p-SCHATTEN NORM INEQUALITIES FOR ČEBYŠEV'S FUNCTIONAL WITH COMPLEX WEIGHTS

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Abstract. An operator $A \in \mathcal{B}(H)$ is said to belong to the *von Neumann-Schatten class* $\mathcal{B}_p(H)$, $1 \leq p < \infty$ if the *p-Schatten norm* is finite

$$\|A\|_p := [\operatorname{tr}(|A|^p)]^{1/p} < \infty.$$

For $p,\ q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, assume that $\{A_i\}_{i=0}^n \subset B_p(H)$, $\{B_i\}_{i=0}^n \subset B_q(H)$ are sequences of operators and $\alpha_i \in \mathbb{C}$ $(i = 1, ..., n)$. In this paper we show among others that

$$\left\| \sum_{i=1}^n \alpha_i \sum_{i=1}^n \alpha_i A_i B_i - \sum_{i=1}^n \alpha_i A_i \sum_{i=1}^n \alpha_i B_i \right\|_1 \leq \frac{1}{2} \left[\left(\sum_{i=1}^n |\alpha_i| \right)^2 - \sum_{i=1}^n |\alpha_i|^2 \right] \sum_{i=1}^{n-1} \|\Delta A_i\|_p \sum_{i=1}^{n-1} \|\Delta B_i\|_q,$$

where $\Delta A_i := A_{i+1} - A_i$ $(i = 1, ..., n-1)$ are the usual forward differences. Applications for operator functions defined by power series with complex coefficients with applications for the inverse and exponential functions are also provided.

1. Introduction

In 1935, G. Grüss [15] proved the following integral inequality which gives an approximation of the integral mean of the product in terms of the product of the integral means integrals as follows:

$$\left| \frac{1}{b-a} \int_a^b f(x) g(x) dx - \frac{1}{b-a} \int_a^b f(x) dx \cdot \frac{1}{b-a} \int_a^b g(x) dx \right| \leq \frac{1}{4} (\Phi - \varphi)(\Gamma - \gamma) \tag{1.1}$$

where $f, g : [a,b] \to \mathbb{R}$ are integrable on $[a,b]$ and satisfying the assumption

$$\varphi \leq f(x) \leq \Phi, \gamma \leq g(x) \leq \Gamma$$

for each $x \in [a,b]$ where $\varphi, \Phi, \gamma, \Gamma$ are given real constants.

Moreover the constant $\frac{1}{4}$ is sharp in the sense that it can not be replaced by a smaller one.

For a simple proof of (1.1) as well as for some other integral inequalities of Grüss' type see the Chapter X of the recent book [16] by Mitrinović, Pečarić and Fink.

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In 1950, M. Biernacki, H. Pidek and C. Ryll-Nardzewski [3] established the following discrete version of Grüss' inequality, see also [16, Ch. X]:

Theorem 1. *Let* $a=(a_1,...,a_n), b=(b_1,...,b_n)$ *be two* n*-tuples of real numbers such that* $r\le a_i\le R$ *and* $s\le b_i\le S$ *for* $i=1,...,n$. *Then one has the inequality:*

$$\left|\frac{1}{n}\sum_{i=1}^{n}a_ib_i-\frac{1}{n}\sum_{i=1}^{n}a_i\cdot\frac{1}{n}\sum_{i=1}^{n}b_i\right| \le \frac{1}{n}\left[\frac{n}{2}\right]\left(1-\frac{1}{n}\left[\frac{n}{2}\right]\right)(R-r)(S-s) \tag{1.2}$$

when $[x]$ *is the integer part of* $x, x\in\mathbb{R}$.

In 1981, A. Lupaş [16, Ch. X] proved some similar results for the first difference of a as follows :

Theorem 2. *Let* a,b *two monotonic* n*-tuples in the same sense and* p *a positive* n*-tuple. Then*

$$\min_{1\le i\le n-1}|a_{i+1}-a_i|\min_{1\le i\le n-1}|b_{i+1}-b_i|\left[\frac{1}{P_n}\sum_{i=1}^{n}i^2p_i-\left(\frac{1}{P_n}\sum_{i=1}^{n}ip_i\right)^2\right] \tag{1.3}$$
$$\le\frac{1}{P_n}\sum_{i=1}^{n}p_ia_ib_i-\frac{1}{P_n}\sum_{i=1}^{n}p_ia_i\cdot\frac{1}{P_n}\sum_{i=1}^{n}p_ib_i$$
$$\le\max_{1\le i\le n-1}|a_{i+1}-a_i|\max_{1\le i\le n-1}|b_{i+1}-b_i|\left[\frac{1}{P_n}\sum_{i=1}^{n}i^2p_i-\left(\frac{1}{P_n}\sum_{i=1}^{n}ip_i\right)^2\right].$$

If there exists the numbers $\bar{a},\bar{a}_1,r,r_1,(rr_1>0)$ *such that* $a_k=\bar{a}+kr$ *and* $b_k=\bar{a}_1+kr_1$, *then in (1.3) the equality holds.*

In order to extend these results for p-Schatten norms we need the following preparations.

Let $(H;\langle.,.\rangle)$ be a complex Hilbert space and $\mathcal{B}(H)$ the Banach algebra of all bounded linear operators on H. If $\{e_i\}_{i\in I}$ an orthonormal basis of H, we say that $A\in\mathcal{B}(H)$ is of *trace class* if

$$\|A\|_1:=\sum_{i\in I}\langle|A|e_i,e_i\rangle<\infty. \tag{1.4}$$

The definition of $\|A\|_1$ does not depend on the choice of the orthonormal basis $\{e_i\}_{i\in I}$. We denote by $\mathcal{B}_1(H)$ the set of trace class operators in $\mathcal{B}(H)$.

