NEW NONCOMMUTATIVE GRÜSS TYPE INEQUALITIES FOR FUNCTIONS IN BANACH ALGEBRAS

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Abstract. Let $\mathcal{B}$ be a complex Banach algebra. Assume that $x, y : [a,b] \to \mathcal{B}$ are continuous and y is strongly differentiable on (a,b). In this paper we show among others that

$$\left\| (b-a) \int_a^b x(t) y(t) dt - \int_a^b x(t) dt \int_a^b y(t) dt \right\|$$
$$\leq \int_a^b (t-a)(b-t) \|\Theta(x,a,b;t)\| \|y'(t)\| dt$$
$$\leq \frac{1}{4}(b-a)^2 \begin{cases} \sup_{t\in[a,b]} \|\Theta(x,a,b;t)\| \int_a^b \|y'(t)\| dt, \\ \left(\int_a^b \|\Theta(x,a,b;t)\|^q dt\right)^{1/q} \left(\int_a^b \|y'(t)\|^p dt\right)^{1/p}, \\ \int_a^b \|\Theta(x,a,b;t)\| dt \sup_{t\in[a,b]} \|y'(t)\|, \end{cases}$$

where

$$\Theta(x,a,b;t) := \frac{1}{b-t} \int_t^b x(s) ds - \frac{1}{t-a} \int_a^t x(s) ds, \ t \in (a,b).$$

Some applications for analytic functions of elements in Banach algebras with examples for exponential function are also given.

1. Introduction

Let $\mathcal{B}$ be an algebra. An *algebra norm* on $\mathcal{B}$ is a map $\|\cdot\| : \mathcal{B} \to [0,\infty)$ such that $(\mathcal{B}, \|\cdot\|)$ is a normed space, and, further: $\|ab\| \leq \|a\| \|b\|$ for any $a, b \in \mathcal{B}$. The normed algebra $(\mathcal{B}, \|\cdot\|)$ is a *Banach algebra* if $\|\cdot\|$ is a *complete norm*. We assume that the Banach algebra is *unital,* this means that $\mathcal{B}$ has an identity 1 and that $\|1\| = 1$.

Let $\mathcal{B}$ be a unital algebra. An element $a \in \mathcal{B}$ is *invertible* if there exists an element $b \in \mathcal{B}$ with $ab = ba = 1$. The element b is unique; it is called the *inverse* of a and written a^{-1} or $\frac{1}{a}$. The set of invertible elements of $\mathcal{B}$ is denoted by $\mathrm{Inv}(\mathcal{B})$. If $a, b \in \mathrm{Inv}(\mathcal{B})$ then $ab \in \mathrm{Inv}(\mathcal{B})$ and $(ab)^{-1} = b^{-1}a^{-1}$.

For a unital Banach algebra we also have:

(i) If $a \in \mathcal{B}$ and $\lim_{n\to\infty} \|a^n\|^{1/n} < 1$, then $1 - a \in \mathrm{Inv}(\mathcal{B})$;
(ii) $\{b \in \mathcal{B}\colon \|1 - b\| < 1\} \subset \mathrm{Inv}(\mathcal{B})$;
(iii) $\mathrm{Inv}(\mathcal{B})$ is an *open subset* of $\mathcal{B}$;
(iv) The map $\mathrm{Inv}(\mathcal{B}) \ni a \longmapsto a^{-1} \in \mathrm{Inv}(\mathcal{B})$ is continuous.

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For simplicity, we denote $z1$, where $z \in \mathbb{C}$ and 1 is the identity of $\mathcal{B}$, by z. The *resolvent set* of $a \in \mathcal{B}$ is defined by

$$\rho(a) := \{z \in \mathbb{C} : \; z - a \in \operatorname{Inv}(\mathcal{B})\};$$

the *spectrum* of a is $\sigma(a)$, the complement of $\rho(a)$ in $\mathbb{C}$, and the *resolvent function* of a is $R_a : \rho(a) \to \operatorname{Inv}(\mathcal{B})$, $R_a(z) := (z-a)^{-1}$. For each $z, w \in \rho(a)$ we have the identity

$$R_a(w) - R_a(z) = (z-w) R_a(z) R_a(w).$$

We also have that

$$\sigma(a) \subset \{z \in \mathbb{C} : \; |z| \leq \|a\|\}.$$

The *spectral radius* of a is defined as

$$\nu(a) = \sup\{|z| : z \in \sigma(a)\}.$$

Let $\mathcal{B}$ a unital Banach algebra and $a \in \mathcal{B}$. Then

(i) The resolvent set $\rho(a)$ is open in $\mathbb{C}$;
(ii) For any *bounded linear functionals* $\lambda : \mathcal{B} \to \mathbb{C}$, the function $\lambda \circ R_a$ is analytic on $\rho(a)$;
(iii) The spectrum $\sigma(a)$ is compact and nonempty in $\mathbb{C}$;
(iv) For each $n \in \mathbb{N}$ and $r > \nu(a)$, we have $a^n = \frac{1}{2\pi i}\int_{|\xi|=r} \xi^n (\xi - a)^{-1} d\xi$;
(v) We have $\nu(a) = \lim_{n\to\infty} \|a^n\|^{1/n}$.

Let $\mathcal{B}$ be a unital Banach algebra, $a \in \mathcal{B}$ and G be a domain of $\mathbb{C}$ with $\sigma(a) \subset G$. If $f : G \to \mathbb{C}$ is analytic on G, we define an element $f(a)$ in $\mathcal{B}$ by

$$f(a) := \frac{1}{2\pi i}\int_\delta f(\xi)(\xi - a)^{-1} d\xi, \tag{1.1}$$

where $\delta \subset G$ is taken to be close rectifiable curve in G and such that $\sigma(a) \subset \operatorname{ins}(\delta)$, the inside of δ.

It is well known (see for instance [3, pp. 201-204]) that $f(a)$ does not depend on the choice of δ and the *Spectral Mapping Theorem* (SMT)

$$\sigma(f(a)) = f(\sigma(a)) \tag{1.2}$$

holds.

Concerning other basic definitions and facts in the theory of Banach algebras, the reader can consult the classical books [14] and [17].

For some recent norm inequalities for functions on Banach algebras, see [8], [2] and [5]-[13].

If p, f are integrable on $[a,b]$ and

$$n \leq p \leq N \text{ and } m \leq f \leq M \text{ on } [a,b]$$

for some constants n, N, m, M, then

$$\left|(b-a)\int_a^b p(t) f(t)\,dt - \int_a^b p(t)\,dt \int_a^b f(t)\,dt\right| \leq \frac{1}{4}(b-a)^2 (N-n)(M-m),$$

which is well known in the literature as *Grüss' inequality*.

The constant $\frac{1}{4}$ is best possible in the sense that it cannot be replaced by a smaller quantity.

