p-SCHATTEN NORM INEQUALITIES FOR ČEBYŠEV'S FUNCTIONAL VIA DOUBLE SUMS IDENTITIES

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Abstract. An operator $A \in \mathcal{B}(H)$ is said to belong to the *von Neumann-Schatten class* $\mathcal{B}_p(H)$, $1 \leq p < \infty$ if the *p-Schatten norm* is finite

$$\|A\|_p := [\operatorname{tr}(|A|^p)]^{1/p} < \infty.$$

Assume that $p,\ q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ and $\mathbf{p} = (p_1, ..., p_n) \in \mathbb{R}^n$, $A= (A_1, ..., A_n) \in (\mathcal{B}_p(H))^n$ and $B= (B_1, ..., B_n) \in (\mathcal{B}_q(H))^n$. In this paper we show among others that

$$\left\| P_n \sum_{i=1}^n p_i A_i B_i - \sum_{i=1}^n p_i A_i \sum_{i=1}^n p_i B_i \right\|_1 \leq \begin{cases} \max_{1\leq i,j\leq n-1} \left\{ \left|P_{\min\{i,j\}}\right| \left|\bar{P}_{\max\{i,j\}}\right| \right\} \sum_{i=1}^{n-1} \|\Delta A_i\|_p \sum_{i=1}^{n-1} \|\Delta B_i\|_q ; \\ \left(\sum_{i=1}^{n-1} \sum_{j=1}^{n-1} \left|P_{\min\{i,j\}}\right|^\beta \left|\bar{P}_{\max\{i,j\}}\right|^\beta \right)^{1/\beta} \\ \times \left(\sum_{i=1}^{n-1} \|\Delta A_i\|_p^\alpha \right)^{1/\alpha} \left(\sum_{i=1}^{n-1} \|\Delta B_i\|_q^\alpha \right)^{1/\alpha} \text{ for } \alpha > 1, \frac{1}{\alpha} + \frac{1}{\beta} = 1; \\ \sum_{i=1}^{n-1} \sum_{j=1}^{n-1} \left|P_{\min\{i,j\}}\right| \left|\bar{P}_{\max\{i,j\}}\right| \\ \times \max_{1\leq i\leq n-1} \|\Delta A_i\|_p \max_{1\leq i\leq n-1} \|\Delta B_i\|_q , \end{cases}$$

where $P_i := \sum_{k=1}^i p_k$, $\bar{P}_i := P_n - P_i$ and $\Delta A_i := A_{i+1} - A_i$ $(i = 1, ..., n-1)$ are the usual forward differences.

1. Introduction

Consider the Čebyšev functional defined for $\mathbf{p} = (p_1, ..., p_n) \in \mathbb{R}^n$, $\boldsymbol{\alpha} = (\alpha_1, ..., \alpha_n) \in \mathbb{K}^n$ and $\mathbf{x} = (x_1, ..., x_n) \in X^n$, where X is a linear space over the real or complex number field $\mathbb{K}$:

$$T_n(\mathbf{p}; \boldsymbol{\alpha}, \mathbf{x}) := P_n \sum_{i=1}^n p_i \alpha_i x_i - \sum_{i=1}^n p_i \alpha_i \sum_{i=1}^n p_i x_i, \tag{1.1}$$

where $P_n := \sum_{i=1}^n p_i$.

The following Grüss type inequalities for sequences in normed linear spaces hold.

Theorem 1. *Let $(X, \|.\|)$ be a normed linear space over the real or complex number field $\mathbb{K}$, $\boldsymbol{\alpha} = (\alpha_1, ..., \alpha_n) \in \mathbb{K}^n$, $\mathbf{p} = (p_1, ..., p_n) \in \mathbb{R}_+^n$ with $\sum_{i=1}^n p_i = 1$ and $\mathbf{x} =$*

