# SEVERAL HERMITE-HADAMARD TYPE INEQUALITIES FOR $M_\varphi A$-S-CONVEX FUNCTIONS

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Abstract. In this paper is used the new concept of $M_\varphi A$-$s$-convexity, which is a generalization of several well-known concepts, of s-convexity, GA-s-convexity, harmonically s-convexity and $(p,s)$-convexity in order to establish some Hermite-Hadamard type inequalities for these functions.

## 1. Introduction

The classical inequality of Hermite-Hadamard was extended and generalized in many directions by many authors, like for example, [5, 4, 1, 12, 14, 3, 7, 15, 17, 8, 9, 10, 11] and the references therein.

We begin by recalling below the classical definition for the convex functions.

**Definition 1.** *A function $f : I \subset \mathbb{R} \to \mathbb{R}$ is said to be convex on an interval $I$ if the inequality*

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) \tag{1}$$

*holds for all $x, y \in I$ and $t \in [0,1]$. The function $f$ is said to be concave on $I$ if the inequality (1) takes place in reversed direction.*

For other type of convexity see also [16, 13].

**Definition 2.** *([17]) We consider $I$ a real interval, $\varphi : I \to \mathbb{R}$ a continuous function and strictly monotonic function and $s \in (0,1]$.*

*(a) A function $f : I \to \mathbb{R}$ is said to be $M_\varphi A$-$s$-convex in the first sense, if*

$$f\left(\varphi^{-1}(t\varphi(x) + (1-t)\varphi(y))\right) \leq t^s f(x) + (1-t^s)f(y)$$

*for all $x, y \in I$ and $t \in [0,1]$. If the above inequality is reversed then $f$ is said to be $M_\varphi A$-$s$-concave in the first sense.*

*(b) A function $f : I \to \mathbb{R}$ is said to be $M_\varphi A$-$s$-convex in the second sense, if*

$$f\left(\varphi^{-1}(t\varphi(x) + (1-t)\varphi(y))\right) \leq t^s f(x) + (1-t)^s f(y)$$

*for all $x, y \in I$ and $t \in [0,1]$. If the above inequality is reversed then $f$ is said to be $M_\varphi A$-$s$-concave in the second sense.*

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Let $f$ be a $M_{\varphi}A$-$s$-convex function. It is well-known that ( see [17]) :

(i) If we take $\varphi : I \subset \mathbb{R} \to \mathbb{R}$, $\varphi(x) = x$, then $M_{\varphi}A$-convexity means GA-convexity.

(ii) If we consider $\varphi : I \subset (0,\infty) \to \mathbb{R}$, $\varphi(x) = lnx$, then $M_{\varphi}A$-convexity means convexity.

(iii) If we have $\varphi : I \subset (0,\infty) \to \mathbb{R}$, $\varphi(x) = x^{-1}$, then $M_{\varphi}A$-convexity means Harmonically-convexity.

(iv) If we have $\varphi : I \subset (0,\infty) \to \mathbb{R}$, $\varphi(x) = x^{p}$, $p \in \mathbb{R} - \{0\}$ then $M_{\varphi}A$-convexity means p-convexity.

The classical Hermite-Hadamard's inequality for convex functions is

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{3}$$

Moreover, if the function $f$ is concave then the inequality (2) hold in reversed direction.

**Lemma 1.** *(see [2]) Let $f : I^o \to \mathbb{R}$, $: I^o \subset [0,\infty)$ be a twice differentiable function on $: I^o$ where $a, b \in I$, $a < b$. If $f^{''} \in L[a,b]$, then the following equality holds:*

$$-f\left(\frac{a+b}{2}\right) + \frac{1}{b-a}\int_a^b f(x)dx =$$

$$= \frac{(b-a)^2}{16}[\int_0^1 t^2 f^{''}\left(t\frac{a+b}{2} + (1-t)a\right)dt + \int_0^1 (t-1)^2 f^{''}\left(tb + (1-t)\frac{a+b}{2}\right)].$$

In this paper the concept of $M_{\varphi}A$-$s$-convexity in the second sense and respectively in the first sense is used, in order to establish several Hermite-Hadamard type inequalities for these functions.

## 2. Some Hermite-Hadamard type inequalities for $M_{\varphi}A$-$s$-convex functions in the second sense and first sense

The aim of this section is to present new inequalities that refine Hermite-Hadamard inequality for functions whose derivatives are $M_{\varphi}A$-$s$-convex functions in the second sense or in the first sense respectively.