We define the *trace* of a trace class operator $A\in\mathcal{B}_1(H)$ to be

$$\operatorname{tr}(A):=\sum_{i\in I}\langle Ae_i,e_i\rangle, \tag{1.5}$$

where $\{e_i\}_{i\in I}$ an orthonormal basis of H. Note that this coincides with the usual definition of the trace if H is finite-dimensional. We observe that the series (1.5) converges absolutely and it is independent from the choice of basis.

The following result collects some properties of the trace:

Theorem 3. *We have:*

(i) If $A \in \mathcal{B}_1(H)$ *then* $A^* \in \mathcal{B}_1(H)$ *and*

$$\operatorname{tr}(A^*) = \overline{\operatorname{tr}(A)}; \tag{1.6}$$

(ii) If $A \in \mathcal{B}_1(H)$ *and* $T \in \mathcal{B}(H)$, *then* AT, $TA \in \mathcal{B}_1(H)$ *and*

$$\operatorname{tr}(AT) = \operatorname{tr}(TA) \text{ and } |\operatorname{tr}(AT)| \le \|A\|_1 \|T\|; \tag{1.7}$$

(iii) $\operatorname{tr}(\cdot)$ *is a bounded linear functional on* $\mathcal{B}_1(H)$ *with* $\|\operatorname{tr}\| = 1$;
(iv) If $A, B \in \mathcal{B}_2(H)$ *then* AB, $BA \in \mathcal{B}_1(H)$ *and* $\operatorname{tr}(AB) = \operatorname{tr}(BA)$;
(v) $\mathcal{B}_{fin}(H)$, *the space of operators of finite rank, is a dense subspace of* $\mathcal{B}_1(H)$.

An operator $A \in \mathcal{B}(H)$ is said to belong to the *von Neumann-Schatten class* $\mathcal{B}_p(H)$, $1 \le p < \infty$ if the *p-Schatten norm* is finite [20, p. 60-64]

$$\|A\|_p := [\operatorname{tr}(|A|^p)]^{1/p} < \infty.$$

For $1 < p < q < \infty$ we have that

$$\mathcal{B}_1(H) \subset \mathcal{B}_p(H) \subset \mathcal{B}_q(H) \subset \mathcal{B}(H) \tag{1.8}$$

and

$$\|A\|_1 \ge \|A\|_p \ge \|A\|_q \ge \|A\|. \tag{1.9}$$

For $p \ge 1$ the functional $\|\cdot\|_p$ is a *norm* on the $*$-ideal $\mathcal{B}_p(H)$ and $\left(\mathcal{B}_p(H), \|\cdot\|_p\right)$ is a Banach space.

Also, see for instance [20, p. 60-64],

$$\|A\|_p = \|A^*\|_p, \ A \in \mathcal{B}_p(H) \tag{1.10}$$

$$\|AB\|_p \le \|A\|_p \|B\|_p, \ A, B \in \mathcal{B}_p(H) \tag{1.11}$$

and

$$\|AB\|_p \le \|A\|_p \|B\|, \ \|BA\|_p \le \|B\| \|A\|_p, \ A \in \mathcal{B}_p(H), \ B \in \mathcal{B}(H). \tag{1.12}$$

This implies that

$$\|CAB\|_p \le \|C\| \|A\|_p \|B\|, \ A \in \mathcal{B}_p(H), \ B, \ C \in \mathcal{B}(H). \tag{1.13}$$

In terms of *p-Schatten norm* we have the *Hölder inequality* for $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$

$$(|\operatorname{tr}(AB)| \le) \|AB\|_1 \le \|A\|_p \|B\|_q, \ A \in \mathcal{B}_p(H), \ B \in \mathcal{B}_q(H). \tag{1.14}$$

For the theory of trace functionals and their applications the reader is referred to [19] and [20].

For some classical trace inequalities see [7], [8], and [18], which are continuations of the work of Bellman [1].

2. A Discrete Inequality of Grüss Type for p-Schatten Norm

The following inequality of Grüss type holds.

Theorem 4. *For $p,\ q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, assume that $\{A_i\}_{i=0}^{n}\subset B_p(H)$, $\{B_i\}_{i=0}^{n}\subset B_q(H)$ are sequences of operators and $\alpha_i\in\mathbb{C}\ \ (i=1,...,n)$. Then we have the inequality:*

$$\left\|\sum_{i=1}^{n}\alpha_i\sum_{i=1}^{n}\alpha_iA_iB_i-\sum_{i=1}^{n}\alpha_iA_i\sum_{i=1}^{n}\alpha_iB_i\right\|_1 \leq\frac{1}{2}\left[\left(\sum_{i=1}^{n}|\alpha_i|\right)^2-\sum_{i=1}^{n}|\alpha_i|^2\right]\sum_{i=1}^{n-1}\|\Delta A_i\|_p\sum_{i=1}^{n-1}\|\Delta B_i\|_q, \tag{2.1}$$

where $\Delta A_i:=A_{i+1}-A_i\ \ (i=1,...,n-1)$ and $\Delta B_i:=B_{i+1}-B_i\ \ (i=1,...,n-1)$ are the usual forward differences.