Assume that $x,\, y:[a,b]\rightarrow\mathcal{B}$ are continuous and y is strongly differentiable on (a,b). In the recent paper [9] we obtained among others the following *noncommutative* Grüss' type inequalities

$$\begin{aligned}
&\left\|(b-a)\int_a^b x(t)\,y(t)\,dt-\int_a^b x(t)\,dt\int_a^b y(t)\,dt\right\| \\
&\leq \int_a^b \|x(t)-v\|\left[\int_a^t (s-a)\,\|y'(s)\|\,ds+\int_t^b (b-s)\,\|y'(s)\|\,ds\right]dt \\
&\leq \begin{cases} \int_a^b \|x(t)-v\|\max\{t-a,b-t\}\,dt\int_a^b \|y'(s)\|\,ds, \\ \\ \frac{1}{(q+1)^{1/q}}\int_a^b \|x(t)-v\|\left[(t-a)^{q+1}+(b-t)^{q+1}\right]^{1/q}dt \\ \times\left(\int_a^b \|y'(s)\|^p\,ds\right)^{1/p}, \\ \\ \frac{1}{2}\int_a^b \|x(t)-v\|\left[(t-a)^2+(b-t)^2\right]dt\sup_{s\in[a,b]}\|y'(s)\| \end{cases}
\end{aligned}$$

for all $v\in\mathcal{B}$.

Motivated by the above result, we establish in this paper other upper bounds for the quantity

$$\left\|(b-a)\int_a^b x(t)\,y(t)\,dt-\int_a^b x(t)\,dt\int_a^b y(t)\,dt\right\|$$

for various assumptions on the functions $x,\, y:[a,b]\rightarrow\mathcal{B}$. Some applications for analytic functions of elements in Banach algebras with examples for exponential function are also given.

2. Main Results

We have the following equality:

Lemma 1. *Let $y:[a,b]\rightarrow\mathcal{B}$ be a strongly differentiable function on the interval (a,b) and $x:[a,b]\rightarrow\mathcal{B}$ a continuous function, then*

$$\begin{aligned}
(2.1)\qquad &(b-a)\int_a^b x(t)\,y(t)\,dt-\int_a^b x(t)\,dt\int_a^b y(t)\,dt \\
&=\int_a^b (t-a)(b-t)\,\Theta(x,a,b;t)\,y'(t)\,dt,
\end{aligned}$$

where

$$\Theta(x,a,b;t):=\frac{1}{b-t}\int_t^b x(s)\,ds-\frac{1}{t-a}\int_a^t x(s)\,ds,\ \ t\in(a,b).$$

Proof. We start to the Montgomery identity for a strongly differentiable function $y:[a,b]\rightarrow\mathcal{B}$

$$(2.2)\qquad y(t)(b-a)-\int_a^b y(s)\,ds=\int_a^t (s-a)\,y'(s)\,ds+\int_t^b (s-b)\,y'(s)\,ds$$

that holds for all $t\in[a,b]$.

Indeed, integrating by parts in Bochner's integral [15], we have

$$\int_a^t (s-a)\, y'(s)\, ds = (t-a)\, y(t) - \int_a^t y(s)\, ds$$

and

$$\int_t^b (s-b)\, y'(s)\, ds = (b-t)\, y(t) - \int_t^b y(s)\, ds,$$

which, by addition, gives (2.2).

If we multiply this identity at right by $x(t)$ and integrate over t in $[a,b]$, then we get

$$\begin{aligned}(2.3)\qquad & (b-a)\int_a^b x(t)\, y(t)\, dt - \int_a^b x(t)\, dt \int_a^b y(s)\, ds \\ &= \int_a^b x(t)\left(\int_a^t (s-a)\, y'(s)\, ds + \int_t^b (s-b)\, y'(s)\, ds\right) dt \\ &= \int_a^b x(t)\left(\int_a^t (s-a)\, y'(s)\, ds\right) dt + \int_a^b x(t)\left(\int_t^b (s-b)\, y'(s)\, ds\right) dt.\end{aligned}$$

Using integration by parts for the Bochner's integral we also have

$$\begin{aligned}&\int_a^b x(t)\left(\int_a^t (s-a)\, y'(s)\, ds\right) dt \\ &= \int_a^b \left(\int_a^t x(s)\, ds\right)'\left(\int_a^t (s-a)\, y'(s)\, ds\right) dt \\ &= \left(\int_a^t x(s)\, ds\right)\left(\int_a^t (s-a)\, y'(s)\, ds\right)\Bigg|_a^b - \int_a^b (t-a)\left(\int_a^t x(s)\, ds\right) y'(t)\, dt \\ &= \left(\int_a^b x(s)\, ds\right)\left(\int_a^b (s-a)\, y'(s)\, ds\right) - \int_a^b (t-a)\left(\int_a^t x(s)\, ds\right) y'(t)\, dt \\ &= \int_a^b (t-a)\left(\int_a^b x(s)\, ds - \int_a^t x(s)\, ds\right) y'(t)\, dt \\ &= \int_a^b (t-a)\left(\int_t^b x(s)\, ds\right) y'(t)\, dt\end{aligned}$$

and

$$\begin{aligned}&\int_a^b x(t)\left(\int_t^b (s-b)\, y'(s)\, ds\right) dt \\ &= \int_a^b \left(\int_a^t x(s)\, ds\right)'\left(\int_t^b (s-b)\, y'(s)\, ds\right) dt \\ &= \left(\int_a^t x(s)\, ds\right)\left(\int_t^b (s-b)\, y'(s)\, ds\right)\Bigg|_a^b + \int_a^b (t-b)\left(\int_a^t x(s)\, ds\right) y'(t)\, dt \\ &= \int_a^b (t-b)\left(\int_a^t x(s)\, ds\right) y'(t)\, dt.\end{aligned}$$

If we add these two equalities, then we get

$$\begin{aligned}
&\int_a^b x(t)\left(\int_a^t (s-a)\,y'(s)\,ds\right)dt+\int_a^b x(t)\left(\int_t^b (s-b)\,y'(s)\,ds\right)dt\\
&=\int_a^b (t-a)\left(\int_t^b x(s)\,ds\right)y'(t)\,dt-\int_a^b (b-t)\left(\int_a^t x(s)\,ds\right)y'(t)\,dt\\
&=\int_a^b \left[(t-a)\int_t^b x(s)\,ds-(b-t)\int_a^t x(s)\,ds\right]y'(t)\,dt\\
&=\int_a^b \left[(t-a)(b-t)\left(\frac{1}{b-t}\int_t^b x(s)\,ds-\frac{1}{t-a}\int_a^t x(s)\,ds\right)\right]y'(t)\,dt\\
&=\int_a^b (t-a)(b-t)\,\Theta(x,a,b;t)\,y'(t)\,dt
\end{aligned}$$

and by (2.3), we derive (2.1). □

Lemma 2. *With the assumptions of Lemma 1 we have for all $w\in\mathcal{B}$ that*

$$\begin{aligned}
&(b-a)\int_a^b x(t)\,y(t)\,dt-\int_a^b x(t)\,dt\int_a^b y(t)\,dt\\
&-(b-a)\left(\int_a^b\left(t-\frac{a+b}{2}\right)x(t)\,dt\right)w\\
&=\int_a^b (t-a)(b-t)\,\Theta(x,a,b;t)\,[y'(t)-w]\,dt.
\end{aligned}\tag{2.4}$$