**Lemma 2.** *Let $f$ and $f^{'}$ be two differentiable functions on $I^{\circ}$, where $f : I \to \mathbb{R}$ and $a, b \in I$ with $a < b$ and $\varphi : I \to \mathbb{R}$ be a continuous and strictly monotonic function such that $\varphi^{-1} : \varphi(I^{\circ}) \to I^{\circ}$ is continuously differentiable and $(\varphi^{-1})^{'}$ also continuously differentiable. If $f^{'}, f^{''} \in L[a,b]$ then we have*

$$\frac{1}{\varphi(b)-\varphi(a)}\int_a^b f(x)\varphi^{'}(x)dx - f\left(\frac{a+b}{2}\right) + \frac{f^{'}}{\varphi^{'}}\left(\frac{a+b}{2}\right)\left(\varphi\left(\frac{a+b}{2}\right) - \frac{\varphi(a)+\varphi(b)}{2}\right) =$$

$$= \frac{1}{2\left(\varphi(b)-\varphi(a)\right)}\left[\left(\varphi\left(\frac{a+b}{2}\right) - \varphi(a)\right)^3 I_1 + \left(\varphi(b) - \varphi\left(\frac{a+b}{2}\right)\right)^3 I_2\right],$$

*where*

$$I_1=\int_0^1\{t^2\left[(\varphi^{-1})'\left(t\varphi\left(\frac{a+b}{2}\right)+(1-t)\varphi(a)\right)\right]^2 f''\left(\varphi^{-1}\left(t\varphi\left(\frac{a+b}{2}\right)+(1-t)\varphi(a)\right)\right)+$$

$$+t^2(\varphi^{-1})''\left(t\varphi\left(\frac{a+b}{2}\right)+(1-t)\varphi(a)\right)f'\left(\varphi^{-1}\left(t\varphi\left(\frac{a+b}{2}\right)+(1-t)\varphi(a)\right)\right)\}dt,$$

*and*

$$I_2=\int_0^1\{(1-t)^2\left[(\varphi^{-1})'\left(t\varphi(b)+(1-t)\varphi(\frac{a+b}{2})\right)\right]^2 f''\left(\varphi^{-1}\left(t\varphi(b)+(1-t)\varphi(\frac{a+b}{2})\right)\right)+$$

$$+(1-t)^2(\varphi^{-1})''\left(t\varphi(b)+(1-t)\varphi(\frac{a+b}{2})\right)f'\left(\varphi^{-1}\left(t\varphi(b)+(1-t)\varphi(\frac{a+b}{2})\right)\right)\}dt.$$

*Proof.* By integrating by parts two times $I_1$ and $I_2$ we get respectively,

$$I_1=\frac{t^2(f\circ\varphi^{-1})'\left(t\varphi\left(\frac{a+b}{2}\right)+(1-t)\varphi(a)\right)}{\varphi\left(\frac{a+b}{2}\right)-\varphi(a)}|_0^1-2\int_0^1\frac{t(f\circ\varphi^{-1})'\left(t\varphi\left(\frac{a+b}{2}\right)+(1-t)\varphi(a)\right)}{\varphi\left(\frac{a+b}{2}\right)-\varphi(a)}dt=$$

$$=\frac{(f\circ\varphi^{-1})'\left(\varphi\left(\frac{a+b}{2}\right)\right)}{\varphi\left(\frac{a+b}{2}\right)-\varphi(a)}-2\frac{f\left(\frac{a+b}{2}\right)}{\left(\varphi\left(\frac{a+b}{2}\right)-\varphi(a)\right)^2}+2\int_0^1\frac{(f\circ\varphi^{-1})\left(t\varphi\left(\frac{a+b}{2}\right)+(1-t)\varphi(a)\right)}{\left(\varphi\left(\frac{a+b}{2}\right)-\varphi(a)\right)^2}dt$$

$$I_2=\frac{(1-t)^2(f\circ\varphi^{-1})'\left(t\varphi(b)+(1-t)\varphi(\frac{a+b}{2})\right)}{\varphi(b)-\varphi(\frac{a+b}{2})}|_0^1-2\int_0^1\frac{(t-1)(f\circ\varphi^{-1})'\left(t\varphi(b)+(1-t)\varphi(\frac{a+b}{2})\right)}{\varphi(b)-\varphi(\frac{a+b}{2})}dt=$$