In particular, if $\{A_i\}_{i=0}^{n},\ \{B_i\}_{i=0}^{n}\subset B_2(H)$, then for $p=q=2$ we have

$$\left\|\sum_{i=1}^{n}\alpha_i\sum_{i=1}^{n}\alpha_iA_iB_i-\sum_{i=1}^{n}\alpha_iA_i\sum_{i=1}^{n}\alpha_iB_i\right\|_1 \leq\frac{1}{2}\left[\left(\sum_{i=1}^{n}|\alpha_i|\right)^2-\sum_{i=1}^{n}|\alpha_i|^2\right]\sum_{i=1}^{n-1}\|\Delta A_i\|_2\sum_{i=1}^{n-1}\|\Delta B_i\|_2. \tag{2.2}$$

Proof. Let us start with the following identity in Banach algebras which can be proved by direct computation

$$\begin{aligned}\sum_{i=1}^{n}\alpha_i\sum_{i=1}^{n}\alpha_iA_iB_i-\sum_{i=1}^{n}\alpha_iA_i\sum_{i=1}^{n}\alpha_iB_i&=\frac{1}{2}\sum_{i,j=1}^{n}\alpha_i\alpha_j\left(A_j-A_i\right)\left(B_j-B_i\right)\\&=\sum_{1\leq i<j\leq n}\alpha_i\alpha_j\left(A_j-A_i\right)\left(B_j-B_i\right).\end{aligned}$$

As $i<j$, we can write $A_j-A_i=\sum_{k=i}^{j-1}\Delta A_k$ and $B_j-B_i=\sum_{l=i}^{j-1}\Delta B_l$. Using the generalized triangle inequality and the property (1.14), we have successively:

$$\begin{aligned}&\left\|\sum_{i=1}^{n}\alpha_i\sum_{i=1}^{n}\alpha_iA_iB_i-\sum_{i=1}^{n}\alpha_iA_i\sum_{i=1}^{n}\alpha_iB_i\right\|_1\\&=\left\|\sum_{1\leq i<j\leq n}\alpha_i\alpha_j\sum_{k=i}^{j-1}\Delta A_k\sum_{l=i}^{j-1}\Delta B_l\right\|_1\\&\leq\sum_{1\leq i<j\leq n}|\alpha_i|\,|\alpha_j|\left\|\sum_{k=i}^{j-1}\Delta A_k\sum_{l=i}^{j-1}\Delta B_l\right\|_1\\&\leq\sum_{1\leq i<j\leq n}|\alpha_i|\,|\alpha_j|\left\|\sum_{k=i}^{j-1}\Delta A_k\right\|_p\left\|\sum_{l=i}^{j-1}\Delta B_l\right\|_q\\&\leq\sum_{1\leq i<j\leq n}|\alpha_i|\,|\alpha_j|\sum_{k=i}^{j-1}\|\Delta A_k\|_p\sum_{l=i}^{j-1}\|\Delta B_l\|_q=:\Gamma.\end{aligned} \tag{2.3}$$

It is obvious for all $1 \le i < j \le n-1$, we have that

$$\sum_{k=i}^{j-1} \|\Delta A_k\|_p \le \sum_{k=1}^{n-1} \|\Delta A_k\|_p$$

and

$$\sum_{l=i}^{j-1} \|\Delta B_l\|_q \le \sum_{l=1}^{n-1} \|\Delta B_l\|_q$$

and then

$$\Gamma \le \sum_{k=1}^{n-1} \|\Delta A_k\|_p \sum_{l=1}^{n-1} \|\Delta B_l\|_q \sum_{1\le i<j\le n} |\alpha_i| |\alpha_j| . \tag{2.4}$$

Now, let us observe that

$$\begin{aligned} \sum_{1\le i<j\le n} |\alpha_i| |\alpha_j| &= \frac{1}{2}\left[\sum_{i,j=1}^{n} |\alpha_i| |\alpha_j| - \sum_{i=j} |\alpha_i| |\alpha_j|\right] \\ &= \frac{1}{2}\left[\sum_{i=1}^{n} |\alpha_i| \sum_{j=1}^{n} |\alpha_j| - \sum_{i=1}^{n} |\alpha_i|^2\right] \\ &= \frac{1}{2}\left[\left(\sum_{i=1}^{n} |\alpha_i|\right)^2 - \sum_{i=1}^{n} |\alpha_i|^2\right]. \end{aligned} \tag{2.5}$$

Using (2.3)-(2.5) , we deduce the desired inequality (2.1) . □

Remark 1. *For $p,\, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, assume that $\{A_i\}_{i=0}^{n} \subset B_p(H) \cap B_q(H)$ are sequences of operators and $\alpha_i \in \mathbb{C}$ $(i = 1, ..., n)$. Then for $B_i = A_i$ in (2.1) we have the inequality:*

$$\begin{aligned} &\left\| \sum_{i=1}^{n} \alpha_i \sum_{i=1}^{n} \alpha_i A_i^2 - \left(\sum_{i=1}^{n} \alpha_i A_i\right)^2 \right\|_1 \\ &\le \frac{1}{2}\left[\left(\sum_{i=1}^{n} |\alpha_i|\right)^2 - \sum_{i=1}^{n} |\alpha_i|^2\right] \sum_{i=1}^{n-1} \|\Delta A_i\|_p \sum_{i=1}^{n-1} \|\Delta A_i\|_q . \end{aligned} \tag{2.6}$$

In particular, for $p = q = 2$ we deduce for $\{A_i\}_{i=0}^{n} \subset B_2(H)$ that

$$\begin{aligned} &\left\| \sum_{i=1}^{n} \alpha_i \sum_{i=1}^{n} \alpha_i A_i^2 - \left(\sum_{i=1}^{n} \alpha_i A_i\right)^2 \right\|_1 \\ &\le \frac{1}{2}\left[\left(\sum_{i=1}^{n} |\alpha_i|\right)^2 - \sum_{i=1}^{n} |\alpha_i|^2\right] \left(\sum_{i=1}^{n-1} \|\Delta A_i\|_2\right)^2 . \end{aligned} \tag{2.7}$$

