p-SCHATTEN NORM HÖLDER'S TYPE INEQUALITIES FOR ČEBYŠEV'S FUNCTIONAL WITH COMPLEX WEIGHTS

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Abstract. An operator $A \in \mathcal{B}(H)$ is said to belong to the *von Neumann-Schatten class* $\mathcal{B}_p(H)$, $1 \leq p < \infty$ if the *p-Schatten norm* is finite

$$\|A\|_p := [\operatorname{tr}(|A|^p)]^{1/p} < \infty.$$

For $p,\ q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, assume that $\{A_i\}_{i=0}^n \subset B_p(H)$, $\{B_i\}_{i=0}^n \subset B_q(H)$ are sequences of operators and $\alpha_i \in \mathbb{C}$ $(i = 1, ..., n)$. Then we have the inequality:

$$\left\|\sum_{i=1}^n \alpha_i \sum_{i=1}^n \alpha_i A_i B_i - \sum_{i=1}^n \alpha_i A_i \sum_{i=1}^n \alpha_i B_i\right\|_1 \leq \sum_{1\leq j<i\leq n} (i-j)|\alpha_i||\alpha_j| \left(\sum_{k=1}^{n-1} \|\Delta A_k\|_p^\beta\right)^{\frac{1}{\beta}} \left(\sum_{k=1}^{n-1} \|\Delta B_k\|_q^\gamma\right)^{\frac{1}{\gamma}},$$

where $\beta > 1$, $\frac{1}{\beta} + \frac{1}{\gamma} = 1$ and $\Delta A_i := A_{i+1} - A_i$ $(i = 1, ..., n-1)$ are the usual forward differences. Applications for operator functions defined by power series with complex coefficients with applications for the inverse and exponential functions are also provided.

1. Introduction

In 1935, G. Grüss [15] proved the following integral inequality which gives an approximation of the integral mean of the product in terms of the product of the integral means integrals as follows:

$$\left|\frac{1}{b-a}\int_a^b f(x)\,g(x)\,dx - \frac{1}{b-a}\int_a^b f(x)\,dx \cdot \frac{1}{b-a}\int_a^b g(x)\,dx\right| \leq \frac{1}{4}(\Phi-\varphi)(\Gamma-\gamma) \tag{1.1}$$

where $f, g : [a,b] \to \mathbb{R}$ are integrable on $[a,b]$ and satisfying the assumption

$$\varphi \leq f(x) \leq \Phi, \gamma \leq g(x) \leq \Gamma$$

for each $x \in [a,b]$ where $\varphi, \Phi, \gamma, \Gamma$ are given real constants.

Moreover the constant $\frac{1}{4}$ is sharp in the sense that it can not be replaced by a smaller one.

For a simple proof of (1.1) as well as for some other integral inequalities of Grüss' type see the Chapter X of the recent book [16] by Mitrinović, Pečarić and Fink.

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In 1950, M. Biernacki, H. Pidek and C. Ryll-Nardzewski [3] established the following discrete version of Grüss' inequality, see also [16, Ch. X]:

Theorem 1. *Let* $a=(a_1,...,a_n)$, $b=(b_1,...,b_n)$ *be two* n*-tuples of real numbers such that* $r\le a_i\le R$ *and* $s\le b_i\le S$ *for* $i=1,...,n$. *Then one has the inequality:*

$$\left|\frac{1}{n}\sum_{i=1}^{n}a_ib_i-\frac{1}{n}\sum_{i=1}^{n}a_i\cdot\frac{1}{n}\sum_{i=1}^{n}b_i\right| \le\frac{1}{n}\left[\frac{n}{2}\right]\left(1-\frac{1}{n}\left[\frac{n}{2}\right]\right)(R-r)(S-s) \tag{1.2}$$

when $[x]$ *is the integer part of* $x, x\in\mathbb{R}$.

In 1981, A. Lupaş [16, Ch. X] proved some similar results for the first difference of a as follows :

Theorem 2. *Let* a,b *two monotonic* n*-tuples in the same sense and* p *a positive* n*-tuple. Then*

$$\min_{1\le i\le n-1}|a_{i+1}-a_i|\min_{1\le i\le n-1}|b_{i+1}-b_i|\left[\frac{1}{P_n}\sum_{i=1}^{n}i^2p_i-\left(\frac{1}{P_n}\sum_{i=1}^{n}ip_i\right)^2\right] \le\frac{1}{P_n}\sum_{i=1}^{n}p_ia_ib_i-\frac{1}{P_n}\sum_{i=1}^{n}p_ia_i\cdot\frac{1}{P_n}\sum_{i=1}^{n}p_ib_i \le\max_{1\le i\le n-1}|a_{i+1}-a_i|\max_{1\le i\le n-1}|b_{i+1}-b_i|\left[\frac{1}{P_n}\sum_{i=1}^{n}i^2p_i-\left(\frac{1}{P_n}\sum_{i=1}^{n}ip_i\right)^2\right]. \tag{1.3}$$

If there exists the numbers $\bar{a},\bar{a}_1,r,r_1,(rr_1>0)$ *such that* $a_k=\bar{a}+kr$ *and* $b_k=\bar{a}_1+kr_1$, *then in (1.3) the equality holds.*

In order to extend these results for p-Schatten norms we need the following preparations.

Let $(H;\langle .,.\rangle)$ be a complex Hilbert space and $\mathcal{B}(H)$ the Banach algebra of all bounded linear operators on H. If $\{e_i\}_{i\in I}$ an orthonormal basis of H, we say that $A\in\mathcal{B}(H)$ is of *trace class* if

$$\|A\|_1:=\sum_{i\in I}\langle |A|e_i,e_i\rangle<\infty. \tag{1.4}$$

The definition of $\|A\|_1$ does not depend on the choice of the orthonormal basis $\{e_i\}_{i\in I}$. We denote by $\mathcal{B}_1(H)$ the set of trace class operators in $\mathcal{B}(H)$.

We define the *trace* of a trace class operator $A\in\mathcal{B}_1(H)$ to be

$$\operatorname{tr}(A):=\sum_{i\in I}\langle Ae_i,e_i\rangle, \tag{1.5}$$

where $\{e_i\}_{i\in I}$ an orthonormal basis of H. Note that this coincides with the usual definition of the trace if H is finite-dimensional. We observe that the series (1.5) converges absolutely and it is independent from the choice of basis.