$$=-\frac{(f\circ\varphi^{-1})'\left(\varphi\left(\frac{a+b}{2}\right)\right)}{\varphi(b)-\varphi(\frac{a+b}{2})}-2\frac{f\left(\frac{a+b}{2}\right)}{\left(\varphi(b)-\varphi(\frac{a+b}{2})\right)^2}+2\int_0^1\frac{(f\circ\varphi^{-1})\left(t\varphi(b)+(1-t)\varphi(\frac{a+b}{2})\right)}{\left(\varphi(b)-\varphi(\frac{a+b}{2})\right)^2}dt.$$

Setting now, $x=\varphi^{-1}\left(t\varphi\left(\frac{a+b}{2}\right)+(1-t)\varphi(a)\right)$ in $I_1$ and $y=\varphi^{-1}\left(t\varphi(b)+(1-t)\varphi(\frac{a+b}{2})\right)$ in $I_2$ we have respectively, $dt=\frac{\varphi'(x)}{\varphi(\frac{a+b}{2})-\varphi(a)}dx$, $dt=\frac{\varphi'(y)}{\varphi(b)-\varphi(\frac{a+b}{2})}dy$ and then we obtain respectively,

$$I_1=\frac{(f\circ\varphi^{-1})'\left(\varphi\left(\frac{a+b}{2}\right)\right)}{\varphi\left(\frac{a+b}{2}\right)-\varphi(a)}-2\frac{f\left(\frac{a+b}{2}\right)}{\left(\varphi\left(\frac{a+b}{2}\right)-\varphi(a)\right)^2}+2\int_a^{\frac{a+b}{2}}\frac{f(x)\varphi'(x)}{(\varphi\left(\frac{a+b}{2}\right)-\varphi(a))^3}dx,$$

$$I_2=-\frac{(f\circ\varphi^{-1})'\left(\varphi\left(\frac{a+b}{2}\right)\right)}{\varphi(b)-\varphi(\frac{a+b}{2})}-2\frac{f\left(\frac{a+b}{2}\right)}{\left(\varphi(b)-\varphi(\frac{a+b}{2})\right)^2}+2\int_{\frac{a+b}{2}}^b\frac{f(y)\varphi'(y)}{\left(\varphi(b)-\varphi(\frac{a+b}{2})\right)^3}dy.$$

Now multiplying $I_1$ by $\left(\varphi\left(\frac{a+b}{2}\right)-\varphi(a)\right)^3$ and $I_2$ by $\left(\varphi(b)-\varphi(\frac{a+b}{2})\right)^3$ and adding then adding the obtained expressions we get,

$$\left(\varphi\left(\frac{a+b}{2}\right)-\varphi(a)\right)^3 I_1+\left(\varphi(b)-\varphi(\frac{a+b}{2})\right)^3 I_2=$$

$$=2\int_a^b f(x)\varphi'(x)dx-2f\left(\frac{a+b}{2}\right)(\varphi(b)-\varphi(a))+\frac{f'}{\varphi'}\left(\frac{a+b}{2}\right)\left(2\varphi\left(\frac{a+b}{2}\right)-\varphi(a)-\varphi(b)\right)$$

and from here the desired inequality. We used above also the properties of the derivatives of inverse of the function $\varphi$.

□

**Remark 1.** *In previous lemma*

*(a) if we take* $\varphi(x) = mx + n$ *then we obtain the inequality from Lemma 1( see [2]).*

*(b) if we take* $\varphi(x) = lnx$ *then we obtain the inequality*

$$\frac{1}{lnb-lna}\int_a^b \frac{f(x)}{x}dx - f\left(\frac{a+b}{2}\right) + \frac{a+b}{2}f^{'}\left(\frac{a+b}{2}\right)\left(ln\left(\frac{a+b}{2}\right) - ln(ab)^{\frac{1}{2}}\right) =$$

$$= \frac{1}{2(lnb-lna)}[\left(ln\frac{a+b}{2a}\right)^3\int_0^1 t^2(\left(\frac{a+b}{2}\right)^{2t}a^{2(1-t)}f^{''}\left((\frac{a+b}{2})^t a^{1-t}\right) +$$

$$+\left(\frac{a+b}{2}\right)^t a^{1-t}f^{'}\left((\frac{a+b}{2})^t a^{1-t}\right))dt+$$

$$+\left(ln\frac{2b}{a+b}\right)^3\int_0^1 (1-t)^2(b^{2t}\left(\frac{a+b}{2}\right)^{2(1-t)}f^{''}\left(b^t(\frac{a+b}{2})^{1-t}\right) +$$

$$+b^t\left(\frac{a+b}{2}\right)^{1-t}f^{'}\left(b^t(\frac{a+b}{2})^{1-t}\right))dt].$$