For $p,\, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, assume that $\{B_i\}_{i=0}^{n} \subset B_p(H) \cap B_q(H)$ are sequences of operators and $\alpha_i \in \mathbb{C}$ $(i = 1, ..., n)$. Then by taking $A_i = B_i^*$ in (2.1) we have *the inequality:*

$$\left\| \sum_{i=1}^{n} \alpha_i \sum_{i=1}^{n} \alpha_i |B_i|^2 - \left| \sum_{i=1}^{n} \alpha_i B_i \right|^2 \right\|_1 \leq \frac{1}{2} \left[\left(\sum_{i=1}^{n} |\alpha_i| \right)^2 - \sum_{i=1}^{n} |\alpha_i|^2 \right] \sum_{i=1}^{n-1} \|\Delta B_i\|_p \sum_{i=1}^{n-1} \|\Delta B_i\|_q , \tag{2.8}$$

where the modulus of the operator C is defined as $|C| := (C^*C)^{1/2}$.

In particular, if $\{B_i\}_{i=0}^n \subset B_2(H)$, then

$$\left\| \sum_{i=1}^{n} \alpha_i \sum_{i=1}^{n} \alpha_i |B_i|^2 - \left| \sum_{i=1}^{n} \alpha_i B_i \right|^2 \right\|_1 \leq \frac{1}{2} \left[\left(\sum_{i=1}^{n} |\alpha_i| \right)^2 - \sum_{i=1}^{n} |\alpha_i|^2 \right] \left(\sum_{i=1}^{n-1} \|\Delta B_i\|_2 \right)^2 . \tag{2.9}$$

Corollary 1. *With the assumptions of Theorem 4 and if $p_i \geq 0$, $(i = 1, ..., n)$ with $\sum_{i=1}^n p_i = 1$, then*

$$\begin{aligned} \left\| \sum_{i=1}^{n} p_i A_i B_i - \sum_{i=1}^{n} p_i A_i \sum_{i=1}^{n} p_i B_i \right\|_1 &\leq \frac{1}{2} \left(1 - \sum_{i=1}^{n} p_i^2 \right) \sum_{i=1}^{n-1} \|\Delta A_i\|_p \sum_{i=1}^{n-1} \|\Delta B_i\|_q \\ &= \frac{1}{2} \left[\sum_{i=1}^{n} p_i (1 - p_i) \right] \sum_{i=1}^{n-1} \|\Delta A_i\|_p \sum_{i=1}^{n-1} \|\Delta B_i\|_q \end{aligned} \tag{2.10}$$

and

$$\left\| \sum_{i=1}^{n} p_i A_i B_i - \sum_{i=1}^{n} p_i A_i \sum_{i=1}^{n} p_i B_i \right\|_1 \leq \frac{1}{2} \left[\sum_{i=1}^{n} p_i (1 - p_i) \right] \sum_{i=1}^{n-1} \|\Delta A_i\|_2 \sum_{i=1}^{n-1} \|\Delta B_i\|_2 \tag{2.11}$$

The following corollary holds.

Corollary 2. *For p, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, assume that $\{A_i\}_{i=0}^n \subset B_p(H)$, $\{B_i\}_{i=0}^n \subset B_q(H)$ are sequences of operators, then we have the inequality*

$$\left\| \frac{1}{n} \sum_{i=1}^{n} A_i B_i - \frac{1}{n} \sum_{i=1}^{n} A_i \frac{1}{n} \sum_{i=1}^{n} B_i \right\|_1 \leq \frac{1}{2} \left(1 - \frac{1}{n} \right) \sum_{i=1}^{n-1} \|\Delta A_i\|_p \sum_{i=1}^{n-1} \|\Delta B_i\|_q . \tag{2.12}$$

In particular, if $\{A_i\}_{i=0}^n$, $\{B_i\}_{i=0}^n \subset B_2(H)$, then

$$\left\| \frac{1}{n} \sum_{i=1}^{n} A_i B_i - \frac{1}{n} \sum_{i=1}^{n} A_i \frac{1}{n} \sum_{i=1}^{n} B_i \right\|_1 \leq \frac{1}{2} \left(1 - \frac{1}{n} \right) \sum_{i=1}^{n-1} \|\Delta A_i\|_2 \sum_{i=1}^{n-1} \|\Delta B_i\|_2 . \tag{2.13}$$

We also have the following result:

Theorem 5. *For $p \geq 1$, assume that $\{A_i\}_{i=0}^n, \{B_i\}_{i=0}^n \subset B_p(H)$ are sequences of operators and $\alpha_i \in \mathbb{C}$ $(i = 1, ..., n)$. Then we have the inequality:*

$$\left\| \sum_{i=1}^n \alpha_i \sum_{i=1}^n \alpha_i A_i B_i - \sum_{i=1}^n \alpha_i A_i \sum_{i=1}^n \alpha_i B_i \right\|_p \leq \frac{1}{2} \left[\left(\sum_{i=1}^n |\alpha_i| \right)^2 - \sum_{i=1}^n |\alpha_i|^2 \right] \sum_{i=1}^{n-1} \|\Delta A_i\|_p \sum_{i=1}^{n-1} \|\Delta B_i\|_p . \tag{2.14}$$