The following result collects some properties of the trace:

Theorem 3. *We have:*

(i) If $A\in\mathcal{B}_1(H)$ *then* $A^*\in\mathcal{B}_1(H)$ *and*

$$\operatorname{tr}(A^*)=\overline{\operatorname{tr}(A)}; \tag{1.6}$$

(ii) If $A\in\mathcal{B}_1(H)$ *and* $T\in\mathcal{B}(H)$, *then* $AT,\ TA\in\mathcal{B}_1(H)$ *and*

$$\operatorname{tr}(AT)=\operatorname{tr}(TA)\ \text{ and }\ |\operatorname{tr}(AT)|\le\|A\|_1\|T\|; \tag{1.7}$$

(iii) $\operatorname{tr}(\cdot)$ *is a bounded linear functional on* $\mathcal{B}_1(H)$ *with* $\|\operatorname{tr}\|=1$;
(iv) If $A,\ B\in\mathcal{B}_2(H)$ *then* $AB,\ BA\in\mathcal{B}_1(H)$ *and* $\operatorname{tr}(AB)=\operatorname{tr}(BA)$;
(v) $\mathcal{B}_{fin}(H)$, *the space of operators of finite rank, is a dense subspace of* $\mathcal{B}_1(H)$.

An operator $A\in\mathcal{B}(H)$ is said to belong to the *von Neumann-Schatten class* $\mathcal{B}_p(H)$, $1\le p<\infty$ if the *p-Schatten norm* is finite [20, p. 60-64]

$$\|A\|_p:=[\operatorname{tr}(|A|^p)]^{1/p}<\infty.$$

For $1<p<q<\infty$ we have that

$$\mathcal{B}_1(H)\subset\mathcal{B}_p(H)\subset\mathcal{B}_q(H)\subset\mathcal{B}(H) \tag{1.8}$$

and

$$\|A\|_1\ge\|A\|_p\ge\|A\|_q\ge\|A\|. \tag{1.9}$$

For $p\ge1$ the functional $\|\cdot\|_p$ is a *norm* on the $*$-ideal $\mathcal{B}_p(H)$ and $\left(\mathcal{B}_p(H),\|\cdot\|_p\right)$ is a Banach space.

Also, see for instance [20, p. 60-64],

$$\|A\|_p=\|A^*\|_p,\ A\in\mathcal{B}_p(H) \tag{1.10}$$

$$\|AB\|_p\le\|A\|_p\|B\|_p,\ A,B\in\mathcal{B}_p(H) \tag{1.11}$$

and

$$\|AB\|_p\le\|A\|_p\|B\|,\ \|BA\|_p\le\|B\|\|A\|_p,\ A\in\mathcal{B}_p(H),\ B\in\mathcal{B}(H). \tag{1.12}$$

This implies that

$$\|CAB\|_p\le\|C\|\|A\|_p\|B\|,\ A\in\mathcal{B}_p(H),\ B,\ C\in\mathcal{B}(H). \tag{1.13}$$

In terms of *p-Schatten norm* we have the *Hölder inequality* for $p,\ q>1$ with $\frac{1}{p}+\frac{1}{q}=1$

$$(|\operatorname{tr}(AB)|\le)\,\|AB\|_1\le\|A\|_p\|B\|_q,\ A\in\mathcal{B}_p(H),\ B\in\mathcal{B}_q(H). \tag{1.14}$$

For the theory of trace functionals and their applications the reader is referred to [19] and [20].

For some classical trace inequalities see [7], [8], and [18], which are continuations of the work of Bellman [1].

2. An Inequality of Grüss Type for β-Norm

The following result that provides a version for the β-norm, $\beta>1$ of the forward difference also holds.

Theorem 4. *For $p,\ q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, assume that $\{A_i\}_{i=0}^{n}\subset B_p(H)$, $\{B_i\}_{i=0}^{n}\subset B_q(H)$ are sequences of operators and $\alpha_i\in\mathbb{C}\quad(i=1,...,n)$. Then we have the inequality:*

$$\left\|\sum_{i=1}^{n}\alpha_i\sum_{i=1}^{n}\alpha_iA_iB_i-\sum_{i=1}^{n}\alpha_iA_i\sum_{i=1}^{n}\alpha_iB_i\right\|_1 \leq \sum_{1\leq j<i\leq n}(i-j)\left|\alpha_i\right|\left|\alpha_j\right|\left(\sum_{k=1}^{n-1}\left\|\Delta A_k\right\|_p^{\beta}\right)^{\frac{1}{\beta}}\left(\sum_{k=1}^{n-1}\left\|\Delta B_k\right\|_q^{\gamma}\right)^{\frac{1}{\gamma}}, \tag{2.1}$$

where $\beta>1$, $\frac{1}{\beta}+\frac{1}{\gamma}=1$ and $\Delta A_i:=A_{i+1}-A_i\quad(i=1,...,n-1)$ are the usual forward differences.

In particular, if $\{A_i\}_{i=0}^{n},\ \{B_i\}_{i=0}^{n}\subset B_2(H)$, then for $p=q=2$ we have

$$\left\|\sum_{i=1}^{n}\alpha_i\sum_{i=1}^{n}\alpha_iA_iB_i-\sum_{i=1}^{n}\alpha_iA_i\sum_{i=1}^{n}\alpha_iB_i\right\|_1 \leq \sum_{1\leq j<i\leq n}(i-j)\left|\alpha_i\right|\left|\alpha_j\right|\left(\sum_{k=1}^{n-1}\left\|\Delta A_k\right\|_2^{\beta}\right)^{\frac{1}{\beta}}\left(\sum_{k=1}^{n-1}\left\|\Delta B_k\right\|_2^{\gamma}\right)^{\frac{1}{\gamma}}. \tag{2.2}$$

Proof. Let us start with the following identity which can be proved by direct computation:

$$\sum_{i=1}^{n}\alpha_i\sum_{i=1}^{n}\alpha_iA_iB_i-\alpha_iA_i\sum_{i=1}^{n}\alpha_iB_i\sum_{i=1}^{n} = \frac{1}{2}\sum_{i,j=1}^{n}\alpha_i\alpha_j\left(A_j-A_i\right)\left(B_j-B_i\right)=\sum_{1\leq i<j\leq n}^{n}\alpha_i\alpha_j\left(A_j-A_i\right)\left(B_j-B_i\right). \tag{2.3}$$

