1 \leq \max_{1\leq j\leq n-1}\|\Delta B_j\|_p\max_{1\leq j\leq n-1}\|\Delta C_j\|_q\left[\sum_{i=1}^{n}|\alpha_i|\sum_{i=1}^{n}i^2|\alpha_i|-\left(\sum_{i=1}^{n}i|\alpha_i|\right)^2\right], \tag{2.1}$$

where $\Delta B_i:=B_{i+1}-B_i$ $(i=1,...,n-1)$ are the usual forward differences.

In particular, if $\{B_i\}_{i=0}^{n},\ \{C_i\}_{i=0}^{n}\subset B_2(H)$, then for $p=q=2$ we have

$$\left\|\sum_{i=1}^{n}\alpha_i\sum_{i=1}^{n}\alpha_iB_iC_i-\sum_{i=1}^{n}\alpha_iB_i\sum_{i=1}^{n}\alpha_iC_i\right\|_1 \leq \max_{1\leq j\leq n-1}\|\Delta B_j\|_2\max_{1\leq j\leq n-1}\|\Delta C_j\|_2\left[\sum_{i=1}^{n}|\alpha_i|\sum_{i=1}^{n}i^2|\alpha_i|-\left(\sum_{i=1}^{n}i|\alpha_i|\right)^2\right]. \tag{2.2}$$

Proof. Let us start with the following identity which can be proved by direct computation:

$$\sum_{i=1}^{n}\alpha_i\sum_{i=1}^{n}\alpha_iB_iC_i-\sum_{i=1}^{n}\alpha_iB_i\sum_{i=1}^{n}\alpha_iC_i =\frac{1}{2}\sum_{i,j=1}^{n}\alpha_i\alpha_j(B_j-B_i)(C_j-C_i)=\sum_{1\leq i<j\leq n}^{n}\alpha_i\alpha_j(B_j-B_i)(C_j-C_i). \tag{2.3}$$

As $i<j$ we can write that $B_j-B_i=\sum_{k=i}^{j-1}\Delta B_k$ and $C_j-C_i=\sum_{k=i}^{j-1}\Delta C_k$. Using the generalized triangle inequality and the property (1.14) we have successively from

(2.3) :

$$\left\| \sum_{i=1}^{n} \alpha_i \sum_{i=1}^{n} \alpha_i B_i C_i \sum_{i=1}^{n} \alpha_i B_i C_i - \sum_{i=1}^{n} \alpha_i B_i \sum_{i=1}^{n} \alpha_i C_i \right\|_1 \tag{2.4}$$

$$= \left\| \sum_{1\le i<j\le n}^{n} \alpha_i \alpha_j \sum_{k=i}^{j-1} \Delta B_k \sum_{k=i}^{j-1} \Delta C_k \right\|_1$$

$$\le \sum_{1\le i<j\le n}^{n} |\alpha_i \alpha_j| \left\| \sum_{k=i}^{j-1} \Delta B_k \sum_{k=i}^{j-1} \Delta C_k \right\|_1$$

$$\le \sum_{1\le i<j\le n}^{n} |\alpha_i \alpha_j| \left\| \sum_{k=i}^{j-1} \Delta B_k \right\|_p \left\| \sum_{k=i}^{j-1} \Delta C_k \right\|_q$$

$$\le \sum_{1\le i<j\le n}^{n} |\alpha_i \alpha_j| \sum_{k=i}^{j-1} \|\Delta B_k\|_p \sum_{k=i}^{j-1} \|\Delta C_k\|_q =: \Gamma.$$

Now, we have

$$\|\Delta B_k\|_p \le \max_{1\le s\le n-1} \|\Delta B_s\|_p \text{ and } \|\Delta C_k\|_q \le \max_{1\le s\le n-1} \|\Delta C_s\|_q$$

for all $k = i, ..., j-1$ and then by summation,

$$\sum_{k=i}^{j-1} \|\Delta B_k\|_p \le (j-i) \max_{1\le s\le n-1} \|\Delta B_s\|_p$$

and

$$\sum_{k=i}^{j-1} \|\Delta C_k\|_q \le (j-i) \max_{1\le s\le n-1} \|\Delta C_s\|_q .$$

Taking into account the above estimations, we can write

$$\Gamma \le \left[\sum_{1\le i<j\le n}^{n} |\alpha_i \alpha_j| (j-i)^2 \right] \max_{1\le s\le n-1} \|\Delta B_s\|_p \max_{1\le s\le n-1} \|\Delta C_s\|_q .$$

As a simple calculation shows that

$$\sum_{1\le i<j\le n}^{n} |\alpha_i| |\alpha_j| (j-i)^2 = \frac{1}{2} \sum_{i,j=1}^{n} |\alpha_i| |\alpha_j| (j-i)^2$$

$$= \frac{1}{2} \sum_{i,j=1}^{n} |\alpha_i| |\alpha_j| \left(i^2 - 2ij + j^2\right)$$

$$= \sum_{i=1}^{n} |\alpha_i| \sum_{i=1}^{n} i^2 |\alpha_i| - \left(\sum_{i=1}^{n} i |\alpha_i| \right)^2$$

and the inequality (2.1) is proved. □

Remark 1. *For p, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, assume that $\{B_i\}_{i=0}^{n} \subset B_p(H) \cap B_q(H)$ is a sequence of operators and $\alpha_i \in \mathbb{C}$ $(i = 1, ..., n)$. Then by taking $C_i = B_i$ in*