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$(x_1, ..., x_n) \in X^n$. Then one has the inequalities

$$\|T_n(\mathbf{p};\boldsymbol{\alpha},\mathbf{x})\| \leq \begin{cases} \left[\sum_{i=1}^n i^2 p_i - \left(\sum_{i=1}^n i p_i\right)^2\right] \\ \times \max_{1\leq j\leq n-1} |\Delta\alpha_j| \max_{1\leq j\leq n-1} \|\Delta x_j\|, \ [5]; \\ \\ \frac{1}{2}\sum_{i=1}^n p_i(1-p_i)\sum_{j=1}^{n-1}|\Delta\alpha_j|\sum_{j=1}^{n-1}\|\Delta x_j\|, \ [7]; \\ \\ \sum_{1\leq i<j\leq n} p_i p_j (j-i)\left(\sum_{j=1}^{n-1}|\Delta\alpha_j|^p\right)^{1/p}\left(\sum_{j=1}^{n-1}\|\Delta x_j\|^q\right)^{1/q}, \\ \\ p>1, \ \frac{1}{p}+\frac{1}{q}=1, \ [6]. \end{cases} \tag{1.2}$$

The constant 1 in the first branch, $\frac{1}{2}$ in the second branch and 1 in the third branch are best possible in the sense that they cannot be replaced by smaller constants.

The following particular inequalities for unweighted means hold as well, where $T_n(\boldsymbol{\alpha},\mathbf{x})$ is defined as follows:

$$T_n(\boldsymbol{\alpha},\mathbf{x}) := \frac{1}{n}\sum_{i=1}^n \alpha_i x_i - \frac{1}{n}\sum_{i=1}^n a_i \frac{1}{n}\sum_{i=1}^n x_i.$$

Corollary 1. *With the assumptions of Theorem 1 for X, $\boldsymbol{\alpha}$ and $\mathbf{x}$, we have*

$$\|T_n(\boldsymbol{\alpha},\mathbf{x})\| \leq \begin{cases} \frac{1}{12}\left(n^2-1\right)\max_{1\leq j\leq n-1}|\Delta\alpha_j|\max_{1\leq j\leq n-1}\|\Delta x_j\|, \ [5]; \\ \\ \frac{1}{2}\left(1-\frac{1}{n}\right)\sum_{j=1}^{n-1}|\Delta\alpha_j|\sum_{j=1}^{n-1}\|\Delta x_j\|, \ [7]; \\ \\ \frac{1}{6}\frac{n^2-1}{n}\left(\sum_{j=1}^{n-1}|\Delta\alpha_j|^p\right)^{1/p}\left(\sum_{j=1}^{n-1}\|\Delta x_j\|^q\right)^{1/q}, \\ \\ p>1, \frac{1}{p}+\frac{1}{q}=1, \ [6]. \end{cases} \tag{1.3}$$

Here the constants $\frac{1}{12}, \frac{1}{2}$ and $\frac{1}{6}$ are best possible in the sense that they cannot be replaced by smaller constants.

For applications to estimate the p-moments of guessing mappings, see [5]. For applications in approximating the discrete Fourier transform, the discrete Mellin transform as well as some applications for polynomials and Lipschitzian mappings, see [6] and [7].

For classical results related the Čebyšev functional, see [8], [9], [10], [11], [12], [14] and [16]. For more recent results, see [16], [17], [18], [20], [13] and [15].

In order to extend these results for p-Schatten norms we need the following preparations.

Let $(H; \langle ., . \rangle)$ be a complex Hilbert space and $\mathcal{B}(H)$ the Banach algebra of all bounded linear operators on H. If $\{e_i\}_{i\in I}$ an orthonormal basis of H, we say that $A \in \mathcal{B}(H)$ is of *trace class* if

$$\|A\|_1 := \sum_{i\in I}\langle |A| e_i, e_i\rangle < \infty. \tag{1.4}$$

The definition of $\|A\|_1$ does not depend on the choice of the orthonormal basis $\{e_i\}_{i\in I}$. We denote by $\mathcal{B}_1(H)$ the set of trace class operators in $\mathcal{B}(H)$.

We define the *trace* of a trace class operator $A\in\mathcal{B}_1(H)$ to be

$$\operatorname{tr}(A):=\sum_{i\in I}\langle Ae_i,e_i\rangle, \tag{1.5}$$

where $\{e_i\}_{i\in I}$ an orthonormal basis of H. Note that this coincides with the usual definition of the trace if H is finite-dimensional. We observe that the series (1.5) converges absolutely and it is independent from the choice of basis.