As $i<j$ we can write that $B_j-B_i=\sum_{k=i}^{j-1}\Delta B_k$ and $A_j-A_i=\sum_{k=i}^{j-1}\Delta A_k$. Using the generalized triangle inequality and the property (1.14) we have successively from (2.3) :

$$\left\|\sum_{i=1}^{n}\alpha_i\sum_{i=1}^{n}\alpha_iA_iB_i-\sum_{i=1}^{n}\alpha_iA_i\sum_{i=1}^{n}\alpha_iB_i\right\|_1 = \left\|\sum_{1\leq i<j\leq n}^{n}\alpha_i\alpha_j\sum_{k=i}^{j-1}\Delta A_k\sum_{k=i}^{j-1}\Delta B_k\right\|_1 \tag{2.4}$$

$$\leq \sum_{1\leq i<j\leq n}^{n} |\alpha_i\alpha_j| \left\| \sum_{k=i}^{j-1} \Delta A_k \sum_{k=i}^{j-1} \Delta B_k \right\|_1$$
$$\leq \sum_{1\leq i<j\leq n}^{n} |\alpha_i\alpha_j| \left\| \sum_{k=i}^{j-1} \Delta A_k \right\|_p \left\| \sum_{k=i}^{j-1} \Delta B_k \right\|_q$$
$$\leq \sum_{1\leq i<j\leq n}^{n} |\alpha_i\alpha_j| \sum_{k=i}^{j-1} \|\Delta A_k\|_p \sum_{k=i}^{j-1} \|\Delta B_k\|_q =: \Gamma.$$

Using Hölder's discrete inequality, we can state that

$$\sum_{k=j}^{i-1} \|\Delta A_k\|_p \leq (i-j)^{\frac{1}{\gamma}} \left(\sum_{k=j}^{i-1} \|\Delta A_k\|_p^\beta \right)^{\frac{1}{\beta}}$$

and

$$\sum_{k=j}^{i-1} \|\Delta B_k\|_q \leq (i-j)^{\frac{1}{\beta}} \left(\sum_{k=j}^{i-1} \|\Delta B_k\|_q^\gamma \right)^{\frac{1}{\gamma}}$$

where $\beta, \gamma > 1$ and $\frac{1}{\beta} + \frac{1}{\gamma} = 1$, and then, by multiplication, we have

$$\Gamma \leq \sum_{1\leq j<i\leq n} |\alpha_i| |\alpha_j| (i-j) \left(\sum_{k=j}^{i-1} \|\Delta A_k\|_p^\beta \right)^{\frac{1}{\beta}} \left(\sum_{k=j}^{i-1} \|\Delta B_k\|_q^\gamma \right)^{\frac{1}{\gamma}}. \tag{2.5}$$

As

$$\sum_{k=j}^{i-1} \|\Delta A_k\|_p^\beta \leq \sum_{k=1}^{n-1} \|\Delta A_k\|_p^\beta$$

and

$$\sum_{k=j}^{i-1} \|\Delta B_k\|_q^\gamma \leq \sum_{k=1}^{n-1} \|\Delta B_k\|_q^\gamma,$$

for all $1 \leq j < i \leq n$, then by (2.4) and (2.5), we get the desired inequality (2.1). □

If we take in (2.1) $\beta = p$ and $\gamma = q$ and using the trace property (1.7) we get

$$\left| \operatorname{tr}\left(\sum_{i=1}^n \alpha_i \sum_{i=1}^n \alpha_i A_i B_i - \sum_{i=1}^n \alpha_i A_i \sum_{i=1}^n \alpha_i B_i \right) \right| \tag{2.6}$$
$$\leq \sum_{1\leq j<i\leq n} (i-j) |\alpha_i| |\alpha_j| \left[\operatorname{tr}\left(\sum_{k=1}^{n-1} |\Delta A_k|^p \right) \right]^{\frac{1}{p}} \left[\operatorname{tr}\left(\sum_{k=1}^{n-1} |\Delta B_k|^q \right) \right]^{\frac{1}{q}},$$

for p, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ and $\{A_i\}_{i=0}^n \subset B_p(H)$, $\{B_i\}_{i=0}^n \subset B_q(H)$.

In particular, if $\{B_i\}_{i=0}^{n}, \{A_i\}_{i=0}^{n} \subset B_2(H)$ then

$$\left|\operatorname{tr}\left(\sum_{i=1}^{n}\alpha_i\sum_{i=1}^{n}\alpha_iA_iB_i-\sum_{i=1}^{n}\alpha_iA_i\sum_{i=1}^{n}\alpha_iB_i\right)\right| \leq \sum_{1\leq j<i\leq n}(i-j)|\alpha_i||\alpha_j|\operatorname{tr}\left(\sum_{k=1}^{n-1}|\Delta A_k|^2\right)^{1/2}\operatorname{tr}\left(\sum_{k=1}^{n-1}|\Delta B_k|^2\right)^{1/2}. \tag{2.7}$$

Remark 1. *A coarser upper bound, which can be more useful may be obtained by applying Cauchy-Schwarz's inequality:*

$$\sum_{1\leq j<i\leq n}(i-j)|\alpha_i||\alpha_j| \leq \left(\sum_{1\leq j<i\leq n}|\alpha_i||\alpha_j|\right)^{\frac{1}{2}}\left(\sum_{1\leq j<i\leq n}|\alpha_i||\alpha_j|(i-j)^2\right)^{\frac{1}{2}}$$

and taking into account that

$$\sum_{1\leq j<i\leq n}|\alpha_i||\alpha_j| = \frac{1}{2}\left[\left(\sum_{i=1}^{n}|\alpha_i|\right)^2-\sum_{i=1}^{n}|\alpha_i|^2\right]$$

and

$$\begin{aligned}\sum_{1\leq i<j\leq n}^{n}|\alpha_i||\alpha_j|(j-i)^2 &= \frac{1}{2}\sum_{i,j=1}^{n}|\alpha_i||\alpha_j|(j-i)^2\\ &= \frac{1}{2}\sum_{i,j=1}^{n}|\alpha_i||\alpha_j|\left(i^2-2ij+j^2\right)\\ &= \sum_{i=1}^{n}|\alpha_i|\sum_{i=1}^{n}i^2|\alpha_i|-\left(\sum_{i=1}^{n}i|\alpha_i|\right)^2.\end{aligned}$$