Proof. If we replace $y(t)$ with $y(t)-tw$ in (2.1), then we get

$$\begin{aligned}
&(b-a)\int_a^b x(t)\,[y(t)-tw]\,dt-\int_a^b x(t)\,dt\int_a^b (y(t)-tw)\,dt\\
&=\int_a^b (t-a)(b-t)\,\Theta(x,a,b;t)\,[y'(t)-w]\,dt.
\end{aligned}\tag{2.5}$$

Observe that

$$\begin{aligned}
&(b-a)\int_a^b x(t)\,[y(t)-tw]\,dt-\int_a^b x(t)\,dt\int_a^b (y(t)-tw)\,dt\\
&=(b-a)\left[\int_a^b x(t)\,y(t)\,dt-\int_a^b tx(t)\,wdt\right]\\
&-\int_a^b x(t)\,dt\left[\int_a^b y(t)\,dt-\frac{1}{2}\left(b^2-a^2\right)w\right]
\end{aligned}\tag{2.6}$$

$$
\begin{aligned}
&= (b-a)\int_a^b x(t)\,y(t)\,dt - (b-a)\int_a^b tx(t)\,wdt \\
&- \int_a^b x(t)\,dt \int_a^b y(t)\,dt + \frac{1}{2}\left(b^2-a^2\right)\int_a^b x(t)\,dtw \\
&= (b-a)\int_a^b x(t)\,y(t)\,dt - \int_a^b x(t)\,dt \int_a^b y(t)\,dt \\
&- (b-a)\int_a^b \left(t-\frac{a+b}{2}\right) x(t)\,wdt
\end{aligned}
$$

and by (2.5) we get (2.4). □

Remark 1. *If x is symmetric on $[a,b]$, namely $x(a+b-t) = x(t)$ for all $t \in [a,b]$, then $h(t) := \left(t - \frac{a+b}{2}\right) x(t)$ is antisymmetric on $[a,b]$, which gives that*

$$
\int_a^b \left(t-\frac{a+b}{2}\right) x(t)\,dt = 0,
$$

and by (2.4) we derive

$$
\begin{aligned}
(2.7) \qquad & (b-a)\int_a^b x(t)\,y(t)\,dt - \int_a^b x(t)\,dt \int_a^b y(t)\,dt \\
&= \int_a^b (t-a)(b-t)\,\Theta(x,a,b;t)\left[y'(t) - w\right] dt.
\end{aligned}
$$

for all $w \in \mathcal{B}$.

We have the following inequalities:

Theorem 1. *Let $y : [a,b] \to \mathcal{B}$ be a strongly differentiable function on the interval (a,b) and $x : [a,b] \to \mathcal{B}$ a continuous function on $[a,b]$, then*

$$
\begin{aligned}
(2.8) \qquad & \left\| (b-a)\int_a^b x(t)\,y(t)\,dt - \int_a^b x(t)\,dt \int_a^b y(t)\,dt \right\| \\
&\leq \int_a^b (t-a)(b-t)\,\|\Theta(x,a,b;t)\|\,\|y'(t)\|\,dt =: M(x,y,a,b).
\end{aligned}
$$

We also have the bounds for p, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$,

$$
(2.9) \qquad M(x,y,a,b) \leq \frac{1}{4}(b-a)^2 \times \begin{cases}
\sup_{t\in[a,b]} \|\Theta(x,a,b;t)\| \int_a^b \|y'(t)\|\,dt, \\[2mm]
\left(\int_a^b \|\Theta(x,a,b;t)\|^q\,dt\right)^{1/q} \left(\int_a^b \|y'(t)\|^p\,dt\right)^{1/p}, \\[2mm]
\int_a^b \|\Theta(x,a,b;t)\|\,dt \sup_{t\in[a,b]} \|y'(t)\|,
\end{cases}
$$

$$\begin{aligned}
(2.10)\qquad & M(x,y,a,b) \\
& \leq \sup_{t\in[a,b]} \|\Theta(x,a,b;t)\| \\
& \times \begin{cases} (b-a)^{2+1/q}\left[B(q+1,q+1)\right]^{1/q}\left(\int_a^b \|y'(t)\|^p\,dt\right)^{1/p}, \\ \\ \frac{1}{6}(b-a)^3 \sup_{t\in[a,b]} \|y'(t)\|, \end{cases}
\end{aligned}$$

and

(2.11)

$$\begin{aligned}
M(x,y,a,b) \leq{}& \sup_{t\in[a,b]} \|y'(t)\| \\
& \times (b-a)^{2+1/q}\left[B(q+1,q+1)\right]^{1/q}\left(\int_a^b \|\Theta(x,a,b;t)\|^p\,dt\right)^{1/p},
\end{aligned}$$

where $B(\cdot,\cdot)$ is Beta function.

Proof. By taking the norm in (2.1) we get

$$\begin{aligned}
& \left\|(b-a)\int_a^b x(t)\,y(t)\,dt - \int_a^b x(t)\,dt \int_a^b y(t)\,dt\right\| \\
& \leq \int_a^b (t-a)(b-t)\,\|\Theta(x,a,b;t)\,y'(t)\|\,dt \\
& \leq \int_a^b (t-a)(b-t)\,\|\Theta(x,a,b;t)\|\,\|y'(t)\|\,dt =: M(x,y,a,b),
\end{aligned}$$

which proves (2.8).

Observe also that for all $t\in[a,b]$

$$(t-a)(b-t) \leq \frac{1}{4}(b-a)^2,$$

which together with Hölder's inequality prove the other inequalities above. □

Theorem 2. *With the assumptions of Theorem 1 and if there exists $w\in\mathcal{B}$ and $L>0$ such that*

$$\|y'(t)-w\| \leq L \text{ for all } t\in(a,b), \qquad (2.12)$$

then

$$\begin{aligned}
(2.13)\qquad & \Bigg\|(b-a)\int_a^b x(t)\,y(t)\,dt - \int_a^b x(t)\,dt \int_a^b y(t)\,dt \\
& -(b-a)\left(\int_a^b \left(t-\frac{a+b}{2}\right)x(t)\,dt\right)w\Bigg\|
\end{aligned}$$

$$\leq L \int_a^b (t-a)(b-t) \|\Theta(x,a,b;t)\| \, dt$$

$$\leq L \times \begin{cases} \frac{1}{4}(b-a)^2 \int_a^b \|\Theta(x,a,b;t)\|, \\ \\ (b-a)^{2+1/q} [B(q+1,q+1)]^{1/q} \left(\int_a^b \|\Theta(x,a,b;t)\|^p \, dt\right)^{1/p}, \\ \\ \frac{1}{6}(b-a)^3 \sup_{t\in[a,b]} \|\Theta(x,a,b;t)\| \end{cases}$$

for p, $q > 1$ *with* $\frac{1}{p} + \frac{1}{q} = 1$.