**Theorem 1.** *Let* $f : I \subset \mathbb{R}_+ \to \mathbb{R}$ *be differentiable and* $f^{'}$ *differentiable on* $I^\circ$*, and* $a, b \in I^\circ$ *with* $a < b$*,* $\varphi : I \to \mathbb{R}$ *be a continuous and strictly monotonic function so that* $\varphi^{-1} : \varphi(I^\circ) \to I^\circ$ *is continuously differentiable,* $(\varphi^{-1})^{'}$ *continuously differentiable and* $f^{'}, f^{''} \in L[a, b]$*.*

*If* $|f^{'}|$ *and* $|f^{''}|$ *are* $M_\varphi A$*-s-convex function in the second sense on* $[a, b]$ *and* $s \in (0, 1]$*, then*

$$|\frac{1}{\varphi(b)-\varphi(a)}\int_a^b f(x)\varphi^{'}(x)dx - f\left(\frac{a+b}{2}\right) + \frac{f^{'}}{\varphi^{'}}\left(\frac{a+b}{2}\right)\left(\varphi\left(\frac{a+b}{2}\right) - \frac{\varphi(a)+\varphi(b)}{2}\right)| \leq$$

$$\leq \frac{|\varphi\left(\frac{a+b}{2}\right) - \varphi(a)|^3}{2|\varphi(b)-\varphi(a)|}\{|f^{''}\left(\frac{a+b}{2}\right)|A_\varphi(a,b) + |f^{''}(a)|B_\varphi(a,b) + |f^{'}\left(\frac{a+b}{2}\right)|C_\varphi(a,b)+$$

$$+|f^{'}(a)|D_\varphi(a,b)\} + \frac{|\varphi(b) - \varphi(\frac{a+b}{2})|^3}{2|\varphi(b)-\varphi(a)|}\{|f^{''}(b)|E_\varphi(a,b) + |f^{''}(\frac{a+b}{2})|F_\varphi(a,b)+$$

$$+|f^{'}(b)|G_\varphi(a,b) + |f^{'}(\frac{a+b}{2})|H_\varphi(a,b)\},$$

*where* $A_\varphi(a,b) = \int_0^1 t^{s+2}[(\varphi^{-1})^{'}(t\varphi\left(\frac{a+b}{2}\right) + (1-t)\varphi(a))]^2dt,$

$B_\varphi(a,b) = \int_0^1 t^2(1-t)^s[(\varphi^{-1})^{'}(t\varphi\left(\frac{a+b}{2}\right) + (1-t)\varphi(a))]^2dt,$

$C_\varphi(a,b) = \int_0^1 t^{s+2}|(\varphi^{-1})^{''}(t\varphi\left(\frac{a+b}{2}\right) + (1-t)\varphi(a))|dt,$

$D_\varphi(a,b) = \int_0^1 t^2(1-t)^s|(\varphi^{-1})^{''}(t\varphi\left(\frac{a+b}{2}\right) + (1-t)\varphi(a))|dt,$

$E_\varphi(a,b) = \int_0^1 (1-t)^2t^s[(\varphi^{-1})^{'}(t\varphi(b) + (1-t)\varphi\left(\frac{a+b}{2}\right))]^2dt,$

$F_\varphi(a,b) = \int_0^1 (1-t)^{s+2}[(\varphi^{-1})^{'}(t\varphi(b) + (1-t)\varphi\left(\frac{a+b}{2}\right))]^2dt,$

$G_\varphi(a,b) = \int_0^1 (1-t)^2t^s|(\varphi^{-1})^{''}(t\varphi(b) + (1-t)\varphi\left(\frac{a+b}{2}\right))|dt,$

$H_\varphi(a,b) = \int_0^1 (1-t)^{s+2}|(\varphi^{-1})^{''}(t\varphi(b) + (1-t)\varphi\left(\frac{a+b}{2}\right))|dt.$