Thus, from (2.1)*, we can state the inequality*

$$\begin{aligned}&\left\|\sum_{i=1}^{n}\alpha_i\sum_{i=1}^{n}\alpha_iA_iB_i-\sum_{i=1}^{n}\alpha_iA_i\sum_{i=1}^{n}\alpha_iB_i\right\|_1\\ &\leq \frac{\sqrt{2}}{2}\left[\left(\sum_{i=1}^{n}|\alpha_i|\right)^2-\sum_{i=1}^{n}|\alpha_i|^2\right]^{\frac{1}{2}}\left[\sum_{i=1}^{n}|\alpha_i|\sum_{i=1}^{n}i^2|\alpha_i|-\left(\sum_{i=1}^{n}i|\alpha_i|\right)^2\right]^{\frac{1}{2}}\\ &\times\left(\sum_{k=1}^{n-1}\|\Delta A_k\|_p^{\beta}\right)^{\frac{1}{\beta}}\left(\sum_{k=1}^{n-1}\|\Delta B_k\|_q^{\gamma}\right)^{\frac{1}{\gamma}},\end{aligned} \tag{2.8}$$

where $\beta > 1,\ \frac{1}{\beta}+\frac{1}{\gamma}=1.$

The following corollary holds.

Corollary 1. *For $p,\ q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, assume that $\{A_i\}_{i=0}^{n}\subset B_p(H)$, $\{B_i\}_{i=0}^{n}\subset B_q(H)$, then we have*

$$\left\|\frac{1}{n}\sum_{i=1}^{n}A_iB_i-\frac{1}{n}\sum_{i=1}^{n}A_i\frac{1}{n}\sum_{i=1}^{n}B_i\right\|_1 \leq \frac{n^2-1}{6n}\left(\sum_{k=1}^{n-1}\|\Delta A_k\|_p^{\beta}\right)^{\frac{1}{\beta}}\left(\sum_{k=1}^{n-1}\|\Delta B_k\|_q^{\gamma}\right)^{\frac{1}{\gamma}} \tag{2.9}$$

where $\beta>1$, $\frac{1}{\beta}+\frac{1}{\gamma}=1$.

In particular,

$$\left\|\frac{1}{n}\sum_{i=1}^{n}A_iB_i-\frac{1}{n}\sum_{i=1}^{n}A_i\frac{1}{n}\sum_{i=1}^{n}B_i\right\|_1 \leq \frac{n^2-1}{6n}\left(\sum_{k=1}^{n-1}\|\Delta A_k\|_2^{\beta}\right)^{\frac{1}{\beta}}\left(\sum_{k=1}^{n-1}\|\Delta B_k\|_2^{\gamma}\right)^{\frac{1}{\gamma}}, \tag{2.10}$$

where $\{B_i\}_{i=0}^{n},\ \{A_i\}_{i=0}^{n}\subset B_2(H)$.

Proof. The proof follows by (2.1), putting $\alpha_i=\frac{1}{n}$ and taking into account that

$$\begin{aligned}
&\sum_{1\leq j<i\leq n}(i-j)\\
&=\sum_{1\leq j\leq 2}(2-j)+\sum_{1\leq j\leq 3}(3-j)+...+\sum_{1\leq j\leq n}(n-j)\\
&=1^2+2^2+...+n^2-1-(1+2)-(1+2+3)-...-(1+2+...+n)\\
&=\sum_{k=1}^{n}k^2-\sum_{k=1}^{n}\frac{k(k+1)}{2}=\frac{1}{2}\left(\sum_{k=1}^{n}k^2-\sum_{k=1}^{n}k\right)=\frac{n\left(n^2-1\right)}{6},
\end{aligned}$$

and the corollary is thus proved. □

Remark 2. *Let $p,\ q>1$ with $\frac{1}{p}+\frac{1}{q}=1$. If in* (2.1) *and* (2.9) *we assume that $\beta=\gamma=2$, then we get the inequalities:*

$$\left\|\sum_{i=1}^{n}\alpha_i\sum_{i=1}^{n}\alpha_iA_iB_i-\sum_{i=1}^{n}\alpha_iA_i\sum_{i=1}^{n}\alpha_iB_i\right\|_1 \leq \sum_{1\leq j<i\leq n}(i-j)\left|\alpha_i\right|\left|\alpha_j\right|\left(\sum_{k=1}^{n-1}\|\Delta A_k\|_p^{2}\right)^{\frac{1}{2}}\left(\sum_{k=1}^{n-1}\|\Delta B_k\|_q^{2}\right)^{\frac{1}{2}} \tag{2.11}$$

and

$$\left\|\frac{1}{n}\sum_{i=1}^{n}A_iB_i-\frac{1}{n}\sum_{i=1}^{n}A_i\frac{1}{n}\sum_{i=1}^{n}B_i\right\|_1 \leq \frac{n^2-1}{6n}\left(\sum_{k=1}^{n-1}\|\Delta A_k\|_p^{2}\right)^{\frac{1}{2}}\left(\sum_{k=1}^{n-1}\|\Delta B_k\|_q^{2}\right)^{\frac{1}{2}}, \tag{2.12}$$

respectively.