The proof follows by Lemma 2 and by using Hölder's inequality.

Remark 2. *With the assumptions of Theorem 2 and if* x *is symmetric on* $[a,b]$, *then we have*

$$\left\| (b-a) \int_a^b x(t) y(t) \, dt - \int_a^b x(t) \, dt \int_a^b y(t) \, dt \right\| \tag{2.14}$$

$$\leq L \int_a^b (t-a)(b-t) \|\Theta(x,a,b;t)\| \, dt$$

$$\leq L \times \begin{cases} \frac{1}{4}(b-a)^2 \int_a^b \|\Theta(x,a,b;t)\|, \\ \\ (b-a)^{2+1/q} [B(q+1,q+1)]^{1/q} \left(\int_a^b \|\Theta(x,a,b;t)\|^p \, dt\right)^{1/p}, \\ \\ \frac{1}{6}(b-a)^3 \sup_{t\in[a,b]} \|\Theta(x,a,b;t)\|. \end{cases}$$

3. Applications for Analytic Functions

Assume that $f : G \to \mathbb{C}$ is analytic on the convex domain G and x, $y \in \mathcal{B}$, $x \neq y$ with $\sigma(x)$, $\sigma(y) \subset G$. By the convexity of G we have that $\sigma((1-t)x + ty) \subset G$ for all $t \in [0,1]$ and we can define the auxiliary function $f_{x,y} : [0,1] \to \mathcal{B}$ by

$$f_{x,y}(t) := f((1-t)x + ty), \ t \in [0,1]. \tag{3.1}$$

Lemma 3. *Assume that* $f : G \to \mathbb{C}$ *is analytic on the convex domain* G *and* x, $y \in \mathcal{B}$, $x \neq y$ *with* $\sigma(x)$, $\sigma(y) \subset G$. *The function* $f_{x,y} : [0,1] \to \mathcal{B}$ *is differentiable on* $(0,1)$ *as a function of* t *and we have*

$$f'_{x,y}(t) = D(f)((1-t)x + ty)(y-x) \tag{3.2}$$

for all $t \in (0,1)$, *where* $D(f)(\cdot)(\cdot)$ *is the Fréchet derivative of function* f *as a function defined on the Banach algebra* $\mathcal{B}$ *by equation (1.1).*

We also have the lateral derivatives

$$f'_{x,y}(0+) = D(f)(x)(y-x) \ \text{ and } f'_{x,y}(1-) = D(f)(y)(y-x). \tag{3.3}$$

Proof. We prove only for the interior points $t \in (0,1)$. Let h be in a small interval around 0 such that $t+h \in (0,1)$. Then for $h \neq 0$,

$$\begin{aligned} &\frac{f_{x,y}(t+h) - f(t)}{h} \\ &= \frac{f((1-(t+h))x + (t+h)y) - f((1-t)x + ty)}{h} \\ &= \frac{f((1-t)x + ty + h(y-x)) - f((1-t)x + ty)}{h} \end{aligned}$$

and by taking the limit over $h \to 0$, we get

$$\begin{aligned} f_{x,y}^{'}(t) &= \frac{df_{x,y}(t)}{dt} = \lim_{h\to 0} \frac{f_{x,y}(t+h) - f(t)}{h} \\ &= \lim_{h\to 0}\left[\frac{f((1-t)x + ty + h(y-x)) - f((1-t)x + ty)}{h}\right] \\ &= D(f)((1-t)x + ty)(y-x), \end{aligned}$$

which proves (3.2).

The proof is similar for the lateral derivatives. □

Lemma 4. *Assume that $f : G \to \mathbb{C}$ is analytic on the domain G and $x \in \mathcal{B}$, with $\sigma(x) \subset G$, then for $v \in \mathcal{B}$ we have*

$$D(f)(x)(v) = \frac{1}{2\pi i}\int_\gamma f(\xi)(\xi - x)^{-1} v (\xi - x)^{-1} d\xi, \tag{3.4}$$

where $\gamma \subset G$ is taken to be close rectifiable curve in G and such that $\sigma(x) \subset \operatorname{ins}(\gamma)$, the inside of γ.

Proof. Let $v \in \mathcal{B}$. Then there exists a small interval around 0 such that for h in this interval $\sigma(x+hv) \subset \operatorname{ins}(\delta) \subset G$. Then

$$\begin{aligned} &f(x+hv) - f(x) \\ &= \frac{1}{2\pi i}\left(\int_\gamma f(\xi)(\xi - x - hv)^{-1} d\xi - \int_\gamma f(\xi)(\xi - x)^{-1} d\xi\right) \\ &= \frac{1}{2\pi i}\int_\gamma f(\xi)\left[(\xi - x - hv)^{-1} - (\xi - x)^{-1}\right] d\xi \\ &= \frac{1}{2\pi i} h \int_\gamma f(\xi)\left[(\xi - x - hv)^{-1} v (\xi - x)^{-1}\right] d\xi, \end{aligned}$$

which gives for $h \neq 0$ that

$$\frac{f(x+hv) - f(x)}{h} = \frac{1}{2\pi i}\int_\gamma f(\xi)(\xi - x - hv)^{-1} v (\xi - x)^{-1} d\xi.$$

By taking the limit over $h \to 0$ and using the properties of the integral, we get (3.4). □

Lemma 5. *Assume that $f : G \to \mathbb{C}$ is analytic on the convex domain G and x, $y \in \mathcal{B}$, $x \neq y$ with $\sigma(x), \sigma(y) \subset G$ while $\gamma \subset G$ is taken to be close rectifiable curve in G and such that $\sigma(x), \sigma(y) \subset \operatorname{ins}(\gamma)$. Then*

$$f_{x,y}^{'}(t) = \frac{1}{2\pi i}\int_\gamma f(\xi)(\xi - (1-t)x - ty)^{-1}(y-x)(\xi - (1-t)x - ty)^{-1} d\xi \tag{3.5}$$

for all $t\in(0,1)$.

We also have the lateral derivatives

$$f_{x,y}^{'}(0+)=\frac{1}{2\pi i}\int_{\gamma}f(\xi)(\xi-x)^{-1}(y-x)(\xi-x)^{-1}d\xi, \tag{3.6}$$

and

$$f_{x,y}^{'}(1-)=\frac{1}{2\pi i}\int_{\gamma}f(\xi)(\xi-y)^{-1}(y-x)(\xi-y)^{-1}d\xi. \tag{3.7}$$

The proof is obvious by Lemmas 3 and 4.