Proof. Using the generalized triangle inequality and the property (1.11), we have successively:

$$\begin{aligned} &\left\| \sum_{i=1}^n \alpha_i \sum_{i=1}^n \alpha_i A_i B_i - \sum_{i=1}^n \alpha_i A_i \sum_{i=1}^n \alpha_i B_i \right\|_p \\ &= \left\| \sum_{1 \leq i < j \leq n} \alpha_i \alpha_j \sum_{k=i}^{j-1} \Delta A_k \sum_{l=i}^{j-1} \Delta B_l \right\|_p \\ &\leq \sum_{1 \leq i < j \leq n} |\alpha_i| |\alpha_j| \left\| \sum_{k=i}^{j-1} \Delta A_k \sum_{l=i}^{j-1} \Delta B_l \right\|_p \\ &\leq \sum_{1 \leq i < j \leq n} |\alpha_i| |\alpha_j| \left\| \sum_{k=i}^{j-1} \Delta A_k \right\|_p \left\| \sum_{l=i}^{j-1} \Delta B_l \right\| p \\ &\leq \sum_{1 \leq i < j \leq n} |\alpha_i| |\alpha_j| \sum_{k=i}^{j-1} \|\Delta A_k\|_p \sum_{l=i}^{j-1} \|\Delta B_l\|_p . \end{aligned}$$

Now, by making use of a similar argument as in the proof of Theorem 4 we obtain the desired result (2.14). □

Remark 2. *For $p \geq 1$, assume that $\{A_i\}_{i=0}^n \subset B_p(H)$ and $\alpha_i \in \mathbb{C}$ $(i = 1, ..., n)$. Then by (2.14) we have the inequality:*

$$\left\| \sum_{i=1}^n \alpha_i \sum_{i=1}^n \alpha_i A_i^2 - \left(\sum_{i=1}^n \alpha_i A_i \right)^2 \right\|_p \leq \frac{1}{2} \left[\left(\sum_{i=1}^n |\alpha_i| \right)^2 - \sum_{i=1}^n |\alpha_i|^2 \right] \left(\sum_{i=1}^{n-1} \|\Delta A_i\|_p \right)^2 . \tag{2.15}$$

For $p \geq 1$, assume that $\{B_i\}_{i=0}^n \subset B_p(H)$ and $\alpha_i \in \mathbb{C}$ $(i = 1, ..., n)$. Then by (2.14) we have the inequality:

$$\left\| \sum_{i=1}^n \alpha_i \sum_{i=1}^n \alpha_i |B_i|^2 - \left| \sum_{i=1}^n \alpha_i B_i \right|^2 \right\|_p \leq \frac{1}{2} \left[\left(\sum_{i=1}^n |\alpha_i| \right)^2 - \sum_{i=1}^n |\alpha_i|^2 \right] \left(\sum_{i=1}^{n-1} \|\Delta B_i\|_p \right)^2 . \tag{2.16}$$

3. Inequalities for Power Series

As some natural examples that are useful for applications, we can point out that, if

$$f(\lambda)=\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\lambda^n=\ln\frac{1}{1+\lambda},\ \ \lambda\in D(0,1);\tag{3.1}$$
$$g(\lambda)=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}\lambda^{2n}=\cos\lambda,\ \ \lambda\in\mathbb{C};$$
$$h(\lambda)=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}\lambda^{2n+1}=\sin\lambda,\ \ \lambda\in\mathbb{C};$$
$$l(\lambda)=\sum_{n=0}^{\infty}(-1)^n\lambda^n=\frac{1}{1+\lambda},\ \ \lambda\in D(0,1);$$

then the corresponding functions constructed by the use of the absolute values of the coefficients are

$$f_a(\lambda)=\sum_{n=1}^{\infty}\frac{1}{n}\lambda^n=\ln\frac{1}{1-\lambda},\ \ \lambda\in D(0,1);\tag{3.2}$$
$$g_a(\lambda)=\sum_{n=0}^{\infty}\frac{1}{(2n)!}\lambda^{2n}=\cosh\lambda,\ \ \lambda\in\mathbb{C};$$
$$h_a(\lambda)=\sum_{n=0}^{\infty}\frac{1}{(2n+1)!}\lambda^{2n+1}=\sinh\lambda,\ \ \lambda\in\mathbb{C};$$
$$l_a(\lambda)=\sum_{n=0}^{\infty}\lambda^n=\frac{1}{1-\lambda},\ \ \lambda\in D(0,1).$$

Other important examples of functions as power series representations with non-negative coefficients are:

$$\exp(\lambda)=\sum_{n=0}^{\infty}\frac{1}{n!}\lambda^n\qquad\lambda\in\mathbb{C},\tag{3.3}$$
$$\frac{1}{2}\ln\left(\frac{1+\lambda}{1-\lambda}\right)=\sum_{n=1}^{\infty}\frac{1}{2n-1}\lambda^{2n-1},\qquad\lambda\in D(0,1);$$
$$\sin^{-1}(\lambda)=\sum_{n=0}^{\infty}\frac{\Gamma\left(n+\frac{1}{2}\right)}{\sqrt{\pi}(2n+1)n!}\lambda^{2n+1},\qquad\lambda\in D(0,1);$$
$$\tanh^{-1}(\lambda)=\sum_{n=1}^{\infty}\frac{1}{2n-1}\lambda^{2n-1},\qquad\lambda\in D(0,1)$$
$${}_2F_1(\alpha,\beta,\gamma,\lambda)=\sum_{n=0}^{\infty}\frac{\Gamma(n+\alpha)\Gamma(n+\beta)\Gamma(\gamma)}{n!\Gamma(\alpha)\Gamma(\beta)\Gamma(n+\gamma)}\lambda^n,\alpha,\beta,\gamma>0,$$
$$\lambda\in D(0,1);$$

where Γ is *Gamma function*.