We also have the inequality

$$\left\|\sum_{i=1}^{n}\alpha_i\sum_{i=1}^{n}\alpha_iA_iB_i-\sum_{i=1}^{n}\alpha_iA_i\sum_{i=1}^{n}\alpha_iB_i\right\|_1 \tag{2.13}$$
$$\leq\frac{\sqrt{2}}{2}\left[\left(\sum_{i=1}^{n}|\alpha_i|\right)^2-\sum_{i=1}^{n}|\alpha_i|^2\right]^{\frac{1}{2}}\left[\sum_{i=1}^{n}|\alpha_i|\sum_{i=1}^{n}i^2|\alpha_i|-\left(\sum_{i=1}^{n}i|\alpha_i|\right)^2\right]^{\frac{1}{2}}$$
$$\times\left(\sum_{k=1}^{n-1}\|\Delta A_k\|_p^2\right)^{\frac{1}{2}}\left(\sum_{k=1}^{n-1}\|\Delta B_k\|_q^2\right)^{\frac{1}{2}}.$$

In the case when $B_i=A_i$, $i\in\{1,...,n\}$ we get from (2.11)

$$\left\|\sum_{i=1}^{n}\alpha_i\sum_{i=1}^{n}\alpha_iA_i^2-\left(\sum_{i=1}^{n}\alpha_iA_i\right)^2\right\| \tag{2.14}$$
$$\leq\sum_{1\leq j<i\leq n}(i-j)|\alpha_i||\alpha_j|\left(\sum_{k=1}^{n-1}\|\Delta A_k\|_p^2\right)^{\frac{1}{2}}\left(\sum_{k=1}^{n-1}\|\Delta A_k\|_q^2\right)^{\frac{1}{2}}$$

and, in particular

$$\left\|\sum_{i=1}^{n}\alpha_i\sum_{i=1}^{n}\alpha_iA_i^2-\left(\sum_{i=1}^{n}\alpha_iA_i\right)^2\right\|_1\leq\sum_{1\leq j<i\leq n}(i-j)|\alpha_i||\alpha_j|\sum_{k=1}^{n-1}\|\Delta A_k\|_2^2 \tag{2.15}$$

From (2.13) we derive

$$\left\|\sum_{i=1}^{n}\alpha_i\sum_{i=1}^{n}\alpha_iA_i^2-\left(\sum_{i=1}^{n}\alpha_iA_i\right)^2\right\|_1 \tag{2.16}$$
$$\leq\frac{\sqrt{2}}{2}\left[\left(\sum_{i=1}^{n}|\alpha_i|\right)^2-\sum_{i=1}^{n}|\alpha_i|^2\right]^{\frac{1}{2}}\left[\sum_{i=1}^{n}|\alpha_i|\sum_{i=1}^{n}i^2|\alpha_i|-\left(\sum_{i=1}^{n}i|\alpha_i|\right)^2\right]^{\frac{1}{2}}$$
$$\times\left(\sum_{k=1}^{n-1}\|\Delta A_k\|_p^2\right)^{\frac{1}{2}}\left(\sum_{k=1}^{n-1}\|\Delta A_k\|_q^2\right)^{\frac{1}{2}}$$

and, in particular

$$\left\|\sum_{i=1}^{n}\alpha_i\sum_{i=1}^{n}\alpha_iA_i^2-\left(\sum_{i=1}^{n}\alpha_iA_i\right)^2\right\|_1 \tag{2.17}$$
$$\leq\frac{\sqrt{2}}{2}\left[\left(\sum_{i=1}^{n}|\alpha_i|\right)^2-\sum_{i=1}^{n}|\alpha_i|^2\right]^{\frac{1}{2}}\left[\sum_{i=1}^{n}|\alpha_i|\sum_{i=1}^{n}i^2|\alpha_i|-\left(\sum_{i=1}^{n}i|\alpha_i|\right)^2\right]^{\frac{1}{2}}$$
$$\times\sum_{k=1}^{n-1}\|\Delta A_k\|_2^2.$$

We also have

Theorem 5. *For $p\geq 1$, assume that $\{A_i\}_{i=0}^{n}, \{B_i\}_{i=0}^{n}\subset B_p(H)$ are sequences of operators and $\alpha_i\in\mathbb{C}$ $(i=1,...,n)$. Then we have the inequality:*

$$\left\|\sum_{i=1}^{n}\alpha_i\sum_{i=1}^{n}\alpha_iA_iB_i-\sum_{i=1}^{n}\alpha_iA_i\sum_{i=1}^{n}\alpha_iB_i\right\|_p \tag{2.18}$$
$$\leq\sum_{1\leq j<i\leq n}(i-j)\left|\alpha_i\right|\left|\alpha_j\right|\left(\sum_{k=1}^{n-1}\left\|\Delta A_k\right\|_p^{\beta}\right)^{\frac{1}{\beta}}\left(\sum_{k=1}^{n-1}\left\|\Delta B_k\right\|_p^{\gamma}\right)^{\frac{1}{\gamma}},$$

where $\beta>1$, $\frac{1}{\beta}+\frac{1}{\gamma}=1$.

The proof follows by the fact that

$$\left\|\sum_{i=1}^{n}\alpha_i\sum_{i=1}^{n}\alpha_iA_iB_i\sum_{i=1}^{n}\alpha_iA_iB_i-\sum_{i=1}^{n}\alpha_iA_i\sum_{i=1}^{n}\alpha_iB_i\right\|_p$$
$$=\left\|\sum_{1\leq i<j\leq n}^{n}\alpha_i\alpha_j\sum_{k=i}^{j-1}\Delta A_k\sum_{k=i}^{j-1}\Delta B_k\right\|_p$$
$$\leq\sum_{1\leq i<j\leq n}^{n}\left|\alpha_i\alpha_j\right|\left\|\sum_{k=i}^{j-1}\Delta A_k\sum_{k=i}^{j-1}\Delta B_k\right\|_p$$
$$\leq\sum_{1\leq i<j\leq n}^{n}\left|\alpha_i\alpha_j\right|\left\|\sum_{k=i}^{j-1}\Delta A_k\right\|_p\left\|\sum_{k=i}^{j-1}\Delta B_k\right\|_p$$
$$\leq\sum_{1\leq i<j\leq n}^{n}\left|\alpha_i\alpha_j\right|\sum_{k=i}^{j-1}\left\|\Delta A_k\right\|_p\sum_{k=i}^{j-1}\left\|\Delta B_k\right\|_p$$

and using a similar argument to the one in the proof of Theorem 4.