*Proof.* By using previous lemma and that $|f^{'}|, |f^{''}|$ are $M_\varphi A$-s-convex function in the second sense on $[a, b]$ we get

$$|\frac{1}{\varphi(b)-\varphi(a)}\int_a^b f(x)\varphi^{'}(x)dx - f\left(\frac{a+b}{2}\right) + \frac{f^{'}}{\varphi^{'}}\left(\frac{a+b}{2}\right)\left(\varphi\left(\frac{a+b}{2}\right) - \frac{\varphi(a)+\varphi(b)}{2}\right)| \leq$$

$$\leq \frac{|\varphi\left(\frac{a+b}{2}\right)-\varphi(a)|^3}{2|\varphi(b)-\varphi(a)|}\int_0^1\{t^2\left[(\varphi^{-1})'\left(t\varphi\left(\frac{a+b}{2}\right)+(1-t)\varphi(a)\right)\right]^2\cdot$$
$$\cdot|f''\left(\varphi^{-1}\left(t\varphi\left(\frac{a+b}{2}\right)+(1-t)\varphi(a)\right)\right)|+$$
$$+t^2|(\varphi^{-1})''\left(t\varphi\left(\frac{a+b}{2}\right)+(1-t)\varphi(a)\right)|\cdot|f'\left(\varphi^{-1}\left(t\varphi\left(\frac{a+b}{2}\right)+(1-t)\varphi(a)\right)\right)|\}dt+$$
$$+\frac{|\varphi(b)-\varphi(\frac{a+b}{2})|^3}{2|\varphi(b)-\varphi(a)|}\int_0^1\{(1-t)^2\left[(\varphi^{-1})'\left(t\varphi(b)+(1-t)\varphi(\frac{a+b}{2})\right)\right]^2\cdot$$
$$\cdot|f''\left(\varphi^{-1}\left(t\varphi(b)+(1-t)\varphi(\frac{a+b}{2})\right)\right)|+$$
$$+(1-t)^2|(\varphi^{-1})''\left(t\varphi(b)+(1-t)\varphi(\frac{a+b}{2})\right)|\cdot|f'\left(\varphi^{-1}\left(t\varphi(b)+(1-t)\varphi(\frac{a+b}{2})\right)\right)|\}dt\leq$$

$$\leq \frac{|\varphi\left(\frac{a+b}{2}\right)-\varphi(a)|^3}{2|\varphi(b)-\varphi(a)|}\{|f''\left(\frac{a+b}{2}\right)|\int_0^1 t^{s+2}\left[(\varphi^{-1})'\left(t\varphi\left(\frac{a+b}{2}\right)+(1-t)\varphi(a)\right)\right]^2dt+$$
$$|f''(a)|\int_0^1 t^2(1-t)^s\left[(\varphi^{-1})'\left(t\varphi\left(\frac{a+b}{2}\right)+(1-t)\varphi(a)\right)\right]^2dt+$$
$$+|f'\left(\frac{a+b}{2}\right)|\int_0^1 t^{s+2}|(\varphi^{-1})''\left(t\varphi\left(\frac{a+b}{2}\right)+(1-t)\varphi(a)\right)|dt+$$
$$+|f'(a)|\int_0^1 t^2(1-t)^s|(\varphi^{-1})''\left(t\varphi\left(\frac{a+b}{2}\right)+(1-t)\varphi(a)\right)|dt\}+$$
$$+\frac{|\varphi(b)-\varphi(\frac{a+b}{2})|^3}{2|\varphi(b)-\varphi(a)|}\{|f''(b)|\int_0^1(1-t)^2t^s\left[(\varphi^{-1})'\left(t\varphi(b)+(1-t)\varphi(\frac{a+b}{2})\right)\right]^2dt+$$
$$+|f''\left(\frac{a+b}{2}\right)|\int_0^1(1-t)^{s+2}\left[(\varphi^{-1})'\left(t\varphi(b)+(1-t)\varphi(\frac{a+b}{2})\right)\right]^2dt+$$
$$+|f'(b)|\int_0^1(1-t)^2t^s|(\varphi^{-1})''\left(t\varphi(b)+(1-t)\varphi\left(\frac{a+b}{2}\right)\right)|dt+$$
$$+|f''\left(\frac{a+b}{2}\right)|\int_0^1(1-t)^{s+2}|\left(\varphi^{-1}\right)''\left(t\varphi(b)+(1-t)\varphi(\frac{a+b}{2})\right)|dt\}.$$

This leads to the desired inequality.