(2.1) we have the inequality:

$$\left\| \sum_{i=1}^{n} \alpha_i \sum_{i=1}^{n} \alpha_i B_i^2 - \left(\sum_{i=1}^{n} \alpha_i B_i \right)^2 \right\|_1 \leq \max_{1\leq j\leq n-1} \|\Delta B_j\|_p \max_{1\leq j\leq n-1} \|\Delta B_j\|_q \left[\sum_{i=1}^{n} |\alpha_i| \sum_{i=1}^{n} i^2 |\alpha_i| - \left(\sum_{i=1}^{n} i |\alpha_i| \right)^2 \right]. \tag{2.5}$$

In particular, if $\{B_i\}_{i=0}^{n} \subset B_2(H)$, *then*

$$\left\| \sum_{i=1}^{n} \alpha_i \sum_{i=1}^{n} \alpha_i B_i^2 - \left(\sum_{i=1}^{n} \alpha_i B_i \right)^2 \right\|_1 \leq \max_{1\leq j\leq n-1} \|\Delta B_j\|_2^2 \left[\sum_{i=1}^{n} |\alpha_i| \sum_{i=1}^{n} i^2 |\alpha_i| - \left(\sum_{i=1}^{n} i |\alpha_i| \right)^2 \right]. \tag{2.6}$$

Also, if we take $B_i = C_i^*$ *in (2.1), then we get*

$$\left\| \sum_{i=1}^{n} \alpha_i \sum_{i=1}^{n} \alpha_i |C_i|^2 - \left| \sum_{i=1}^{n} \alpha_i C_i \right|^2 \right\|_1 \leq \max_{1\leq j\leq n-1} \|\Delta C_j\|_p \max_{1\leq j\leq n-1} \|\Delta C_j\|_q \left[\sum_{i=1}^{n} |\alpha_i| \sum_{i=1}^{n} i^2 |\alpha_i| - \left(\sum_{i=1}^{n} i |\alpha_i| \right)^2 \right], \tag{2.7}$$

for $p,\ q > 1$ *with* $\frac{1}{p} + \frac{1}{q} = 1$ *and* $\{B_i\}_{i=0}^{n} \subset B_p(H) \cap B_q(H)$.

In particular, we have

$$\left\| \sum_{i=1}^{n} \alpha_i \sum_{i=1}^{n} \alpha_i |C_i|^2 - \left| \sum_{i=1}^{n} \alpha_i C_i \right|^2 \right\|_1 \leq \max_{1\leq j\leq n-1} \|\Delta C_j\|_2^2 \left[\sum_{i=1}^{n} |\alpha_i| \sum_{i=1}^{n} i^2 |\alpha_i| - \left(\sum_{i=1}^{n} i |\alpha_i| \right)^2 \right], \tag{2.8}$$

provided that $\{C_i\}_{i=0}^{n} \subset B_2(H)$.

Corollary 1. *For* $p,\ q > 1$ *with* $\frac{1}{p} + \frac{1}{q} = 1$, *assume that* $\{B_i\}_{i=0}^{n} \subset B_p(H)$, $\{C_i\}_{i=0}^{n} \subset B_q(H)$ *are sequences of operators and* $p_i \geq 0$ $(i = 1, ..., n)$ *with* $\sum_{i=1}^{n} p_i = 1$. *Then we have the inequality:*

$$\left\| \sum_{i=1}^{n} p_i B_i C_i - \sum_{i=1}^{n} p_i B_i \sum_{i=1}^{n} p_i C_i \right\|_1 \leq \max_{1\leq j\leq n-1} \|\Delta B_j\|_p \max_{1\leq j\leq n-1} \|\Delta C_j\|_q \left[\sum_{i=1}^{n} i^2 p_i - \left(\sum_{i=1}^{n} i p_i \right)^2 \right]. \tag{2.9}$$

Theorem 5. *In particular, if* $\{B_i\}_{i=0}^n, \{C_i\}_{i=0}^n \subset B_2(H)$, *then for* $p=q=2$ *we have*

$$\left\| \sum_{i=1}^n p_i B_i C_i - \sum_{i=1}^n p_i B_i \sum_{i=1}^n p_i C_i \right\|_1 \leq \max_{1\leq j\leq n-1} \|\Delta B_j\|_2 \max_{1\leq j\leq n-1} \|\Delta C_j\|_2 \left[\sum_{i=1}^n i^2 p_i - \left(\sum_{i=1}^n i p_i \right)^2 \right]. \tag{2.10}$$

The following corollary holds:

Corollary 2. *For* $p, q>1$ *with* $\frac{1}{p}+\frac{1}{q}=1$, *assume that* $\{B_i\}_{i=0}^n \subset B_p(H)$, $\{C_i\}_{i=0}^n \subset B_q(H)$, *then we have the inequality:*

$$\left\| \frac{1}{n}\sum_{i=1}^n B_i C_i - \frac{1}{n}\sum_{i=1}^n B_i \frac{1}{n}\sum_{i=1}^n C_i \right\|_1 \leq \frac{n^2-1}{12} \times \max_{1\leq j\leq n-1} \|\Delta B_j\|_p \max_{1\leq j\leq n-1} \|\Delta C_j\|_q. \tag{2.11}$$

In particular, if $\{B_i\}_{i=0}^n \subset B_p(H) \cap B_q(H)$, *then*

$$\left\| \frac{1}{n}\sum_{i=1}^n B_i^2 - \left(\frac{1}{n}\sum_{i=1}^n B_i \right)^2 \right\|_1 \leq \frac{n^2-1}{12} \max_{1\leq j\leq n-1} \|\Delta B_j\|_p \max_{1\leq j\leq n-1} \|\Delta B_j\|_q. \tag{2.12}$$

Also,

$$\left\| \frac{1}{n}\sum_{i=1}^n |C_i|^2 - \left| \frac{1}{n}\sum_{i=1}^n C_i \right|^2 \right\|_1 \leq \frac{n^2-1}{12} \max_{1\leq j\leq n-1} \|\Delta C_j\|_p \max_{1\leq j\leq n-1} \|\Delta C_j\|_q, \tag{2.13}$$

provided that $\{C_i\}_{i=0}^n \subset B_p(H) \cap B_q(H)$.