3. Inequalities for Power Series

As some natural examples that are useful for applications, we can point out that, if

$$f(\lambda)=\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\lambda^n=\ln\frac{1}{1+\lambda},\ \lambda\in D(0,1); \tag{3.1}$$
$$g(\lambda)=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}\lambda^{2n}=\cos\lambda,\ \lambda\in\mathbb{C};$$
$$h(\lambda)=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}\lambda^{2n+1}=\sin\lambda,\ \lambda\in\mathbb{C};$$
$$l(\lambda)=\sum_{n=0}^{\infty}(-1)^n\lambda^n=\frac{1}{1+\lambda},\ \lambda\in D(0,1);$$

then the corresponding functions constructed by the use of the absolute values of the coefficients are

$$f_a(\lambda) = \sum_{n=1}^{\infty} \frac{1}{n} \lambda^n = \ln \frac{1}{1-\lambda}, \; \lambda \in D(0,1); \tag{3.2}$$
$$g_a(\lambda) = \sum_{n=0}^{\infty} \frac{1}{(2n)!} \lambda^{2n} = \cosh \lambda, \; \lambda \in \mathbb{C};$$
$$h_a(\lambda) = \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} \lambda^{2n+1} = \sinh \lambda, \; \lambda \in \mathbb{C};$$
$$l_a(\lambda) = \sum_{n=0}^{\infty} \lambda^n = \frac{1}{1-\lambda}, \; \lambda \in D(0,1).$$

Other important examples of functions as power series representations with non-negative coefficients are:

$$\exp(\lambda) = \sum_{n=0}^{\infty} \frac{1}{n!} \lambda^n \qquad \lambda \in \mathbb{C}, \tag{3.3}$$
$$\frac{1}{2} \ln\left(\frac{1+\lambda}{1-\lambda}\right) = \sum_{n=1}^{\infty} \frac{1}{2n-1} \lambda^{2n-1}, \qquad \lambda \in D(0,1);$$
$$\sin^{-1}(\lambda) = \sum_{n=0}^{\infty} \frac{\Gamma\left(n+\frac{1}{2}\right)}{\sqrt{\pi}(2n+1)n!} \lambda^{2n+1}, \qquad \lambda \in D(0,1);$$
$$\tanh^{-1}(\lambda) = \sum_{n=1}^{\infty} \frac{1}{2n-1} \lambda^{2n-1}, \qquad \lambda \in D(0,1)$$
$${}_2F_1(\alpha,\beta,\gamma,\lambda) = \sum_{n=0}^{\infty} \frac{\Gamma(n+\alpha)\Gamma(n+\beta)\Gamma(\gamma)}{n!\Gamma(\alpha)\Gamma(\beta)\Gamma(n+\gamma)} \lambda^n, \alpha,\beta,\gamma>0,$$
$$\lambda \in D(0,1);$$

where Γ is *Gamma function*.

We have the following result as well:

Theorem 6. *Let $f(\lambda) = \sum_{n=0}^{\infty} \alpha_n \lambda^n$ be a power series that is convergent on the open disk $D(0,R)$, with $R>0$. Let p, $q>1$ with $\frac{1}{p}+\frac{1}{q}=1$. If $A \in B_p(H)$ and $B \in B_q(H)$ with $AB=BA$ and $\|A\|_p, \|B\|_q < 1$, then we have for $\lambda \in \mathbb{C}$ with $|\lambda|<R$ the inequality:*

$$\begin{aligned} &\|f(\lambda) f(\lambda AB) - f(\lambda A) f(\lambda B)\|_1 \\ &\leq \frac{\sqrt{2}}{2} \frac{\|A-1_H\|_p \|B-1_H\|_q}{\left(1-\|A\|_p^{\beta}\right)^{\frac{1}{\beta}} \left(1-\|B\|_q^{\gamma}\right)^{\frac{1}{\gamma}}} \left[f_a^2(|\lambda|) - f_{a^2}\left(|\lambda|^2\right)\right]^{1/2} \\ &\times \left\{f_a(|\lambda|)\left[|\lambda| f_a'(|\lambda|) + |\lambda|^2 f_a''(|\lambda|)\right] - \left[|\lambda| f_a'(|\lambda|)\right]^2\right\}^{1/2} \end{aligned} \tag{3.4}$$

where $\beta>1$, $\frac{1}{\beta}+\frac{1}{\gamma}=1$.

In particular, we have

$$\begin{aligned}
&\left\| f(\lambda) f(\lambda AB) - f(\lambda A) f(\lambda B)\right\|_1 \\
&\le \frac{\sqrt{2}}{2} \frac{\|A-1_H\|_2 \|B-1_H\|_2}{\left(1-\|A\|_2^\beta\right)^{\frac{1}{\beta}} \left(1-\|B\|_2^\gamma\right)^{\frac{1}{\gamma}}} \left[f_a^2(|\lambda|) - f_{a^2}\left(|\lambda|^2\right)\right]^{1/2} \\
&\times \left\{ f_a(|\lambda|) \left[|\lambda| f_a'(|\lambda|) + |\lambda|^2 f_a''(|\lambda|)\right] - \left[|\lambda| f_a'(|\lambda|)\right]^2 \right\}^{1/2}
\end{aligned} \tag{3.5}$$

provided that $A,\ B \in B_2(H)$ *with* $AB = BA$ *and* $\|A\|_2, \|B\|_2 < 1$.

Proof. Using the inequality (2.8) we have for $n \ge 1$ that

$$\begin{aligned}
&\left\| \sum_{i=0}^n \alpha_i \lambda^i \sum_{i=0}^n \alpha_i \lambda^i (AB)^i - \sum_{i=0}^n \alpha_i \lambda^i A^i \sum_{i=0}^n \alpha_i \lambda^i B^i \right\|_1 \\
&= \left\| \sum_{i=0}^n \alpha_i \lambda^i \sum_{i=0}^n \alpha_i \lambda^i A^i B^i - \sum_{i=0}^n \alpha_i \lambda^i A^i \sum_{i=0}^n \alpha_i \lambda^i B^i \right\|_1 \\
&\le \frac{\sqrt{2}}{2} \left[\left(\sum_{i=0}^n |\alpha_i| |\lambda|^i \right)^2 - \sum_{i=0}^n |\alpha_i|^2 |\lambda|^{2i} \right]^{1/2} \\
&\times \left[\sum_{i=0}^n |\alpha_i| |\lambda|^i \sum_{i=0}^n i^2 |\alpha_i| |\lambda|^i - \left(\sum_{i=0}^n i |\alpha_i| |\lambda|^i \right)^2 \right]^{\frac{1}{2}} \\
&\times \left(\sum_{j=0}^{n-1} \left\| A^{j+1} - A^j \right\|_p^\beta \right)^{\frac{1}{\beta}} \left(\sum_{j=0}^{n-1} \left\| B^{j+1} - B^j \right\|_q^\gamma \right)^{\frac{1}{\gamma}},
\end{aligned} \tag{3.6}$$

where $\beta > 1,\ \frac{1}{\beta} + \frac{1}{\gamma} = 1$.