□

**Theorem 2.** *Let $f:I\subset\mathbb{R}_+\to\mathbb{R}$ be differentiable and $f'$ differentiable on $I^\circ$, and $a,b\in I^\circ$ with $a<b$, $\varphi:I\to\mathbb{R}$ be a continuous and strictly monotonic function so that $\varphi^{-1}:\varphi(I^\circ)\to I^\circ$ is continuously differentiable, $(\varphi^{-1})'$ continuously differentiable and $f',f''\in L[a,b]$.*

*If $|f'|$ and $|f''|$ are $M_\varphi A$-s-convex function in the first sense on $[a,b]$ and $s\in(0,1]$, then*

$$|\frac{1}{\varphi(b)-\varphi(a)}\int_a^b f(x)\varphi'(x)dx-f\left(\frac{a+b}{2}\right)+\frac{f'}{\varphi'}\left(\frac{a+b}{2}\right)\left(\varphi\left(\frac{a+b}{2}\right)-\frac{\varphi(a)+\varphi(b)}{2}\right)|\leq$$
$$\leq \frac{|\varphi\left(\frac{a+b}{2}\right)-\varphi(a)|^3}{2|\varphi(b)-\varphi(a)|}\{|f''\left(\frac{a+b}{2}\right)|A_\varphi(a,b)+|f''(a)|B'_\varphi(a,b)+|f'\left(\frac{a+b}{2}\right)|C_\varphi(a,b)+$$

$$+|f^{'}(a)|D^{'}_{\varphi}(a,b)\} + \frac{|\varphi(b) - \varphi(\frac{a+b}{2})|^3}{2|\varphi(b)-\varphi(a)|}\{|f^{''}(b)|E_{\varphi}(a,b) + |f^{''}(\frac{a+b}{2})|F^{'}_{\varphi}(a,b) +$$

$$+|f^{'}(b)|G_{\varphi}(a,b) + |f^{'}(\frac{a+b}{2})|H^{'}_{\varphi}(a,b)\},$$

*where* $A_{\varphi}(a,b)$, $C_{\varphi}(a,b)$, $E_{\varphi}(a,b)$, $G_{\varphi}(a,b)$ *are like in previous theorem and* $B^{'}_{\varphi}(a,b) = \int_0^1 t^2(1-t^s)[(\varphi^{-1})^{'}(t\varphi\left(\frac{a+b}{2}\right) + (1-t)\varphi(a))]^2dt$,

$D^{'}_{\varphi}(a,b) = \int_0^1 t^2(1-t^s)|(\varphi^{-1})^{''}(t\varphi\left(\frac{a+b}{2}\right) + (1-t)\varphi(a))|dt$,

$F^{'}_{\varphi}(a,b) = \int_0^1 (1-t^s)(1-t)^2[(\varphi^{-1})^{'}(t\varphi(b) + (1-t)\varphi\left(\frac{a+b}{2}\right))]^2dt$,

$H^{'}_{\varphi}(a,b) = \int_0^1 (1-t)^2(1-t^s)|(\varphi^{-1})^{''}(t\varphi(b) + (1-t)\varphi\left(\frac{a+b}{2}\right))|dt$.

*Proof.* We use the definition of $M_{\varphi}A$-s-convexity in the first sense on $[a,b]$ like in previous theorem and the demonstration will be similar. □

**Remark 2.** *If we take* $\varphi(x) = mx + n$ *then we obtain the following inequality*

$$|f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)dx| \le$$

$$\le \frac{(b-a)^2}{16}[\frac{2}{s+3}|f^{''}\left(\frac{a+b}{2}\right)| + |f^{''}(a)|\frac{1}{(s+3)(s+2)(s+1)} + |f^{''}(b)|\frac{1}{(s+3)(s+2)(s+1)}]$$

*under conditions of Theorem 1.*

**Theorem 3.** *Let* $f : I \subset \mathbb{R}_+ \to \mathbb{R}$ *be differentiable and* $f^{'}$ *differentiable on* $I^{\circ}$, *and* $a, b \in I^{\circ}$ *with* $a < b$, $\varphi : I \to \mathbb{R}$ *be a continuous and strictly monotonic function so that* $\varphi^{-1} : \varphi(I^{\circ}) \to I^{\circ}$ *is continuously differentiable,* $(\varphi^{-1})^{'}$ *continuously differentiable and* $f^{'}, f^{''} \in L[a,b]$ *and* $q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$.