The proof follows by the above theorem putting $\alpha_i = \frac{1}{n}$ and taking into account that

$$\begin{aligned} &\sum_{i=1}^n \alpha_i \sum_{i=1}^n i^2 \alpha_i - \left(\sum_{i=1}^n i\alpha_i \right)^2 \\ &= \frac{1}{n^2}\left[n\sum_{i=1}^n i^2 - \left(\sum_{i=1}^n i \right)^2 \right] \\ &= \frac{1}{n^2}\left[\frac{n^2(n+1)(2n+1)}{6} - \frac{n^2(n+1)^2}{4} \right] = \frac{n^2-1}{12}. \end{aligned}$$

If $\{B_i\}_{i=0}^n, \{C_i\}_{i=0}^n \subset B_2(H)$, then by (2.11) we get

$$\left\| \frac{1}{n}\sum_{i=1}^n B_i C_i - \frac{1}{n}\sum_{i=1}^n B_i \frac{1}{n}\sum_{i=1}^n C_i \right\|_1 \leq \frac{n^2-1}{12} \max_{1\leq j\leq n-1} \|\Delta B_j\|_2 \max_{1\leq j\leq n-1} \|\Delta C_j\|_2, \tag{2.14}$$

$$\left\| \frac{1}{n} \sum_{i=1}^{n} B_i^2 - \left(\frac{1}{n} \sum_{i=1}^{n} B_i \right)^2 \right\|_1 \leq \frac{n^2-1}{12} \max_{1 \leq j \leq n-1} \|\Delta B_j\|_2^2 , \tag{2.15}$$

and

$$\left\| \frac{1}{n} \sum_{i=1}^{n} |C_i|^2 - \left| \frac{1}{n} \sum_{i=1}^{n} C_i \right|^2 \right\|_1 \leq \frac{n^2-1}{12} \max_{1 \leq j \leq n-1} \|\Delta C_j\|_2^2 . \tag{2.16}$$

Remark 2. *For* $p, q > 1$ *with* $\frac{1}{p} + \frac{1}{q} = 1$, *assume that* $\{B_i\}_{i=0}^n \subset B_p(H)$, $\{C_i\}_{i=0}^n \subset B_q(H)$ *and* $\alpha_i \in \mathbb{C}$ *with* $|\alpha_i| = 1$, $(i = 1, ..., n)$, *then we have the inequality :*

$$\begin{aligned} & \left\| \sum_{i=1}^{n} \alpha_i \sum_{i=1}^{n} \alpha_i B_i C_i - \sum_{i=1}^{n} \alpha_i B_i \cdot \sum_{i=1}^{n} \alpha_i C_i \right\|_1 \\ & \leq \frac{n^2-1}{12} \max_{1 \leq j \leq n-1} \|\Delta B_j\|_p \max_{1 \leq j \leq n-1} \|\Delta C_j\|_q . \end{aligned} \tag{2.17}$$

We also have

Theorem 6. *For* $p \geq 1$, *assume that* $\{B_i\}_{i=0}^n$, $\{C_i\}_{i=0}^n \subset B_p(H)$ *are sequences of operators and* $\alpha_i \in \mathbb{C}$ $(i = 1, ..., n)$. *Then we have the inequality:*

$$\begin{aligned} & \left\| \sum_{i=1}^{n} \alpha_i \sum_{i=1}^{n} \alpha_i B_i C_i - \sum_{i=1}^{n} \alpha_i B_i \sum_{i=1}^{n} \alpha_i C_i \right\|_p \\ & \leq \max_{1 \leq j \leq n-1} \|\Delta B_j\|_p \max_{1 \leq j \leq n-1} \|\Delta C_j\|_p \left[\sum_{i=1}^{n} |\alpha_i| \sum_{i=1}^{n} i^2 |\alpha_i| - \left(\sum_{i=1}^{n} i |\alpha_i| \right)^2 \right] . \end{aligned} \tag{2.18}$$

In particular,

$$\begin{aligned} & \left\| \sum_{i=1}^{n} \alpha_i \sum_{i=1}^{n} \alpha_i B_i^2 - \left(\sum_{i=1}^{n} \alpha_i B_i \right)^2 \right\|_p \\ & \leq \max_{1 \leq j \leq n-1} \|\Delta B_j\|_p^2 \left[\sum_{i=1}^{n} |\alpha_i| \sum_{i=1}^{n} i^2 |\alpha_i| - \left(\sum_{i=1}^{n} i |\alpha_i| \right)^2 \right] \end{aligned} \tag{2.19}$$

and

$$\begin{aligned} & \left\| \sum_{i=1}^{n} \alpha_i \sum_{i=1}^{n} \alpha_i |C_i|^2 - \left| \sum_{i=1}^{n} \alpha_i C_i \right|^2 \right\|_p \\ & \leq \max_{1 \leq j \leq n-1} \|\Delta C_j\|_p^2 \left[\sum_{i=1}^{n} |\alpha_i| \sum_{i=1}^{n} i^2 |\alpha_i| - \left(\sum_{i=1}^{n} i |\alpha_i| \right)^2 \right] . \end{aligned} \tag{2.20}$$