Observe that

$$\begin{aligned}
\sum_{j=0}^{n-1} \left\| A^{j+1} - A^j \right\|_p^\beta &\le \sum_{j=0}^{n-1} \left\| A^j (A-1) \right\|_p^\beta \le \|A-1_H\|_p^\beta \sum_{j=0}^{n-1} \left\| A^j \right\|_p^\beta \\
&\le \|A-1\|_p^\beta \sum_{j=0}^{n-1} \|A\|_p^{j\beta} = \|A-1_H\|_p^\beta \frac{1-\|A\|_p^{n\beta}}{1-\|A\|_p^\beta},
\end{aligned}$$

which implies that

$$\left(\sum_{j=0}^{n-1} \left\| A^{j+1} - A^j \right\|_p^\beta \right)^{\frac{1}{\beta}} \le \|A-1_H\|_p \left(\frac{1-\|A\|_p^{n\beta}}{1-\|A\|_p^\beta} \right)^{\frac{1}{\beta}}.$$

Similarly,

$$\left(\sum_{j=0}^{n-1} \left\| B^{j+1} - B^j \right\|_q^\gamma \right)^{\frac{1}{\gamma}} \le \|B-1_H\|_q \left(\frac{1-\|B\|_q^{n\gamma}}{1-\|B\|_q^\gamma} \right)^{\frac{1}{\gamma}},$$

where $\beta > 1,\ \frac{1}{\beta} + \frac{1}{\gamma} = 1$.

From (3.6) we get

$$\left\|\sum_{i=0}^{n}\alpha_i\lambda^i\sum_{i=0}^{n}\alpha_i\lambda^i(AB)^i-\sum_{i=0}^{n}\alpha_i\lambda^iA^i\sum_{i=0}^{n}\alpha_i\lambda^iB^i\right\|_1 \tag{3.7}$$
$$\leq\frac{\sqrt{2}}{2}\left[\left(\sum_{i=0}^{n}|\alpha_i|\,|\lambda|^i\right)^2-\sum_{i=0}^{n}|\alpha_i|^2\,|\lambda|^{2i}\right]^{1/2}$$
$$\times\left[\sum_{i=0}^{n}|\alpha_i|\,|\lambda|^i\sum_{i=0}^{n}i^2\,|\alpha_i|\,|\lambda|^i-\left(\sum_{i=0}^{n}i\,|\alpha_i|\,|\lambda|^i\right)^2\right]^{\frac{1}{2}}$$
$$\times\|A-1_H\|_p\,\|B-1_H\|_q\left(\frac{1-\|A\|_p^{n\beta}}{1-\|A\|_p^{\beta}}\right)^{\frac{1}{\beta}}\left(\frac{1-\|B\|_q^{n\gamma}}{1-\|B\|_q^{\gamma}}\right)^{\frac{1}{\gamma}},$$

where $\beta>1,\ \frac{1}{\beta}+\frac{1}{\gamma}=1$.

If we denote $g(u):=\sum_{n=0}^{\infty}\alpha_nu^n$, then for $|u|<R$ we have

$$\sum_{n=0}^{\infty}n\alpha_nu^n=ug'(u)$$

and

$$\sum_{n=0}^{\infty}n^2\alpha_nu^n=u\left(ug'(u)\right)'.$$

However

$$u\left(ug'(u)\right)'=ug'(u)+u^2g''(u)$$

and then

$$\sum_{n=0}^{\infty}n^2\alpha_nu^n=ug'(u)+u^2g''(u).$$

Therefore

$$\sum_{n=0}^{\infty}n^2\,|\alpha_n|\,|\lambda|^n=|\lambda|\,f_a'(|\lambda|)+|\lambda|^2\,f_a''(|\lambda|)$$

and

$$\sum_{n=0}^{\infty}n\,|\alpha_n|\,|\lambda|^n=|\lambda|\,f_a'(|\lambda|)$$

for $|\lambda|<R$.

Since all the series whose partial sums are involved in (3.7) are convergent, then by letting $n\to\infty$ in (3.7) we deduce the desired inequality (3.4). □

Corollary 2. *With the assumptions of Theorem 6 for $p=q=2$, we have*

$$\|f(\lambda)f(\lambda AB)-f(\lambda A)f(\lambda B)\|_1 \tag{3.8}$$
$$\leq\frac{\sqrt{2}}{2}\frac{\|A-1_H\|_2\,\|B-1_H\|_2}{\left(1-\|A\|_2^2\right)^{\frac{1}{2}}\left(1-\|B\|_2^2\right)^{\frac{1}{2}}}\left[f_a^2(|\lambda|)-f_{a^2}\left(|\lambda|^2\right)\right]^{1/2}$$
$$\times\left\{f_a(|\lambda|)\left[|\lambda|\,f_a'(|\lambda|)+|\lambda|^2\,f_a''(|\lambda|)\right]-\left[|\lambda|\,f_a'(|\lambda|)\right]^2\right\}^{1/2}$$

and, in particular

$$\left\| f(\lambda) f\left(\lambda A^2\right) - [f(\lambda A)]^2 \right\|_1 \le \frac{\sqrt{2}}{2} \frac{\|A - 1_H\|_2^2}{1 - \|A\|_2^2} \left[f_a^2(|\lambda|) - f_{a^2}\left(|\lambda|^2\right)\right]^{1/2} \times \left\{ f_a(|\lambda|) \left[|\lambda| f_a'(|\lambda|) + |\lambda|^2 f_a''(|\lambda|)\right] - [|\lambda| f_a'(|\lambda|)]^2 \right\}^{1/2}. \tag{3.9}$$