Lemma 6. *With the assumptions of Lemma 5 we have the bounds*

$$\begin{aligned}
&\left\|f_{x,y}^{'}(t)\right\| \\
&\le\frac{1}{2\pi}\|y-x\|\int_{\gamma}|f(\xi)|\left\|(\xi-(1-t)x-ty)^{-1}\right\|^{2}|d\xi| \\
&\le\frac{1}{2\pi}\|y-x\|\int_{\gamma}|f(\xi)|(|\xi|-\|(1-t)x+ty\|)^{-2}|d\xi| \\
&\le\frac{1}{2\pi}\|y-x\|\int_{\gamma}|f(\xi)|[(1-t)(|\xi|-\|x\|)+t(|\xi|-\|y\|)]^{-2}|d\xi| \\
&\le\frac{1}{2\pi}\|y-x\|\int_{\gamma}|f(\xi)|\left[(1-t)(|\xi|-\|x\|)^{-2}+t(|\xi|-\|y\|)^{-2}\right]|d\xi| \\
&\le\frac{1}{2\pi}\|y-x\|\int_{\gamma}\frac{|f(\xi)|}{\min\left\{(|\xi|-\|x\|)^{2},(|\xi|-\|y\|)^{2}\right\}}|d\xi|
\end{aligned} \tag{3.8}$$

for all $t\in[0,1]$.

Proof. By taking the norm in (3.5) we get

$$\begin{aligned}
&\left\|f_{x,y}^{'}(t)\right\| \\
&\le\frac{1}{2\pi}\int_{\gamma}|f(\xi)|\left\|(\xi-(1-t)x-ty)^{-1}(y-x)(\xi-(1-t)x-ty)^{-1}\right\||d\xi| \\
&\le\frac{1}{2\pi}\|y-x\|\int_{\gamma}|f(\xi)|\left\|(\xi-(1-t)x-ty)^{-1}\right\|^{2}|d\xi| \\
&=\frac{1}{2\pi}\|y-x\|\int_{\gamma}|f(\xi)||\xi|^{-2}\left\|\left(1-(1-t)\frac{x}{\xi}-t\frac{y}{\xi}\right)^{-1}\right\|^{2}|d\xi|
\end{aligned} \tag{3.9}$$

for all $t\in[0,1]$, which proves the first inequality in (3.8).

Since

$$\left\|(1-t)\frac{x}{\xi}+t\frac{y}{\xi}\right\|\le(1-t)\left\|\frac{x}{\xi}\right\|+t\left\|\frac{y}{\xi}\right\|<1-t+t=1$$

for $\xi\in\gamma$, hence

$$\left(1-(1-t)\frac{x}{\xi}-t\frac{y}{\xi}\right)^{-1}=\sum_{k=0}^{\infty}\left[(1-t)\frac{x}{\xi}+t\frac{y}{\xi}\right]^{k}.$$

Therefore

$$\begin{aligned}\left\|\left(1-(1-t)\frac{x}{\xi}-t\frac{y}{\xi}\right)^{-1}\right\| &\leq \sum_{k=0}^{\infty}\left\|(1-t)\frac{x}{\xi}+t\frac{y}{\xi}\right\|^{k} \\ &= \left(1-\left\|(1-t)\frac{x}{\xi}+t\frac{y}{\xi}\right\|\right)^{-1} \\ &= \left(\frac{|\xi|}{|\xi|}-\left\|(1-t)\frac{x}{\xi}+t\frac{y}{\xi}\right\|\right)^{-1} \\ &= |\xi|\left(|\xi|-\|(1-t)x+ty\|\right)^{-1}\end{aligned}$$

for $\xi\in\gamma$, which implies that

$$\left\|\left(1-(1-t)\frac{x}{\xi}-t\frac{y}{\xi}\right)^{-1}\right\|^{2} \leq |\xi|^{2}\left(|\xi|-\|(1-t)x+ty\|\right)^{-2}$$

for $\xi\in\gamma$.

Therefore

$$\begin{aligned}&\int_{\gamma}|f(\xi)|\,|\xi|^{-2}\left\|\left(1-(1-t)\frac{x}{\xi}-t\frac{y}{\xi}\right)^{-1}\right\|^{2}|d\xi| \\ &\leq \int_{\gamma}|f(\xi)|\left(|\xi|-\|(1-t)x+ty\|\right)^{-2}|d\xi|\end{aligned}$$

and we derive the second inequality in (3.8).

By the triangle inequality we have

$$\begin{aligned}|\xi|-\|(1-t)x+ty\| &\geq |\xi|-(1-t)\|x\|-t\|y\| \\ &= (1-t)\left(|\xi|-\|x\|\right)+t\left(|\xi|-\|y\|\right)>0\end{aligned}$$

for $\xi\in\gamma$.

This implies that

$$\left(|\xi|-\|(1-t)x+ty\|\right)^{-1} \leq \left[(1-t)\left(|\xi|-\|x\|\right)+t\left(|\xi|-\|y\|\right)\right]^{-1}$$

and thus

$$\left(|\xi|-\|(1-t)x+ty\|\right)^{-2} \leq \left[(1-t)\left(|\xi|-\|x\|\right)+t\left(|\xi|-\|y\|\right)\right]^{-2}$$

for $\xi\in\gamma$ and $t\in[0,1]$. This proves the third inequality in (3.8).

By the convexity of the power function $(\cdot)^{-2}$ we also have

$$\begin{aligned}&\left[(1-t)\left(|\xi|-\|x\|\right)+t\left(|\xi|-\|y\|\right)\right]^{-2} \\ &\leq (1-t)\left(|\xi|-\|x\|\right)^{-2}+t\left(|\xi|-\|y\|\right)^{-2}\end{aligned}$$

for $t\in[0,1]$, which proves the fourth inequality in (3.8).

Finally, observe that

$$(1-t)\left(|\xi|-\|x\|\right)^{-2}+t\left(|\xi|-\|y\|\right)^{-2} \leq \max\left\{\left(|\xi|-\|x\|\right)^{-2},\left(|\xi|-\|y\|\right)^{-2}\right\}$$

and the last part of (3.8) is thus proved. □

We have the following bounds for the p-norm of $f'_{x,y}$.

Proposition 1. *Assume that $f : G \to \mathbb{C}$ is analytic on the convex domain G and $x, y \in \mathcal{B}$, $x \neq y$ with $\sigma(x), \sigma(y) \subset G$ while $\gamma \subset G$ is taken to be close rectifiable curve in G and such that $\sigma(x), \sigma(y) \subset \operatorname{ins}(\gamma)$. Then*

$$\sup_{t\in[0,1]} \left\| f_{x,y}^{'}(t) \right\| \leq \frac{1}{2\pi} \|y-x\| \int_\gamma \frac{|f(\xi)|}{\min\left\{ (|\xi| - \|x\|)^2, (|\xi| - \|y\|)^2 \right\}} |d\xi|, \tag{3.10}$$

$$\int_0^1 \left\| f_{x,y}^{'}(t) \right\| dt \leq \frac{1}{2\pi} \|y-x\| \int_\gamma \frac{|f(\xi)|}{(|\xi| - \|y\|)(|\xi| - \|x\|)} |d\xi| \tag{3.11}$$

and

$$\begin{aligned} &\left(\int_0^1 \left\| f_{x,y}^{'}(t) \right\|^p dt \right)^{1/p} \\ &\leq \frac{1}{2\pi} \|y-x\| \left(\int_\gamma |f(\xi)|^q |d\xi| \right)^{1/q} \\ &\times \left(\frac{1}{(2p-1)(\|y\| - \|x\|)} \int_\gamma \frac{(|\xi| - \|x\|)^{2p-1} - (|\xi| - \|y\|)^{2p-1}}{(|\xi| - \|x\|)^{2p-1} (|\xi| - \|y\|)^{2p-1}} |d\xi| \right)^{1/p}, \end{aligned} \tag{3.12}$$

for $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. The inequality (3.10) is obvious by (3.8).