The proof follows by the inequality

$$\begin{aligned}
&\left\| \sum_{i=1}^{n} \alpha_i \sum_{i=1}^{n} \alpha_i B_i C_i \sum_{i=1}^{n} \alpha_i B_i C_i - \sum_{i=1}^{n} \alpha_i B_i \sum_{i=1}^{n} \alpha_i C_i \right\|_p \\
&= \left\| \sum_{1\le i<j\le n}^{n} \alpha_i \alpha_j \sum_{k=i}^{j-1} \Delta B_k \sum_{k=i}^{j-1} \Delta C_k \right\|_p \\
&\le \sum_{1\le i<j\le n}^{n} |\alpha_i \alpha_j| \left\| \sum_{k=i}^{j-1} \Delta B_k \sum_{k=i}^{j-1} \Delta C_k \right\|_p \\
&\le \sum_{1\le i<j\le n}^{n} |\alpha_i \alpha_j| \left\| \sum_{k=i}^{j-1} \Delta B_k \right\|_p \left\| \sum_{k=i}^{j-1} \Delta C_k \right\|_p \\
&\le \sum_{1\le i<j\le n}^{n} |\alpha_i \alpha_j| \sum_{k=i}^{j-1} \|\Delta B_k\|_p \sum_{k=i}^{j-1} \|\Delta C_k\|_p
\end{aligned}$$

by utilising a similar argument to the one in the proof of Theorem 4 and we omit the details.

3. Inequalities for Power Series

Let α_n be nonzero complex numbers and let

$$R := \frac{1}{\limsup |\alpha_n|^{\frac{1}{n}}}.$$

Clearly $0 \le R \le \infty$, but we consider only the case $0 < R \le \infty$.

Denote by:

$$D(0,R) = \begin{cases} \{\lambda \in \mathbb{C} : |\lambda| < R\}, & \text{if } R < \infty \\ \mathbb{C}, & \text{if } R = \infty, \end{cases}$$

consider the functions:

$$\lambda \mapsto f(\lambda) : D(0,R) \to \mathbb{C}, f(\lambda) := \sum_{n=0}^{\infty} \alpha_n \lambda^n$$

and

$$\lambda \mapsto f_a(\lambda) : D(0,R) \to \mathbb{C}, f_a(\lambda) := \sum_{n=0}^{\infty} |\alpha_n| \lambda^n.$$

Let $\mathcal{B}$ be a unital Banach algebra and 1 its unity. Denote by

$$B(0,R) = \begin{cases} \{B \in \mathcal{B} : \|B\| < R\}, & \text{if } R < \infty \\ \mathcal{B}, & \text{if } R = \infty. \end{cases}$$

We associate to f the map:

$$B \mapsto f(B) : B(0,R) \to \mathcal{B},\ f(B) := \sum_{n=0}^{\infty} \alpha_n B^n.$$

Obviously, f is correctly defined because the series $\sum_{n=0}^{\infty} \alpha_n B^n$ is absolutely convergent, since $\sum_{n=0}^{\infty} \|\alpha_n B^n\| \le \sum_{n=0}^{\infty} |\alpha_n| \|B\|^n$.

As some natural examples that are useful for applications, we can point out that, if

$$f(\lambda)=\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\lambda^n=\ln\frac{1}{1+\lambda},\ \ \lambda\in D(0,1);\tag{3.1}$$
$$g(\lambda)=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}\lambda^{2n}=\cos\lambda,\ \ \lambda\in\mathbb{C};$$
$$h(\lambda)=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}\lambda^{2n+1}=\sin\lambda,\ \ \lambda\in\mathbb{C};$$
$$l(\lambda)=\sum_{n=0}^{\infty}(-1)^n\lambda^n=\frac{1}{1+\lambda},\ \ \lambda\in D(0,1);$$

then the corresponding functions constructed by the use of the absolute values of the coefficients are

$$f_a(\lambda)=\sum_{n=1}^{\infty}\frac{1}{n}\lambda^n=\ln\frac{1}{1-\lambda},\ \ \lambda\in D(0,1);\tag{3.2}$$
$$g_a(\lambda)=\sum_{n=0}^{\infty}\frac{1}{(2n)!}\lambda^{2n}=\cosh\lambda,\ \ \lambda\in\mathbb{C};$$
$$h_a(\lambda)=\sum_{n=0}^{\infty}\frac{1}{(2n+1)!}\lambda^{2n+1}=\sinh\lambda,\ \ \lambda\in\mathbb{C};$$
$$l_a(\lambda)=\sum_{n=0}^{\infty}\lambda^n=\frac{1}{1-\lambda},\ \ \lambda\in D(0,1).$$

Other important examples of functions as power series representations with non-negative coefficients are:

$$\exp(\lambda)=\sum_{n=0}^{\infty}\frac{1}{n!}\lambda^n\qquad\lambda\in\mathbb{C},\tag{3.3}$$
$$\frac{1}{2}\ln\left(\frac{1+\lambda}{1-\lambda}\right)=\sum_{n=1}^{\infty}\frac{1}{2n-1}\lambda^{2n-1},\qquad\lambda\in D(0,1);$$
$$\sin^{-1}(\lambda)=\sum_{n=0}^{\infty}\frac{\Gamma\left(n+\frac{1}{2}\right)}{\sqrt{\pi}(2n+1)n!}\lambda^{2n+1},\qquad\lambda\in D(0,1);$$
$$\tanh^{-1}(\lambda)=\sum_{n=1}^{\infty}\frac{1}{2n-1}\lambda^{2n-1},\qquad\lambda\in D(0,1)$$
$${}_2F_1(\alpha,\beta,\gamma,\lambda)=\sum_{n=0}^{\infty}\frac{\Gamma(n+\alpha)\Gamma(n+\beta)\Gamma(\gamma)}{n!\Gamma(\alpha)\Gamma(\beta)\Gamma(n+\gamma)}\lambda^n,\alpha,\beta,\gamma>0,$$
$$\lambda\in D(0,1);$$

where Γ is *Gamma function*.