4. Some Particular Cases of Interest

Let p, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$. Consider the function $f : D(0,1) \to \mathbb{C}$ defined by

$$f(\lambda) = (1 - \lambda)^{-1} = \sum_{k=0}^{\infty} \lambda^k = f_a(\lambda).$$

Then

$$f_{a^2}(\lambda) := \sum_{n=0}^{\infty} \lambda^n = (1 - \lambda)^{-1},$$

which implies that

$$f_a^2(|\lambda|) - f_{a^2}\left(|\lambda|^2\right) = \frac{2|\lambda|}{(1 - |\lambda|)^2 (1 + |\lambda|)}, \quad |\lambda| < 1,$$

$$\begin{aligned} & f_a(|\lambda|) \left[|\lambda| f_a'(|\lambda|) + |\lambda|^2 f_a''(|\lambda|)\right] - [|\lambda| f_a'(|\lambda|)]^2 \\ &= (1 - |\lambda|)^{-1} \left[|\lambda| (1 - |\lambda|)^{-2} + 2|\lambda|^2 (1 - |\lambda|)^{-3}\right] - \left[|\lambda| (1 - |\lambda|)^{-2}\right]^2 \\ &= |\lambda| (1 - |\lambda|)^{-3} + |\lambda|^2 (1 - |\lambda|)^{-4} = |\lambda| (1 - |\lambda|)^{-3} \left[1 + |\lambda| (1 - |\lambda|)^{-1}\right] \\ &= |\lambda| (1 - |\lambda|)^{-4} \end{aligned}$$

and by (3.4) we then have for $A \in B_p(H)$ and $B \in B_q(H)$ with $AB = BA$ and $\|A\|_p, \|B\|_q < 1$, that

$$\begin{aligned} & \left\| (1 - \lambda)^{-1} (1_H - \lambda AB)^{-1} - (1_H - \lambda A)^{-1} (1_H - \lambda B)^{-1} \right\|_1 \\ & \le \frac{\sqrt{2}}{2} \frac{\|A - 1_H\|_p \|B - 1_H\|_q}{\left(1 - \|A\|_p^{\beta}\right)^{\frac{1}{\beta}} \left(1 - \|B\|_q^{\gamma}\right)^{\frac{1}{\gamma}}} \left[\frac{2|\lambda|}{(1 - |\lambda|)^2 (1 + |\lambda|)}, \right]^{1/2} \\ & \times \left\{ \frac{|\lambda|}{(1 - |\lambda|)^4} \right\}^{1/2}, \end{aligned}$$

for $\lambda \in \mathbb{C}$ with $|\lambda| < R$, which is equivalent to

$$\left\| (1 - \lambda)^{-1} (1_H - \lambda AB)^{-1} - (1_H - \lambda A)^{-1} (1_H - \lambda B)^{-1} \right\|_1 \le \frac{|\lambda| \|A - 1_H\|_p \|B - 1_H\|_q}{\left(1 - \|A\|_p^{\beta}\right)^{\frac{1}{\beta}} \left(1 - \|B\|_q^{\gamma}\right)^{\frac{1}{\gamma}} (1 - |\lambda|)^3 (1 + |\lambda|)^{1/2}}, \tag{4.1}$$

where $\beta > 1$, $\frac{1}{\beta} + \frac{1}{\gamma} = 1$.

We consider the *modified Bessel function functions of the first kind*

$$I_{\nu}(\lambda) := \left(\frac{1}{2}\lambda\right)^{\nu} \sum_{k=0}^{\infty} \frac{\left(\frac{1}{4}\lambda^2\right)^k}{k!\Gamma(\nu+k+1)},\ \lambda \in C$$

where Γ is the *Gamma function* and ν is a real number. An integral formula to represent I_{ν} is

$$I_{\nu}(\lambda) = \frac{1}{\pi}\int_0^{\pi} e^{\lambda\cos\theta}\cos(\nu\theta)\,d\theta - \frac{\sin(\nu\pi)}{\pi}\int_0^{\infty} e^{-\lambda\cosh t-\nu t}dt,$$

which simplifies for ν an integer n to

$$I_n(\lambda) = \frac{1}{\pi}\int_0^{\pi} e^{\lambda\cos\theta}\cos(n\theta)\,d\theta.$$

For $n=0$ we have

$$I_0(\lambda) = \frac{1}{\pi}\int_0^{\pi} e^{\lambda\cos\theta}d\theta = \sum_{k=0}^{\infty}\frac{\left(\frac{1}{4}\lambda^2\right)^k}{(k!)^2},\ \lambda \in C.$$

Now, if we consider the exponential function

$$f(\lambda) = \exp(\lambda) = \sum_{k=0}^{\infty}\frac{1}{k!}\lambda^k,$$

then for $\rho > 0$ we have

$$f_{a^2}(\rho) = \sum_{k=0}^{\infty}\frac{1}{(k!)^2}\rho^k = I_0(2\sqrt{\rho}),$$

which implies that

$$f_a^2(|\lambda|) - f_{a^2}\left(|\lambda|^2\right) = \exp(2|\lambda|) - I_0(2|\lambda|),\ \lambda \in C.$$

For $f(\lambda) = \exp(\lambda)$ we have

$$f_a(|\lambda|)\left[|\lambda| f_a'(|\lambda|) + |\lambda|^2 f_a''(|\lambda|)\right] - \left[|\lambda| f_a'(|\lambda|)\right]^2 = |\lambda|\exp(2|\lambda|).$$

If $A \in B_p(H)$ and $B \in B_q(H)$ with $AB = BA$ and $\|A\|_p, \|B\|_q < 1$, then from (3.4) we have for $\lambda \in \mathbb{C}$ the inequality:

$$\|\exp(\lambda(AB+1)) - \exp(\lambda(A+B))\|_1 \le \frac{\sqrt{2}}{2}\frac{|\lambda|^{1/2}\exp(|\lambda|)\|A-1_H\|_p\|B-1_H\|_q}{\left(1-\|A\|_p^{\beta}\right)^{\frac{1}{\beta}}\left(1-\|B\|_q^{\gamma}\right)^{\frac{1}{\gamma}}}\left[\exp(2|\lambda|) - I_0(2|\lambda|)\right]^{1/2}, \tag{4.2}$$

where $\beta > 1$, $\frac{1}{\beta} + \frac{1}{\gamma} = 1$.

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