The following new result holds:

Theorem 6. *Let $f(\lambda)=\sum_{n=0}^{\infty}\alpha_n\lambda^n$ be a power series that is convergent on the open disk $D(0,R)$, with $R\geq 1$. For $p,\ q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, assume that $C\in B_p(H)$, $B\in B_q(H)$ with $CB=BC$ and $\|C\|_p, \|B\|_q<1$, then we have for $\lambda\in\mathbb{C}$ with $|\lambda|<R$ the inequality:*

$$\begin{aligned}(3.4)\qquad &\|f(\lambda)f(\lambda CB)-f(\lambda C)f(\lambda B)\|_1\\ &\leq\frac{1}{2}\frac{\|C-1_H\|_p\|B-1_H\|_q}{\left(1-\|C\|_p\right)\left(1-\|B\|_q\right)}\left[f_a^2(|\lambda|)-f_{a^2}\left(|\lambda|^2\right)\right],\end{aligned}$$

where

$$(3.5)\qquad f_{a^2}(\lambda):=\sum_{n=0}^{\infty}|\alpha_n|^2\lambda^n$$

has the radius of convergence R^2.

In particular, if $C,\ B\in B_2(H)$ with $CB=BC$ and $\|C\|_2, \|B\|_2<1$, then we have for $\lambda\in\mathbb{C}$ with $|\lambda|<R$, then

$$\begin{aligned}(3.6)\qquad &\|f(\lambda)f(\lambda CB)-f(\lambda C)f(\lambda B)\|_1\\ &\leq\frac{1}{2}\frac{\|C-1_H\|_2\|B-1_H\|_2}{(1-\|C\|_2)(1-\|B\|_2)}\left[f_a^2(|\lambda|)-f_{a^2}\left(|\lambda|^2\right)\right].\end{aligned}$$

Proof. From the inequality (2.1) and commutativity of B with C we have

$$\begin{aligned}&\left\|\sum_{i=0}^{n}\alpha_i\lambda^i\sum_{i=0}^{n}\alpha_i\lambda^i(CB)^i-\sum_{i=0}^{n}\alpha_i\lambda^iC^i\sum_{i=0}^{n}\alpha_i\lambda^iB^i\right\|_1\\ &=\left\|\sum_{i=0}^{n}\alpha_i\lambda^i\sum_{i=0}^{n}\alpha_i\lambda^iC^iB^i-\sum_{i=0}^{n}\alpha_i\lambda^iC^i\sum_{i=0}^{n}\alpha_i\lambda^iB^i\right\|_1\\ &\leq\frac{1}{2}\left[\left(\sum_{i=0}^{n}|\alpha_i||\lambda|^i\right)^2-\sum_{i=0}^{n}|\alpha_i|^2|\lambda|^{2i}\right]\\ &\times\sum_{j=0}^{n-1}\left\|C^{j+1}-C^j\right\|_p\sum_{j=0}^{n-1}\left\|B^{j+1}-B^j\right\|_q,\\ &=\frac{1}{2}\left[\left(\sum_{i=0}^{n}|\alpha_i||\lambda|^i\right)^2-\sum_{i=0}^{n}|\alpha_i|^2|\lambda|^{2i}\right]\\ &\times\sum_{j=0}^{n-1}\left\|C^j(C-1_H)\right\|_p\sum_{j=0}^{n-1}\left\|B^j(B-1_H)\right\|_q\end{aligned}$$

$$\begin{aligned}&\leq\frac{1}{2}\|C-1_H\|_p\|B-1_H\|_q\\ &\times\left[\left(\sum_{i=0}^{n}|\alpha_i||\lambda|^i\right)^2-\sum_{i=0}^{n}|\alpha_i|^2|\lambda|^{2i}\right]\sum_{j=0}^{n-1}\|C\|_p^j\sum_{j=0}^{n-1}\|B\|_q^j\end{aligned}$$

$$= \frac{1}{2} \|C - 1_H\|_p \|B - 1_H\|_q \left[\left(\sum_{i=0}^{n} |\alpha_i| |\lambda|^i \right)^2 - \sum_{i=0}^{n} |\alpha_i|^2 |\lambda|^{2i} \right] \times \frac{1 - \|C\|_p^n}{1 - \|C\|_p} \frac{1 - \|B\|_q^n}{1 - \|B\|_q}$$

for any $n \geq 1$.

Since all the series whose partial sums are involved in (2.11) are convergent, then by letting $n \to \infty$ in (2.11) we deduce the desired inequality (3.4). □

Corollary 3. *Let $f(\lambda) = \sum_{n=0}^{\infty} \alpha_n \lambda^n$ be a power series that is convergent on the open disk $D(0, R)$, with $R \geq 1$. For p, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, assume that $B \in B_p(H) \cap B_q(H)$ and $\|B\|_p, \|B\|_q < 1$, then we have for $\lambda \in \mathbb{C}$ with $|\lambda| < R$ the inequality:*

$$\left\| f(\lambda) f\left(\lambda B^2\right) - [f(\lambda B)]^2 \right\|_1 \leq \frac{1}{2} \frac{\|B - 1_H\|_p \|B - 1_H\|_q}{\left(1 - \|B\|_p\right)\left(1 - \|B\|_q\right)} \left[f_a^2(|\lambda|) - f_{a^2}\left(|\lambda|^2\right) \right]. \tag{3.7}$$

In particular, if $B \in B_2(H)$ and $\|B\|_2 < 1$, then we have for $\lambda \in \mathbb{C}$ with $|\lambda| < R$ the inequality:

$$\left\| f(\lambda) f\left(\lambda B^2\right) - [f(\lambda B)]^2 \right\|_1 \leq \frac{1}{2} \frac{\|B - 1_H\|_2^2}{\left(1 - \|B\|_2\right)^2} \left[f_a^2(|\lambda|) - f_{a^2}\left(|\lambda|^2\right) \right]. \tag{3.8}$$

We have the following result as well:

Corollary 4. *Let $f(\lambda) = \sum_{n=0}^{\infty} \alpha_n \lambda^n$ be a power series that is convergent on the open disk $D(0, R)$, with $R \geq 1$. For p, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, assume that $B \in B_p(H) \cap B_q(H)$ with $B^*B = BB^*$, namely B is normal, and $\|B\|_p, \|B\|_q < 1$, then we have for $\lambda \in \mathbb{C}$ with $|\lambda| < R$ the inequality:*

$$\left\| f(\lambda) f\left(\lambda |B|^2\right) - |f(\lambda B)|^2 \right\|_1 \leq \frac{1}{2} \frac{\|B - 1_H\|_p \|B - 1_H\|_q}{\left(1 - \|B\|_p\right)\left(1 - \|B\|_q\right)} \left[f_a^2(|\lambda|) - f_{a^2}\left(|\lambda|^2\right) \right]. \tag{3.9}$$

In particular, if $B \in B_2(H)$ and $\|B\|_2 < 1$, then we have for $\lambda \in \mathbb{C}$ with $|\lambda| < R$ the inequality:

$$\left\| f(\lambda) f\left(\lambda |B|^2\right) - |f(\lambda B)|^2 \right\|_1 \leq \frac{1}{2} \frac{\|B - 1_H\|_2^2}{\left(1 - \|B\|_2\right)^2} \left[f_a^2(|\lambda|) - f_{a^2}\left(|\lambda|^2\right) \right]. \tag{3.10}$$

4. Some Examples

Consider the function $f : D(0, 1) \to \mathbb{C}$ defined by

$$f(\lambda) = (1 - \lambda)^{-1} = \sum_{k=0}^{\infty} \lambda^k = f_a(\lambda).$$

Then

$$f_{a^2}(\lambda) := \sum_{n=0}^{\infty} \lambda^n = (1-\lambda)^{-1},$$

which implies that

$$f_a^2(|\lambda|) - f_{a^2}\left(|\lambda|^2\right) = \frac{2|\lambda|}{(1-|\lambda|)^2(1+|\lambda|)}, \quad |\lambda| < 1.$$

For $p,\ q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, assume that $C \in B_p(H),\ B \in B_q(H)$ with $CB = BC$ and $\|C\|_p, \|B\|_q < 1$, then we have for $\lambda \in \mathbb{C}$ with $|\lambda| < 1$ the inequality:

$$\begin{aligned} &\left\|(1-\lambda)^{-1}(1_H - \lambda CB)^{-1} - (1_H - \lambda C)^{-1}(1_H - \lambda B)^{-1}\right\|_1 \\ &\leq \frac{|\lambda|}{(1-|\lambda|)^2(1+|\lambda|)} \frac{\|C - 1_H\|_p \|B - 1_H\|_q}{\left(1 - \|C\|_p\right)\left(1 - \|B\|_q\right)}. \end{aligned} \tag{4.1}$$

In particular, if $C,\ B \in B_2(H)$ with $CB = BC$ and $\|C\|_2, \|B\|_2 < 1$, then we have for $\lambda \in \mathbb{C}$ with $|\lambda| < 1$,

$$\begin{aligned} &\left\|(1-\lambda)^{-1}(1_H - \lambda CB)^{-1} - (1_H - \lambda C)^{-1}(1_H - \lambda B)^{-1}\right\|_1 \\ &\leq \frac{|\lambda|}{(1-|\lambda|)^2(1+|\lambda|)} \frac{\|C - 1_H\|_2 \|B - 1_H\|_2}{(1 - \|C\|_2)(1 - \|B\|_2)}. \end{aligned} \tag{4.2}$$

If $B \in B_p(H) \cap B_q(H)$ with $\|B\|_p, \|B\|_q < 1$, then we have for $\lambda \in \mathbb{C}$ with $|\lambda| < 1$ the inequality:

$$\begin{aligned} &\left\|(1-\lambda)^{-1}\left(1_H - \lambda B^2\right)^{-1} - (1_H - \lambda B)^{-2}\right\|_1 \\ &\leq \frac{|\lambda|}{(1-|\lambda|)^2(1+|\lambda|)} \frac{\|B - 1_H\|_p \|B - 1_H\|_q}{\left(1 - \|B\|_p\right)\left(1 - \|B\|_q\right)}, \end{aligned} \tag{4.3}$$

and, in particular

$$\begin{aligned} &\left\|(1-\lambda)^{-1}\left(1_H - \lambda B^2\right)^{-1} - (1_H - \lambda B)^{-2}\right\|_1 \\ &\leq \frac{|\lambda|}{(1-|\lambda|)^2(1+|\lambda|)} \frac{\|B - 1_H\|_2^2}{(1 - \|B\|_2)^2}, \end{aligned} \tag{4.4}$$

provided that $B \in B_2(H) \cap$ with $\|B\|_2 < 1$.