The following new result holds:

Theorem 7. *Let $f(\lambda)=\sum_{n=0}^{\infty}\alpha_n\lambda^n$ be a power series that is convergent on the open disk $D(0,R)$, with $R>0$. If $B\in B_p(H)$, $C\in B_q(H)$ with $BC=CB$ for p,*

$q>1$ with $\frac{1}{p}+\frac{1}{q}=1$ and $\|B\|_p, \|C\|_q \le 1$, then we have for $\lambda \in \mathbb{C}$ with $|\lambda|<R$ the inequality:

$$
\begin{aligned}
&\|f(\lambda) f(\lambda BC) - f(\lambda B) f(\lambda C)\|_1 \\
&\le \|B-1_H\|_p \|C-1_H\|_q \\
&\times \left\{ f_a(|\lambda|) \left[|\lambda| f_a'(|\lambda|) + |\lambda|^2 f_a''(|\lambda|) \right] - \left[|\lambda| f_a'(|\lambda|) \right]^2 \right\}.
\end{aligned} \tag{3.4}
$$

Proof. From the inequality (2.1) and commutativity of B with C we have

$$
\begin{aligned}
&\left\| \sum_{i=0}^n \alpha_i \lambda^i \sum_{i=0}^n \alpha_i \lambda^i (CB)^i - \sum_{i=0}^n \alpha_i \lambda^i C^i \sum_{i=0}^n \alpha_i \lambda^i B^i \right\|_1 \\
&= \left\| \sum_{i=0}^n \alpha_i \lambda^i \sum_{i=0}^n \alpha_i \lambda^i B^i C^i - \sum_{i=0}^n \alpha_i \lambda^i B^i \sum_{i=0}^n \alpha_i \lambda^i C^i \right\|_1 \\
&\le \max_{0\le j\le n-1} \left\| B^{j+1} - B^j \right\|_p \max_{0\le j\le n-1} \left\| C^{j+1} - C^j \right\|_q \\
&\times \left[\sum_{i=0}^n |\alpha_i| |\lambda|^i \sum_{i=0}^n i^2 |\alpha_i| |\lambda|^i - \left(\sum_{i=0}^n i |\alpha_i| |\lambda|^i \right)^2 \right]
\end{aligned} \tag{3.5}
$$

for all $n \ge 1$.

Observe that, since $\|B\|_p \le 1$, then

$$
\begin{aligned}
\max_{0\le j\le n-1} \left\| B^{j+1} - B^j \right\|_p &= \max_{0\le j\le n-1} \left\| B^j (B-1_H) \right\|_p \\
&\le \max_{0\le j\le n-1} \left\{ \left\| B^j \right\|_p \|B-1_H\|_p \right\} \\
&\le \max_{0\le j\le n-1} \left\{ \|B\|_p^j \|B-1_H\|_p \right\} \le \|B-1_H\|_p .
\end{aligned}
$$

Also,

$$
\max_{0\le j\le n-1} \left\| C^{j+1} - C^j \right\|_q \le \|C-1_H\|_q
$$

for $\|C\|_q \le 1$.

Therefore, since $BC=CB$ from (3.5) then we have

$$
\begin{aligned}
&\left\| \sum_{j=0}^n \alpha_j \lambda^j \sum_{j=0}^n \alpha_j \lambda^j (BC)^j - \sum_{j=0}^n \alpha_j \lambda^j B^j \sum_{j=0}^n \alpha_j \lambda^j C^j \right\|_1 \\
&\le \|B-1_H\|_p \|C-1_H\|_q \\
&\times \left[\sum_{j=0}^n |\alpha_j| |\lambda|^j \sum_{j=0}^n j^2 |\alpha_j| |\lambda|^j - \left(\sum_{j=0}^n j |\alpha_j| |\lambda|^j \right)^2 \right]
\end{aligned} \tag{3.6}
$$

for all $n \ge 1$.

If we denote $f(u) := \sum_{j=0}^{\infty} \alpha_j u^j$, then for $|u|<R$ we have

$$
\sum_{j=0}^{\infty} j \alpha_j u^j = u f'(u)
$$

and

$$\sum_{j=0}^{\infty} j^2 \alpha_j u^j = u \left(u g' \left(u \right) \right)' .$$

However

$$u \left(u g' \left(u \right) \right)' = u g' \left(u \right) + u^2 g'' \left(u \right)$$

and then

$$\sum_{j=0}^{\infty} j^2 \alpha_j u^j = u g' \left(u \right) + u^2 g'' \left(u \right) .$$

Therefore

$$\sum_{j=0}^{\infty} j^2 \left| \alpha_j \right| \left| \lambda \right|^j = \left| \lambda \right| f_a' \left(\left| \lambda \right| \right) + \left| \lambda \right|^2 f_a'' \left(\left| \lambda \right| \right)$$

and

$$\sum_{j=0}^{\infty} j \left| \alpha_j \right| \left| \lambda \right|^j = \left| \lambda \right| f' \left(\left| \lambda \right| \right)$$

for $|\lambda| < R$.