*If* $|f^{'}|^q$ *and* $|f^{''}|^q$ *are* $M_{\varphi}A$*-s-convexity in the second sense on* $[a,b]$ *and* $s \in (0,1]$ *then we have,*

$$|\frac{1}{\varphi(b)-\varphi(a)}\int_a^b f(x)\varphi^{'}(x)dx - f\left(\frac{a+b}{2}\right) + \frac{f^{'}}{\varphi^{'}}\left(\frac{a+b}{2}\right)\left(\varphi\left(\frac{a+b}{2}\right) - \frac{\varphi(a)+\varphi(b)}{2}\right)| \le$$

$$\le \frac{1}{2}\frac{|\varphi\left(\frac{a+b}{2}\right) - \varphi(a)|^3}{|\varphi(b)-\varphi(a)|}\{A^{\frac{1}{p}}_{\varphi}(a,b,p)\left(\frac{|f^{''}\left(\frac{a+b}{2}\right)|^q + |f^{''}(a)|^q}{s+1}\right)^{\frac{1}{q}} +$$

$$+B^{\frac{1}{p}}_{\varphi}(a,b,p)\left(\frac{|f^{'}\left(\frac{a+b}{2}\right)|^q + |f^{'}(a)|^q}{s+1}\right)^{\frac{1}{q}}\} +$$

$$+\frac{1}{2}\frac{|\varphi(b) - \varphi\left(\frac{a+b}{2}\right)|^3}{|\varphi(b)-\varphi(a)|}\{C^{\frac{1}{p}}_{\varphi}(a,b,p)\left(\frac{|f^{''}(b)|^q + |f^{''}\left(\frac{a+b}{2}\right)|^q}{s+1}\right)^{\frac{1}{q}} +$$

$$+D^{\frac{1}{p}}_{\varphi}(a,b,p)\left(\frac{|f^{'}(b)|^q + |f^{'}\left(\frac{a+b}{2}\right)|^q}{s+1}\right)^{\frac{1}{q}}\},$$

*where* $A_{\varphi}(a,b,p) = \int_0^1 t^{2p}[(\varphi^{-1})^{'}\left(t\varphi\left(\frac{a+b}{2}\right) + (1-t)\varphi(a)\right)]^{2p}dt$,

$B_{\varphi}(a,b,p) = \int_0^1 t^{2p}|(\varphi^{-1})^{''}\left(t\varphi\left(\frac{a+b}{2}\right) + (1-t)\varphi(a)\right)|^p dt$,

$C_{\varphi}(a,b,p) = \int_0^1 (1-t)^{2p}[(\varphi^{-1})^{'}\left(t\varphi(b) + (1-t)\varphi\left(\frac{a+b}{2}\right)\right)]^{2p}dt$,

$D_{\varphi}(a,b,p)=\int_0^1(1-t)^{2p}|(\varphi^{-1})^{''}\left(t\varphi(b)+(1-t)\varphi\left(\frac{a+b}{2}\right)\right)|^p dt.$

*Proof.* Using Lemma 2 and Holder's inequality we have succesively

$$|\frac{1}{\varphi(b)-\varphi(a)}\int_a^b f(x)\varphi^{'}(x)dx-f\left(\frac{a+b}{2}\right)+\frac{f^{'}}{\varphi^{'}}\left(\frac{a+b}{2}\right)\left(\varphi\left(\frac{a+b}{2}\right)-\frac{\varphi(a)+\varphi(b)}{2}\right)|\leq$$

$$\leq\frac{1}{2}\frac{|\varphi\left(\frac{a+b}{2}\right)-\varphi(a)|^3}{|\varphi(b)-\varphi(a)|}\{\left(\int_0^1 t^{2p}[(\varphi^{-1})^{'}\left(t\varphi\left(\frac{a+b}{2}\right)+(1-t)\varphi(a)\right)]^{2p}dt\right)^{\frac{1}{p}}\cdot$$

$$\cdot\left(\int_0^1|f^{''}\left(\varphi^{-1}\right)\left(t\varphi\left(\frac{a+b}{2}\right)+(1-t)\varphi(a)\right)|^q dt\right)^{\frac{1}{q}}+$$