The following result collects some properties of the trace:

Theorem 2. *We have:*

(i) If $A\in\mathcal{B}_1(H)$ then $A^*\in\mathcal{B}_1(H)$ and

$$\operatorname{tr}(A^*)=\overline{\operatorname{tr}(A)}; \tag{1.6}$$

(ii) If $A\in\mathcal{B}_1(H)$ and $T\in\mathcal{B}(H)$, then AT, $TA\in\mathcal{B}_1(H)$ and

$$\operatorname{tr}(AT)=\operatorname{tr}(TA) \text{ and } |\operatorname{tr}(AT)|\le\|A\|_1\|T\|; \tag{1.7}$$

(iii) $\operatorname{tr}(\cdot)$ *is a bounded linear functional on $\mathcal{B}_1(H)$ with* $\|\operatorname{tr}\|=1$;

(iv) If A, $B\in\mathcal{B}_2(H)$ then AB, $BA\in\mathcal{B}_1(H)$ and $\operatorname{tr}(AB)=\operatorname{tr}(BA)$;

(v) $\mathcal{B}_{fin}(H)$, the space of operators of finite rank, is a dense subspace of $\mathcal{B}_1(H)$.

An operator $A\in\mathcal{B}(H)$ is said to belong to the *von Neumann-Schatten class* $\mathcal{B}_p(H)$, $1\le p<\infty$ if the *p-Schatten norm* is finite [22, p. 60-64]

$$\|A\|_p:=\left[\operatorname{tr}(|A|^p)\right]^{1/p}<\infty.$$

For $1<p<q<\infty$ we have that

$$\mathcal{B}_1(H)\subset\mathcal{B}_p(H)\subset\mathcal{B}_q(H)\subset\mathcal{B}(H) \tag{1.8}$$

and

$$\|A\|_1\ge\|A\|_p\ge\|A\|_q\ge\|A\|. \tag{1.9}$$

For $p\ge1$ the functional $\|\cdot\|_p$ is a *norm* on the $*$-ideal $\mathcal{B}_p(H)$ and $\left(\mathcal{B}_p(H),\|\cdot\|_p\right)$ is a Banach space.

Also, see for instance [22, p. 60-64],

$$\|A\|_p=\|A^*\|_p,\ A\in\mathcal{B}_p(H) \tag{1.10}$$

$$\|AB\|_p\le\|A\|_p\|B\|_p,\ A,B\in\mathcal{B}_p(H) \tag{1.11}$$

and

$$\|AB\|_p\le\|A\|_p\|B\|,\ \|BA\|_p\le\|B\|\|A\|_p,\ A\in\mathcal{B}_p(H),\ B\in\mathcal{B}(H). \tag{1.12}$$

This implies that

$$\|CAB\|_p\le\|C\|\|A\|_p\|B\|,\ A\in\mathcal{B}_p(H),\ B,\ C\in\mathcal{B}(H). \tag{1.13}$$

In terms of *p-Schatten norm* we have the *Hölder inequality* for p, $q>1$ with $\frac{1}{p}+\frac{1}{q}=1$

$$(|\operatorname{tr}(AB)|\le)\,\|AB\|_1\le\|A\|_p\|B\|_q,\ A\in\mathcal{B}_p(H),\ B\in\mathcal{B}_q(H). \tag{1.14}$$

For the theory of trace functionals and their applications the reader is referred to [21] and [22].

For some classical trace inequalities see [3], [4], and [19], which are continuations of the work of Bellman [1].