Since all the series involved in (3.6) are convergent, then by letting $n \to \infty$ in (3.6) we deduce the desired result (3.4). □

Corollary 3. *Let $f(\lambda) = \sum_{n=0}^{\infty} \alpha_n \lambda^n$ be a power series that is convergent on the open disk $D(0, R)$, with $R > 0$. If $B \in B_p(H) \cap B_q(H)$ for p, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ and $\|B\|_p, \|B\|_q \leq 1$, then we have for $\lambda \in \mathbb{C}$ with $|\lambda| < R$ that*

$$\begin{aligned} (3.7) \qquad & \left\| f(\lambda) f\left(\lambda B^2\right) - \left[f(\lambda B) \right]^2 \right\|_1 \\ & \leq \left\| B - 1_H \right\|_p \left\| B - 1_H \right\|_q \\ & \times \left\{ f_a\left(|\lambda|\right) \left[|\lambda| f_a'\left(|\lambda|\right) + |\lambda|^2 f_a''\left(|\lambda|\right) \right] - \left[|\lambda| f_a'\left(|\lambda|\right) \right]^2 \right\} . \end{aligned}$$

In particular, if $B \in B_2(H)$ and $\|B\|_2 \leq 1$, then we have for $\lambda \in \mathbb{C}$ with $|\lambda| < R$ that

$$\begin{aligned} (3.8) \qquad & \left\| f(\lambda) f\left(\lambda B^2\right) - \left[f(\lambda B) \right]^2 \right\|_1 \\ & \leq \left\| B - 1_H \right\|_p^2 \left\{ f_a\left(|\lambda|\right) \left[|\lambda| f_a'\left(|\lambda|\right) + |\lambda|^2 f_a''\left(|\lambda|\right) \right] - \left[|\lambda| f_a'\left(|\lambda|\right) \right]^2 \right\} . \end{aligned}$$

In the case of normal operators we have

Corollary 4. *Let $f(\lambda) = \sum_{n=0}^{\infty} \alpha_n \lambda^n$ be a power series that is convergent on the open disk $D(0, R)$, with $R > 0$. If $B \in B_p(H) \cap B_q(H)$ for p, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ is normal, i.e. $B^*B = BB^*$ and $\|B\|_p, \|B\|_q \leq 1$, then we have for $\lambda \in \mathbb{C}$ with $|\lambda| < R$ that*

$$\begin{aligned} (3.9) \qquad & \left\| f(\lambda) f\left(\lambda |B|^2\right) - \left| f(\lambda B) \right|^2 \right\|_1 \\ & \leq \left\| B - 1_H \right\|_p \left\| B - 1_H \right\|_q \\ & \times \left\{ f_a\left(|\lambda|\right) \left[|\lambda| f_a'\left(|\lambda|\right) + |\lambda|^2 f_a''\left(|\lambda|\right) \right] - \left[|\lambda| f_a'\left(|\lambda|\right) \right]^2 \right\} . \end{aligned}$$

In particular, if $B \in B_2(H)$ is normal and $\|B\|_2 \leq 1$, then we have for $\lambda \in \mathbb{C}$ with $|\lambda| < R$ that

$$\left\| f(\lambda) f\left(\lambda |B|^2\right) - |f(\lambda B)|^2 \right\|_1 \leq \|B - 1_H\|_p^2 \left\{ f_a(|\lambda|) \left[|\lambda| f_a'(|\lambda|) + |\lambda|^2 f_a''(|\lambda|) \right] - \left[|\lambda| f_a'(|\lambda|) \right]^2 \right\}. \tag{3.10}$$

4. Some Examples

Consider the exponential function $f(\lambda) = \sum_{n=0}^{\infty} \frac{1}{n!} \lambda^n = \exp(\lambda)$, $\lambda \in \mathbb{C}$. Then

$$\begin{aligned} & f_a(|\lambda|) \left[|\lambda| f_a'(|\lambda|) + |\lambda|^2 f_a''(|\lambda|) \right] - \left[|\lambda| f_a'(|\lambda|) \right]^2 \\ & = \exp(|\lambda|) \left[|\lambda| \exp(|\lambda|) + |\lambda|^2 \exp(|\lambda|) \right] - \left[|\lambda| \exp(|\lambda|) \right]^2 \\ & = |\lambda| \exp(2|\lambda|), \ \lambda \in \mathbb{C}. \end{aligned}$$

If we apply the inequality (3.4) to the exponential function, then we have

$$\begin{aligned} & \|\exp[\lambda(1 + BC)] - \exp[\lambda(B + C)]\|_1 \\ & \leq \|B - 1_H\|_p \|C - 1_H\|_q |\lambda| \exp(2|\lambda|) \end{aligned} \tag{4.1}$$

for $B \in B_p(H)$, $C \in B_q(H)$ with $BC = CB$ for p, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ and $\|B\|_p, \|C\|_q \leq 1$ while $\lambda \in \mathbb{C}$.