From (3.8) we get, by taking the integral and by using Fubini's theorem, that

$$\begin{aligned} &\int_0^1 \left\| f_{x,y}^{'}(t) \right\| dt \\ &\leq \frac{1}{2\pi} \|y-x\| \int_\gamma |f(\xi)| \left(\int_0^1 \left[(1-t)(|\xi| - \|x\|) + t(|\xi| - \|y\|) \right]^{-2} dt \right) |d\xi|. \end{aligned} \tag{3.13}$$

Observe that

$$\begin{aligned} &\int_0^1 \left[(1-t)(|\xi| - \|x\|) + t(|\xi| - \|y\|) \right]^{-2} dt \\ &= -\frac{1}{\|x\| - \|y\|} \int_0^1 \frac{d}{dt} \left[(1-t)(|\xi| - \|x\|) + t(|\xi| - \|y\|) \right]^{-1} dt \\ &= -\frac{1}{\|x\| - \|y\|} \left[(1-t)(|\xi| - \|x\|) + t(|\xi| - \|y\|) \right]^{-1} \Big|_0^1 \\ &= \frac{1}{\|y\| - \|x\|} \left[(|\xi| - \|y\|)^{-1} - (|\xi| - \|x\|)^{-1} \right] \\ &= \frac{1}{\|y\| - \|x\|} \frac{\|y\| - \|x\|}{(|\xi| - \|y\|)(|\xi| - \|x\|)} = \frac{1}{(|\xi| - \|y\|)(|\xi| - \|x\|)}, \end{aligned}$$

for $\|y\| \neq \|x\|$, which, by (3.13), proves (3.11).

If $\|y\| = \|x\|$, then (3.11) also holds.

From (3.8) we also have for p, $q>1$ with $\frac{1}{p}+\frac{1}{q}=1$,

$$\begin{aligned}
&\left\| f_{x,y}^{'}(t)\right\| \\
&\leq \frac{1}{2\pi}\left\| y-x\right\| \int_{\gamma}\left| f(\xi)\right| \left[(1-t)(|\xi|-\|x\|)+t(|\xi|-\|y\|)\right]^{-2}|d\xi| \\
&\leq \frac{1}{2\pi}\left\| y-x\right\| \left(\int_{\gamma}\left| f(\xi)\right|^{q}|d\xi|\right)^{1/q} \\
&\times\left(\int_{\gamma}\left[(1-t)(|\xi|-\|x\|)+t(|\xi|-\|y\|)\right]^{-2p}\right)^{1/p}
\end{aligned}$$

and by taking the power p we get

$$\begin{aligned}
\left\| f_{x,y}^{'}(t)\right\|^{p} &\leq \left[\frac{1}{2\pi}\left\| y-x\right\| \left(\int_{\gamma}\left| f(\xi)\right|^{q}|d\xi|\right)^{1/q}\right]^{p} \\
&\times\left(\int_{\gamma}\left[(1-t)(|\xi|-\|x\|)+t(|\xi|-\|y\|)\right]^{-2p}\right)
\end{aligned}$$

for $t\in[0,1]$.

Integrating this inequality on $[0,1]$, we get by Fubini's theorem that

$$\begin{aligned}
\int_{0}^{1}\left\| f_{x,y}^{'}(t)\right\|^{p}dt &\leq \left[\frac{1}{2\pi}\left\| y-x\right\| \left(\int_{\gamma}\left| f(\xi)\right|^{q}|d\xi|\right)^{1/q}\right]^{p} \\
&\times\int_{0}^{1}\left(\int_{\gamma}\left[(1-t)(|\xi|-\|x\|)+t(|\xi|-\|y\|)\right]^{-2p}|d\xi|\right)dt \\
&=\left[\frac{1}{2\pi}\left\| y-x\right\| \left(\int_{\gamma}\left| f(\xi)\right|^{q}|d\xi|\right)^{1/q}\right]^{p} \\
&\times\int_{\gamma}\left(\int_{0}^{1}\left[(1-t)(|\xi|-\|x\|)+t(|\xi|-\|y\|)\right]^{-2p}dt\right)|d\xi| \\
&=\left[\frac{1}{2\pi}\left\| y-x\right\| \left(\int_{\gamma}\left| f(\xi)\right|^{q}|d\xi|\right)^{1/q}\right]^{p} \\
&\int_{\gamma}\left(\int_{0}^{1}\left[(1-t)(|\xi|-\|x\|)+t(|\xi|-\|y\|)\right]^{-2p}dt\right)|d\xi|.
\end{aligned} \tag{3.14}$$

Observe that

$$\begin{aligned}
&\int_{\gamma}\left(\int_{0}^{1}\left[(1-t)(|\xi|-\|x\|)+t(|\xi|-\|y\|)\right]^{-2p}dt\right)|d\xi| \\
&=\int_{\gamma}\left(\left.\frac{\left[(1-t)(|\xi|-\|x\|)+t(|\xi|-\|y\|)\right]^{-2p+1}}{(1-2p)(\|x\|-\|y\|)}\right|_{0}^{1}\right)|d\xi|
\end{aligned}$$

$$= \int_\gamma \frac{(|\xi| - \|y\|)^{-2p+1} - (|\xi| - \|x\|)^{-2p+1}}{(1-2p)(\|x\| - \|y\|)} |d\xi|$$
$$= \int_\gamma \frac{\frac{1}{(|\xi|-\|y\|)^{2p-1}} - \frac{1}{(|\xi|-\|x\|)^{2p-1}}}{(1-2p)(\|x\| - \|y\|)} |d\xi|$$
$$= \frac{1}{(2p-1)(\|y\| - \|x\|)} \int_\gamma \frac{(|\xi| - \|x\|)^{2p-1} - (|\xi| - \|y\|)^{2p-1}}{(|\xi| - \|x\|)^{2p-1}(|\xi| - \|y\|)^{2p-1}} |d\xi|,$$

then by (3.14) we get

$$\int_0^1 \left\| f_{x,y}^{'}(t) \right\|^p dt$$
$$\leq \left[\frac{1}{2\pi} \|y - x\| \left(\int_\gamma |f(\xi)|^q |d\xi| \right)^{1/q} \right]^p$$
$$\times \frac{1}{(2p-1)(\|y\| - \|x\|)} \int_\gamma \frac{(|\xi| - \|x\|)^{2p-1} - (|\xi| - \|y\|)^{2p-1}}{(|\xi| - \|x\|)^{2p-1}(|\xi| - \|y\|)^{2p-1}} |d\xi|,$$

which proves (3.12). □

We can state now the main result of this section:

Theorem 3. *Assume that $f : G \to \mathbb{C}$ is analytic on the convex domain G and x, $y \in \mathcal{B}$, $x \neq y$ with $\sigma(x), \sigma(y) \subset G$. If $g : [0,1] \to \mathcal{B}$ is continuous, then for $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$,*

$$\left\| \int_0^1 g(t) f((1-t)x + ty) dt - \int_0^1 g(t) dt \int_0^1 f((1-t)x + ty) dt \right\| \tag{3.15}$$
$$\leq \frac{1}{2\pi} \|y - x\| \int_\gamma \frac{|f(\xi)|}{\min\left\{ (|\xi| - \|x\|)^2, (|\xi| - \|y\|)^2 \right\}} |d\xi|$$
$$\times \begin{cases} \frac{1}{4} \int_0^1 \|\Theta(g;t)\| dt, \\ \frac{1}{6} \sup_{t \in [0,1]} \|\Theta(g;t)\|, \\ [B(q+1, q+1)]^{1/q} \left(\int_0^1 \|\Theta(g;t)\|^p dt \right)^{1/p}, \end{cases}$$

where

$$\Theta(g;t) := \frac{1}{1-t} \int_t^1 g(s) ds - \frac{1}{t} \int_0^t g(s) ds, \; t \in (0,1).$$

We also have

$$\left\|\int_0^1 g(t) f((1-t)x+ty)\,dt-\int_0^1 g(t)\,dt\int_0^1 f((1-t)x+ty)\,dt\right\| \tag{3.16}$$
$$\le \frac{1}{2\pi}\|y-x\|\left(\int_\gamma |f(\xi)|^q |d\xi|\right)^{1/q}$$
$$\times\left(\frac{1}{(2p-1)(\|y\|-\|x\|)}\int_\gamma \frac{(|\xi|-\|x\|)^{2p-1}-(|\xi|-\|y\|)^{2p-1}}{(|\xi|-\|x\|)^{2p-1}(|\xi|-\|y\|)^{2p-1}}|d\xi|\right)^{1/p}$$
$$\times\begin{cases}\frac{1}{4}\left(\int_0^1 \|\Theta(g,t)\|^q\,dt\right)^{1/q},\\ \\ [B(q+1,q+1)]^{1/q}\sup_{t\in[0,1]}\|\Theta(g,t)\|\end{cases}$$

and

$$\left\|\int_0^1 g(t) f((1-t)x+ty)\,dt-\int_0^1 g(t)\,dt\int_0^1 f((1-t)x+ty)\,dt\right\| \tag{3.17}$$
$$\le \frac{1}{8\pi}\|y-x\|\int_\gamma \frac{|f(\xi)|}{(|\xi|-\|y\|)(|\xi|-\|x\|)}|d\xi|\sup_{t\in[0,1]}\|\Theta(g,t)\|.$$

Proof. From Theorem 1 we have for $y(t)=f_{x,y}(t)$ and $x(t)=g(t)$, $t\in[0,1]$ that

$$\left\|\int_0^1 g(t) f((1-t)x+ty)\,dt-\int_0^1 g(t)\,dt\int_0^1 f((1-t)x+ty)\,dt\right\| \tag{3.18}$$
$$\le \sup_{t\in[a,b]}\left\|f'_{x,y}(t)\right\|\times\begin{cases}\frac{1}{4}\int_0^1\|\Theta(g;t)\|\,dt,\\ \\ \frac{1}{6}\sup_{t\in[0,1]}\|\Theta(g;t)\|,\\ \\ [B(q+1,q+1)]^{1/q}\left(\int_0^1\|\Theta(g;t)\|^p\,dt\right)^{1/p}\end{cases}$$

and by (3.10) we derive (3.15).

By Theorem 1 we have for $y(t)=f_{x,y}(t)$ and $x(t)=g(t)$, $t\in[0,1]$ that

$$\left\|\int_0^1 g(t) f((1-t)x+ty)\,dt-\int_0^1 g(t)\,dt\int_0^1 f((1-t)x+ty)\,dt\right\|$$
$$\le\left(\int_0^1\left\|f'_{x,y}(t)\right\|^p dt\right)^{1/p}\times\begin{cases}\frac{1}{4}\left(\int_0^1\|\Theta(g;t)\|^q\,dt\right)^{1/q},\\ \\ [B(q+1,q+1)]^{1/q}\sup_{t\in[0,1]}\|\Theta(g;t)\|\end{cases}$$

and by (3.12) we get (3.16).

The inequality (3.17) follows by (2.9) and (3.11). □

4. The Case of Circular Paths

We consider the circular path $\xi(s)=Re^{2\pi is}$ where $s\in[0,1]$, then $d\xi(s)=2\pi iRe^{2\pi is}ds$, $|d\xi(s)|=2\pi Rds$ and $|\xi|=R$.

Assume that $f:G\to\mathbb{C}$ is analytic on the convex domain G and $x, y\in\mathcal{B}$, $x\neq y$ with $\sigma(x), \sigma(y)\subset D(0,R)\subset G$. Then by Proposition 1 we derive the simpler inequalities

$$\sup_{t\in[0,1]}\left\|f_{x,y}^{\prime}(t)\right\| \leq \frac{R\|y-x\|}{\min\left\{(R-\|x\|)^{2},(R-\|y\|)^{2}\right\}}\int_{0}^{1}\left|f\left(Re^{2\pi is}\right)\right|ds, \tag{4.1}$$

$$\int_{0}^{1}\left\|f_{x,y}^{\prime}(t)\right\|dt \leq \frac{R\|y-x\|}{(R-\|y\|)(R-\|x\|)}\int_{0}^{1}\left|f\left(Re^{2\pi is}\right)\right|ds, \tag{4.2}$$

and

$$\begin{aligned}&\left(\int_{0}^{1}\left\|f_{x,y}^{\prime}(t)\right\|^{p}dt\right)^{1/p} \\ &\leq R\|y-x\|\left(\int_{0}^{1}\left|f\left(Re^{2\pi is}\right)\right|^{q}ds\right)^{1/q} \\ &\times\left(\frac{1}{(2p-1)(\|y\|-\|x\|)}\frac{(R-\|x\|)^{2p-1}-(R-\|y\|)^{2p-1}}{(R-\|x\|)^{2p-1}(R-\|y\|)^{2p-1}}\right)^{1/p},\end{aligned} \tag{4.3}$$

for $p,\ q>1$ with $\frac{1}{p}+\frac{1}{q}=1$.