$$+\left(\int_0^1 t^{2p}|(\varphi^{-1})^{''}\left(t\varphi\left(\frac{a+b}{2}\right)+(1-t)\varphi(a)\right)|^p dt\right)^{\frac{1}{p}}\cdot$$

$$\cdot\left(\int_0^1|f^{'}\left(\varphi^{-1}\right)\left(t\varphi\left(\frac{a+b}{2}\right)+(1-t)\varphi(a)\right)|^q dt\right)^{\frac{1}{q}}\}+$$

$$+\frac{1}{2}\frac{|\varphi(b)-\varphi\left(\frac{a+b}{2}\right)|^3}{|\varphi(b)-\varphi(a)|}\{\left(\int_0^1(1-t)^{2p}[(\varphi^{-1})^{'}\left(t\varphi(b)+(1-t)\varphi\left(\frac{a+b}{2}\right)\right)]^{2p}dt\right)^{\frac{1}{p}}\cdot$$

$$\cdot\left(\int_0^1|f^{''}\left(\varphi^{-1}\right)\left(t\varphi(b)+(1-t)\varphi\left(\frac{a+b}{2}\right)\right)|^q dt\right)^{\frac{1}{q}}+$$

$$+\left(\int_0^1(1-t)^{2p}|(\varphi^{-1})^{''}\left(t\varphi(b)+(1-t)\varphi\left(\frac{a+b}{2}\right)\right)|^p dt\right)^{\frac{1}{p}}\cdot$$

$$\cdot\left(\int_0^1|f^{'}\left(\varphi^{-1}\right)\left(t\varphi(b)+(1-t)\varphi\left(\frac{a+b}{2}\right)\right)|^q dt\right)^{\frac{1}{q}}\}\leq$$

and from here using that $|f^{'}|^q$ and $|f^{''}|^q$ are $M_{\varphi}A$-s-convexity in the second sense on $[a,b]$ we get the desired inequality.

□

**Theorem 4.** *Under conditions of Theorem 3, if* $|f^{'}|^q$ *and* $|f^{''}|^q$ *are* $M_{\varphi}A$*-s-convexity in the first sense on* $[a,b]$ *and* $s\in(0,1]$ *then we have,*

$$|\frac{1}{\varphi(b)-\varphi(a)}\int_a^b f(x)\varphi^{'}(x)dx-f\left(\frac{a+b}{2}\right)+\frac{f^{'}}{\varphi^{'}}\left(\frac{a+b}{2}\right)\left(\varphi\left(\frac{a+b}{2}\right)-\frac{\varphi(a)+\varphi(b)}{2}\right)|\leq$$

$$\leq\frac{1}{2}\frac{|\varphi\left(\frac{a+b}{2}\right)-\varphi(a)|^3}{|\varphi(b)-\varphi(a)|}\{A_{\varphi}^{\frac{1}{p}}(a,b,p)\left(\frac{|f^{''}\left(\frac{a+b}{2}\right)|^q+s|f^{''}(a)|^q}{s+1}\right)^{\frac{1}{q}}+$$

$$+B_{\varphi}^{\frac{1}{p}}(a,b,p)\left(\frac{|f^{'}\left(\frac{a+b}{2}\right)|^q+s|f^{'}(a)|^q}{s+1}\right)^{\frac{1}{q}}\}+$$

$$+\frac{1}{2}\frac{|\varphi(b)-\varphi\left(\frac{a+b}{2}\right)|^3}{|\varphi(b)-\varphi(a)|}\{C_{\varphi}^{\frac{1}{p}}(a,b,p)\left(\frac{|f^{''}(b)|^q+s|f^{''}\left(\frac{a+b}{2}\right)|^q}{s+1}\right)^{\frac{1}{q}}+$$

$$+D_{\varphi}^{\frac{1}{p}}(a,b,p)\left(\frac{|f^{'}(b)|^q+s|f^{'}\left(\frac{a+b}{2}\right)|^q}{s+1}\right)^{\frac{1}{q}}\},$$

*where $A_\varphi(a,b,p)$, $B_\varphi(a,b,p)$, $C_\varphi(a,b,p)$, and $D_\varphi(a,b,p)$ are like in previous theorem.*

*Proof.* This inequality can be easily obtained by the same method as in previous theorem. □

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