For $p = q = 2$ and if B, $C \in B_2(H)$ with $BC = CB$, then by (4.1) we get

$$\begin{aligned} & \|\exp[\lambda(1 + BC)] - \exp[\lambda(B + C)]\|_1 \\ & \leq \|B - 1_H\|_2 \|C - 1_H\|_2 |\lambda| \exp(2|\lambda|) \end{aligned} \tag{4.2}$$

In particular, we have

$$\left\| \exp\left[\lambda\left(1 + B^2\right)\right] - \exp[2\lambda(B)] \right\|_1 \leq \|B - 1_H\|_p \|B - 1_H\|_q |\lambda| \exp(2|\lambda|) \tag{4.3}$$

for any $B \in B_p(H) \cap B_q(H)$with $\|B\|_p, \|B\|_q \leq 1$ and $\lambda \in \mathbb{C}$. For $p = q = 2$ we derive

$$\left\| \exp\left[\lambda\left(1 + B^2\right)\right] - \exp[2\lambda(B)] \right\|_1 \leq \|B - 1_H\|_2^2 |\lambda| \exp(2|\lambda|), \tag{4.4}$$

provided that $B \in B_2(H)$.

Also we have

$$\begin{aligned} & \left\| \exp\left[\lambda\left(1 + |C|^2\right)\right] - \exp[\lambda(C^* + C)] \right\|_1 \\ & \leq \|C - 1_H\|_p \|C - 1_H\|_q |\lambda| \exp(2|\lambda|) \end{aligned} \tag{4.5}$$

for any normal operator $C \in B_p(H) \cap B_q(H)$ with $\|C\|_p, \|C\|_q \leq 1$ and $\lambda \in \mathbb{C}$. For $p = q = 2$ we obtain

$$\left\| \exp\left[\lambda\left(1 + |C|^2\right)\right] - \exp[\lambda(C^* + C)] \right\|_1 \leq \|C - 1_H\|_2^2 |\lambda| \exp(2|\lambda|), \tag{4.6}$$

provided that $C \in B_2(H)$ and normal.

Now, consider the function $f(\lambda) := \sum_{n=0}^{\infty} \lambda^n = \frac{1}{1-\lambda}$, $\lambda \in D(0,1)$. Then

$$
\begin{aligned}
& f_a(|\lambda|) \left[|\lambda| f_a'(|\lambda|) + |\lambda|^2 f_a''(|\lambda|) \right] - \left[|\lambda| f_a'(|\lambda|) \right]^2 \\
& = (1-|\lambda|)^{-1} \left[|\lambda| (1-|\lambda|)^{-2} + 2|\lambda|^2 (1-|\lambda|)^{-3} \right] - \left[|\lambda| (1-|\lambda|)^{-2} \right]^2 \\
& = |\lambda| (1-|\lambda|)^{-3} + 2|\lambda|^2 (1-|\lambda|)^{-4} - |\lambda|^2 (1-|\lambda|)^{-4} \\
& = |\lambda| (1-|\lambda|)^{-3} + |\lambda|^2 (1-|\lambda|)^{-4} = |\lambda| (1-|\lambda|)^{-3} \left[1 + |\lambda| (1-|\lambda|)^{-1} \right] \\
& = |\lambda| (1-|\lambda|)^{-4}
\end{aligned}
$$

If $B \in B_p(H)$, $C \in B_q(H)$ with $BC = CB$ for $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ and $\|B\|_p, \|C\|_q \le 1$, then we have for $\lambda \in \mathbb{C}$ with $|\lambda| < 1$ the inequality:

$$
\begin{aligned}
& \left\| (1-\lambda)^{-1} (1_H - \lambda BC)^{-1} - (1_H - \lambda B)^{-1} (1_H - \lambda C)^{-1} \right\|_1 \\
& \le \frac{|\lambda|}{(1-|\lambda|)^4} \|B - 1_H\|_p \|C - 1_H\|_q .
\end{aligned} \tag{4.7}
$$

For $p = q = 2$ and if $B, C \in B_2(H)$ with $BC = CB$, then by (4.1) we get

$$
\begin{aligned}
& \left\| (1-\lambda)^{-1} (1_H - \lambda BC)^{-1} - (1_H - \lambda B)^{-1} (1_H - \lambda C)^{-1} \right\|_1 \\
& \le \frac{|\lambda|}{(1-|\lambda|)^4} \|B - 1_H\|_2 \|C - 1_H\|_2 ,
\end{aligned} \tag{4.8}
$$

provided that $\|B\|_2, \|C\|_2 \le 1$ and $\lambda \in \mathbb{C}$ with $|\lambda| < 1$.

If $B \in B_p(H) \cap B_q(H)$ for $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ and $\|B\|_p, \|B\|_q \le 1$, then we have for $\lambda \in \mathbb{C}$ with $|\lambda| < 1$ the inequality:

$$
\begin{aligned}
& \left\| (1-\lambda)^{-1} \left(1_H - \lambda B^2\right)^{-1} - (1_H - \lambda B)^{-2} \right\|_1 \\
& \le \frac{|\lambda|}{(1-|\lambda|)^4} \|B - 1_H\|_p \|B - 1_H\|_q
\end{aligned} \tag{4.9}
$$

and in the case $p = q = 2$ that

$$
\left\| (1-\lambda)^{-1} \left(1_H - \lambda B^2\right)^{-1} - (1_H - \lambda B)^{-2} \right\|_1 \le \frac{|\lambda|}{(1-|\lambda|)^4} \|B - 1_H\|_2^2 , \tag{4.10}
$$

provided that $B \in B_2(H)$ and $\|B\|_2 \le 1$.