For $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, assume that $B \in B_p(H) \cap B_q(H)$ with B is normal, and $\|B\|_p, \|B\|_q < 1$, then we have for $\lambda \in \mathbb{C}$ with $|\lambda| < R$ the inequality:

$$\begin{aligned} &\left\|(1-\lambda)^{-1}\left(1_H - \lambda|B|^2\right)^{-1} - |1_H - \lambda B|^{-2}\right\|_1 \\ &\leq \frac{1}{2} \frac{\|B - 1_H\|_p \|B - 1_H\|_q}{\left(1 - \|B\|_p\right)\left(1 - \|B\|_q\right)} \left[f_a^2(|\lambda|) - f_{a^2}\left(|\lambda|^2\right)\right]. \end{aligned} \tag{4.5}$$

In particular, if $B \in B_2(H)$ and $\|B\|_2 < 1$, then we have for $\lambda \in \mathbb{C}$ with $|\lambda| < R$ the inequality:

$$\left\| (1-\lambda)^{-1} \left(1_H - \lambda |B|^2\right)^{-1} - |1_H - \lambda B|^{-2} \right\|_1 \leq \frac{1}{2} \frac{\|B - 1_H\|_2^2}{(1 - \|B\|_2)^2} \left[f_a^2(|\lambda|) - f_{a^2}\left(|\lambda|^2\right) \right]. \tag{4.6}$$

We consider the *modified Bessel function functions of the first kind*

$$I_\nu(\lambda) := \left(\frac{1}{2}\lambda\right)^\nu \sum_{k=0}^\infty \frac{\left(\frac{1}{4}\lambda^2\right)^k}{k!\Gamma(\nu + k + 1)},\ \lambda \in C$$

where Γ is the *Gamma function* and ν is a real number. An integral formula to represent I_ν is

$$I_\nu(\lambda) = \frac{1}{\pi} \int_0^\pi e^{\lambda \cos\theta} \cos(\nu\theta)\, d\theta - \frac{\sin(\nu\pi)}{\pi} \int_0^\infty e^{-\lambda \cosh t - \nu t} dt,$$

which simplifies for ν an integer n to

$$I_n(\lambda) = \frac{1}{\pi} \int_0^\pi e^{\lambda \cos\theta} \cos(n\theta)\, d\theta.$$

For $n = 0$ we have

$$I_0(\lambda) = \frac{1}{\pi} \int_0^\pi e^{\lambda \cos\theta} d\theta = \sum_{k=0}^\infty \frac{\left(\frac{1}{4}\lambda^2\right)^k}{(k!)^2},\ \lambda \in C.$$

Now, if we consider the exponential function

$$f(\lambda) = \exp(\lambda) = \sum_{k=0}^\infty \frac{1}{k!}\lambda^k,$$

then for $\rho > 0$ we have

$$f_{a^2}(\rho) = \sum_{k=0}^\infty \frac{1}{(k!)^2}\rho^k = I_0(2\sqrt{\rho}),$$

which implies that

$$f_a^2(|\lambda|) - f_{a^2}\left(|\lambda|^2\right) = \exp(2|\lambda|) - I_0(2|\lambda|),\ \lambda \in C.$$

For $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, assume that $C \in B_p(H)$, $B \in B_q(H)$ with $CB = BC$, then we have for $\lambda \in \mathbb{C}$ the inequality:

$$\|\exp[\lambda(1_H + CB)] - \exp[\lambda(C + B)]\|_1 \leq \frac{1}{2} \frac{\|C - 1_H\|_p \|B - 1_H\|_q}{\left(1 - \|C\|_p\right)\left(1 - \|B\|_q\right)} [\exp(2|\lambda|) - I_0(2|\lambda|)]. \tag{4.7}$$

In particular,

$$\|\exp[\lambda(1_H + CB)] - \exp[\lambda(C + B)]\|_1 \leq \frac{1}{2} \frac{\|C - 1_H\|_2 \|B - 1_H\|_2}{(1 - \|C\|_2)(1 - \|B\|_2)} [\exp(2|\lambda|) - I_0(2|\lambda|)], \tag{4.8}$$

provided that $C, B \in B_2(H)$ with $CB = BC$.

If $B\in B_p(H)\cap B_q(H)$, then we have for $\lambda\in\mathbb{C}$ the inequality

$$\left\|\exp\left[\lambda\left(1_H+B^2\right)\right]-\exp\left(2\lambda B\right)\right\|_1 \le \frac{1}{2}\frac{\left\|B-1_H\right\|_p\left\|B-1_H\right\|_q}{\left(1-\left\|B\right\|_p\right)\left(1-\left\|B\right\|_q\right)}\left[\exp\left(2\left|\lambda\right|\right)-I_0\left(2\left|\lambda\right|\right)\right]. \tag{4.9}$$

In particular,

$$\left\|\exp\left[\lambda\left(1_H+B^2\right)\right]-\exp\left(2\lambda B\right)\right\|_1 \le \frac{1}{2}\frac{\left\|B-1_H\right\|_2^2}{\left(1-\left\|B\right\|_2\right)^2}\left[\exp\left(2\left|\lambda\right|\right)-I_0\left(2\left|\lambda\right|\right)\right], \tag{4.10}$$

provided that $B\in B_2(H)$.

For $p,q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, assume that $B\in B_p(H)\cap B_q(H)$ with B is normal, then we have for $\lambda\in\mathbb{C}$

$$\left\|\exp\left[\lambda\left(1_H+\left|B\right|^2\right)\right]-\exp\left[\lambda\left(B+B^*\right)\right]\right\|_1 \le \frac{1}{2}\frac{\left\|B-1_H\right\|_p\left\|B-1_H\right\|_q}{\left(1-\left\|B\right\|_p\right)\left(1-\left\|B\right\|_q\right)}\left[\exp\left(2\left|\lambda\right|\right)-I_0\left(2\left|\lambda\right|\right)\right]. \tag{4.11}$$

Finally, we also have for $\lambda\in\mathbb{C}$ that

$$\left\|\exp\left[\lambda\left(1_H+\left|B\right|^2\right)\right]-\exp\left[\lambda\left(B+B^*\right)\right]\right\|_1 \le \frac{1}{2}\frac{\left\|B-1_H\right\|_2^2}{\left(1-\left\|B\right\|_2\right)^2}\left[\exp\left(2\left|\lambda\right|\right)-I_0\left(2\left|\lambda\right|\right)\right], \tag{4.12}$$

provided that $B\in B_2(H)$.

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