Assume that $f:G\to\mathbb{C}$ is analytic on the convex domain G and $x,\ y\in\mathcal{B}$, $x\neq y$ with $\sigma(x),\ \sigma(y)\subset D(0,R)\subset G$. If $g:[0,1]\to\mathcal{B}$ is continuous, then by Theorem 3

$$\begin{aligned}&\left\|\int_{0}^{1}g(t)f((1-t)x+ty)dt-\int_{0}^{1}g(t)dt\int_{0}^{1}f((1-t)x+ty)dt\right\| \\ &\leq\frac{R\|y-x\|}{\min\left\{(R-\|x\|)^{2},(R-\|y\|)^{2}\right\}}\int_{0}^{1}\left|f\left(Re^{2\pi is}\right)\right|ds \\ &\times\begin{cases}\frac{1}{4}\int_{0}^{1}\|\Theta(g;t)\|dt, \\ \frac{1}{6}\sup_{t\in[0,1]}\|\Theta(g;t)\|, \\ [B(q+1,q+1)]^{1/q}\left(\int_{0}^{1}\|\Theta(g;t)\|^{p}dt\right)^{1/p}.\end{cases}\end{aligned} \tag{4.4}$$

We also have

$$\begin{aligned}&\left\|\int_{0}^{1}g(t)f((1-t)x+ty)dt-\int_{0}^{1}g(t)dt\int_{0}^{1}f((1-t)x+ty)dt\right\| \\ &\leq R\|y-x\|\left(\int_{0}^{1}\left|f\left(Re^{2\pi is}\right)\right|^{q}ds\right)^{1/q} \\ &\times\left(\frac{1}{(2p-1)(\|y\|-\|x\|)}\frac{(R-\|x\|)^{2p-1}-(R-\|y\|)^{2p-1}}{(R-\|x\|)^{2p-1}(R-\|y\|)^{2p-1}}\right)^{1/p} \\ &\times\begin{cases}\frac{1}{4}\left(\int_{0}^{1}\|\Theta(g,t)\|^{q}dt\right)^{1/q}, \\ [B(q+1,q+1)]^{1/q}\sup_{t\in[0,1]}\|\Theta(g,t)\|\end{cases}\end{aligned} \tag{4.5}$$

and

$$\left\| \int_0^1 g(t) f((1-t)x + ty)\, dt - \int_0^1 g(t)\, dt \int_0^1 f((1-t)x+ty)\, dt \right\| \le \frac{1}{4} \frac{R\|y-x\|}{(R-\|y\|)(R-\|x\|)} \int_0^1 \left| f\left(Re^{2\pi i s}\right)\right| ds \sup_{t\in[0,1]} \|\Theta(g,t)\| . \tag{4.6}$$

The *modified Bessel function of the first kind* $I_\nu(z)$ for real number ν can be defined by the power series as [1, p. 376]

$$I_\nu(z) = \left(\frac{1}{2}z\right)^\nu \sum_{k=0}^\infty \frac{\left(\frac{1}{4}z^2\right)^k}{k!\Gamma(\nu+k+1)},$$

where Γ is the *gamma function*. For $n=0$ we have $I_0(z)$ given by

$$I_0(z) = \sum_{k=0}^\infty \frac{\left(\frac{1}{4}z^2\right)^k}{(k!)^2}.$$

An integral formula for real number ν is

$$I_\nu(z) = \frac{1}{\pi}\int_0^\pi e^{z\cos\theta}\cos(\nu\theta)\, d\theta - \frac{\sin(\nu\pi)}{\pi}\int_0^\infty e^{-z\cosh t - \nu t} dt,$$

which simplifies for ν an integer n to

$$I_n(z) = \frac{1}{\pi}\int_0^\pi e^{z\cos\theta}\cos(n\theta)\, d\theta.$$

For $n=0$ we have

$$I_0(z) = \frac{1}{\pi}\int_0^\pi e^{z\cos\theta} d\theta.$$

Consider the exponential function $f(a) = \exp a$, $a \in \mathcal{B}$. Assume that $x, y \in \mathcal{B}$ and $\|x\|, \|y\| < R$ for some $R > 0$. Observe that

$$\left|\exp\left(Re^{2\pi i t}\right)\right| = \left|\exp\left[R\left(\cos(2\pi t) + i\sin(2\pi t)\right)\right]\right| = \exp\left[R\cos(2\pi t)\right]$$

and, by changing the variable $\theta = 2\pi t$, we get $dt = \frac{1}{2\pi}d\theta$ and

$$\begin{aligned} &\int_0^1 \exp[R\cos(2\pi t)]\, dt \\ &= \frac{1}{2\pi}\int_0^{2\pi} \exp[R\cos\theta]\, d\theta \\ &= \frac{1}{2}\left(\frac{1}{\pi}\int_0^\pi \exp[R\cos\theta]\, d\theta + \frac{1}{\pi}\int_\pi^{2\pi}\exp[R\cos\theta]\, d\theta\right) \\ &= \frac{1}{2}\left(I_0(R) + I_0(-R)\right) = I_0(R). \end{aligned} \tag{4.7}$$

If $g:[0,1]\to\mathcal{B}$ is continuous, then for all $u\in[0,1]$ we have by (4.4) for the exponential function, that

$$\left\|\int_0^1 g(t)\exp((1-t)x+ty)\,dt-\int_0^1 g(t)\,dt\int_0^1\exp((1-t)x+ty)\,dt\right\| \tag{4.8}$$
$$\leq\frac{RI_0(R)\|y-x\|}{\min\left\{(R-\|x\|)^2,(R-\|y\|)^2\right\}}$$
$$\times\begin{cases}\frac{1}{4}\int_0^1\|\Theta(g;t)\|\,dt,\\ \\ \frac{1}{6}\sup_{t\in[0,1]}\|\Theta(g;t)\|,\\ \\ [B(q+1,q+1)]^{1/q}\left(\int_0^1\|\Theta(g;t)\|^p\,dt\right)^{1/p}.\end{cases}$$

We also have by (4.5)-(4.6) that

$$\left\|\int_0^1 g(t)\exp((1-t)x+ty)\,dt-\int_0^1 g(t)\,dt\int_0^1\exp((1-t)x+ty)\,dt\right\| \tag{4.9}$$
$$\leq R\|y-x\|[I_0(qR)]^{1/q}$$
$$\times\left(\frac{1}{(2p-1)(\|y\|-\|x\|)}\frac{(R-\|x\|)^{2p-1}-(R-\|y\|)^{2p-1}}{(R-\|x\|)^{2p-1}(R-\|y\|)^{2p-1}}\right)^{1/p}$$
$$\times\begin{cases}\frac{1}{4}\left(\int_0^1\|\Theta(x,t)\|^q\,dt\right)^{1/q},\\ \\ [B(q+1,q+1)]^{1/q}\sup_{t\in[0,1]}\|\Theta(g,t)\|\end{cases}$$

and

$$\left\|\int_0^1 g(t)\exp((1-t)x+ty)\,dt-\int_0^1 g(t)\,dt\int_0^1\exp((1-t)x+ty)\,dt\right\| \tag{4.10}$$
$$\leq\frac{1}{4}\frac{RI_0(R)\|y-x\|}{(R-\|y\|)(R-\|x\|)}\sup_{t\in[0,1]}\|\Theta(g,t)\|.$$

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