Finally, if $C \in B_p(H) \cap B_q(H)$ for $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, normal and $\|C\|_p, \|C\|_q \le 1$, then we have for $\lambda \in \mathbb{C}$ with $|\lambda| < 1$ that

$$
\begin{aligned}
& \left\| (1-\lambda)^{-1} \left(1_H - \lambda |C|^2\right)^{-1} - |1_H - \lambda C|^{-2} \right\|_1 \\
& \le \frac{|\lambda|}{(1-|\lambda|)^4} \|C - 1_H\|_p \|C - 1_H\|_q
\end{aligned} \tag{4.11}
$$

and, in the case $p = q = 2$, that

$$
\left\| (1-\lambda)^{-1} \left(1_H - \lambda |C|^2\right)^{-1} - |1_H - \lambda C|^{-2} \right\|_1 \le \frac{|\lambda|}{(1-|\lambda|)^4} \|C - 1_H\|_2^2 , \tag{4.12}
$$

provided that $C \in B_2(H)$ is normal and $\|C\|_2 \le 1$.

References

[1] R. Bellman, Some inequalities for positive definite matrices, in: E.F. Beckenbach (Ed.), *General Inequalities 2*, Proceedings of the 2nd International Conference on General Inequalities, Birkhäuser, Basel, 1980, pp. 89–90.

[2] R. Bhatia, *Matrix Analysis.* Graduate Texts in Mathematics, 169. Springer-Verlag, New York, 1997. xii+347 pp.

[3] M. Biernacki, Sur une inégalité entre les intégrales due à Tchebyscheff. *Ann. Univ. Mariae Curie-Sklodowska* (Poland), **A5**(1951), 23-29.

[4] K. Boukerrioua, and A. Guezane-Lakoud, On generalization of Čebyšev type inequalities. *J. Inequal. Pure Appl. Math.* **8** (2007), no. 2, Article 55, 4 pp.

[5] P. L. Čebyšev, O približennyh vyraženijah odnih integralov čerez drugie. *Soobšćenija i protokoly zasedaniĭ Matemmatičeskogo občestva pri Imperatorskom Har'kovskom Universitete* No. 2, 93–98; *Polnoe sobranie sočineniĭ P. L. Čebyševa.* Moskva–Leningrad, 1948a, (1882), 128-131.

[6] P. L. Čebyšev, Ob odnom rjade, dostavljajušćem predel'nye veličiny integralov pri razloženii podintegral'noĭ funkcii na množeteli. *Priloženi k 57 tomu Zapisok Imp. Akad. Nauk,* No. 4; *Polnoe sobranie sočineniĭ P. L. Čebyševa.* Moskva–Leningrad, 1948b, (1883),157-169.

[7] D. Chang, A matrix trace inequality for products of Hermitian matrices, *J. Math. Anal. Appl.* **237** (1999) 721–725.

[8] I. D. Coop, On matrix trace inequalities and related topics for products of Hermitian matrix, *J. Math. Anal. Appl.* **188** (1994) 999–1001.3.

[9] S. S. Dragomir, *Operator Inequalities of the Jensen, Čebyšev and Grüss Type.* Springer Briefs in Mathematics. Springer, New York, 2012. xii+121 pp. ISBN: 978-1-4614-1520-6.

[10] S. S. Dragomir, *Operator Inequalities of Ostrowski and Trapezoidal Type.* Springer Briefs in Mathematics. Springer, New York, 2012. x+112 pp. ISBN: 978-1-4614-1778-1.

[11] S. S. Dragomir, M. V. Boldea, C. Buşe and M. Megan, Norm inequalities of Čebyšev type for power series in Banach algebras, Journal of Inequalities and Applications **2014**, 2014:294 Online http://www.journalofinequalitiesandapplications.com/content/2014/1/294.

[12] S. S. Dragomir, M. V. Boldea and M. Megan, New norm inequalities of Čebyšev type for power series in Banach algebras, Preprint *RGMIA Res. Rep. Coll.*, **17** (2014), Art. 65. Online http://rgmia.org/papers/v17/v17a65.pdf.

[13] S. S. Dragomir, M. V. Boldea and M. Megan, New bounds for Čebyšev functional for power series in Banach algebras via a Grüss'-Lupaş type inequality, Preprint *RGMIA Res. Rep. Coll.*, **17** (2014), Art. 104. Online http://rgmia.org/papers/v17/v17a104.pdf.

[14] T. Furuta, J. Mićić Hot, J. Pečarić and Y. Seo, *Mond-Pečarić Method in Operator Inequalities. Inequalities for Bounded Selfadjoint Operators on a Hilbert Space*, Element, Zagreb, 2005..

[15] G. Grüss, Über das Maximum des absoluten Betrages von $\frac{1}{b-a}\int_a^b f(x)g(x)dx - \frac{1}{(b-a)^2}\int_a^b f(x)dx\int_a^b g(x)dx$, *Math. Z.* , **39**(1935), 215-226.

[16] D. S. Mitrinović, J. E. Pečarić and A. M. Fink, *Classical and New Inequalities in Analysis*, Kluwer Academic Publishers, Dordrecht, 1993.

[17] D. S. Mitrinović and P. M. Vasić, History, variations and generalisations of the Čebyšev inequality and the question of some priorities. *Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz.* **No. 461–497** (1974), 1–30.

[18] H. Neudecker, A matrix trace inequality, *J. Math. Anal. Appl.* **166** (1992) 302–303.

[19] B. Simon, *Trace Ideals and Their Applications,* Cambridge University Press, Cambridge, 1979.

[20] V. A. Zagrebnov, *Gibbs Semigroups,* Operator Theory: Advances and Applications, Volume 273, Birkhäuser, 2019

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