2. Some Identities

The first result is embodied in the following

Theorem 3. *Let* $\mathbf{p}=(p_1,...,p_n)$ *, be an* n*-tuples of real numbers and* $A=(A_1,...,A_n)$ *,* $B=(B_1,...,B_n)$ *be* n*-tuples of operators in the Hilbert space* H*. If we define*

$$\begin{aligned} P_i &:= \sum_{k=1}^{i} p_k,\ \bar{P}_i := P_n - P_i,\ i\in\{1,...,n-1\}, \\ S_i(\mathbf{p}) &:= \sum_{k=1}^{i} p_k A_k,\ \bar{S}_i(\mathbf{p}) := S_n(\mathbf{p}) - S_i(\mathbf{p}),\ i\in\{1,...,n-1\}, \end{aligned}$$

then we have the identity

$$\begin{aligned} &T_n(\mathbf{p};\mathbf{A},\mathbf{B}) \\ &= \sum_{i=1}^{n-1} \det\begin{pmatrix} P_i & P_n \\ S_i(\mathbf{p}) & S_n(\mathbf{p}) \end{pmatrix} \Delta B_i \\ &= P_n \sum_{i=1}^{n-1} P_i \left(\frac{S_n(\mathbf{p})}{P_n} - \frac{S_i(\mathbf{p})}{P_i}\right)\Delta B_i \ \ (\text{if } P_i\neq 0,\ i\in\{1,...,n\}) \\ &= \sum_{i=1}^{n-1} P_i \bar{P}_i \left(\frac{\bar{S}_i(\mathbf{p})}{\bar{P}_i} - \frac{S_i(\mathbf{p})}{P_i}\right)\Delta B_i \ \ \big(\text{if } P_i,\ \bar{P}_i\neq 0,\ i\in\{1,...,n-1\}\big); \end{aligned} \tag{2.1}$$

where $\Delta B_i := B_{i+1} - B_i\ (i\in\{1,...,n-1\})$ *is the forward difference.*

Proof. We use the following well known summation by parts formula

$$\sum_{l=p}^{q-1} d_l \Delta v_l = d_l v_l\big|_p^q - \sum_{l=p}^{q-1} v_{l+1}\Delta d_l, \tag{2.2}$$

where d_l are real or complex numbers, and v_l are vectors in a linear space, $l=p,...,q$ $(q>p;\ p,\ q$ are natural numbers$)$.

If we choose in (2.2), $p=1$, $q=n$, $d_i = P_i S_n(\mathbf{p}) - P_n S_i(\mathbf{p})$ and $v_i = B_i$ $(i \in \{1,...,n-1\})$, then we get

$$\begin{aligned}
&\sum_{i=1}^{n-1} \left(P_i S_n(\mathbf{p}) - P_n S_i(\mathbf{p})\right) \Delta B_i \\
&= \left[P_i S_n(\mathbf{p}) - P_n S_i(\mathbf{p})\right] B_i\big|_1^n - \sum_{i=1}^{n-1} \Delta\left(P_i S_n(\mathbf{p}) - P_n S_i(\mathbf{p})\right) B_{i+1} \\
&= \left[P_n S_n(\mathbf{p}) - P_n S_n(\mathbf{p})\right] B_n - \left[P_1 S_n(\mathbf{p}) - P_n S_1(\mathbf{p})\right] B_1 \\
&- \sum_{i=1}^{n-1} \left[P_{i+1} S_n(\mathbf{p}) - P_n S_{i+1}(\mathbf{p}) - P_i S_n(\mathbf{p}) + P_n S_i(\mathbf{p})\right] B_{i+1} \\
&= P_n p_1 A_1 B_1 - p_1 S_n(\mathbf{p}) B_1 - \sum_{i=1}^{n-1} \left(p_{i+1} S_n(\mathbf{p}) - P_n p_{i+1} A_{i+1}\right) B_{i+1} \\
&= P_n p_1 A_1 B_1 - p_1 S_n(\mathbf{p}) B_1 - S_n(\mathbf{p}) \sum_{i=1}^{n-1} p_{i+1} B_{i+1} + P_n \sum_{i=1}^{n-1} p_{i+1} A_{i+1} B_{i+1} \\
&= P_n \sum_{i=1}^{n} p_i A_i B_i - \sum_{i=1}^{n} p_i A_i \sum_{i=1}^{n} p_i B_i \\
&= T_n(\mathbf{p}; \mathbf{A}, \mathbf{B}),
\end{aligned}$$

which produce the first identity in (2.1).

The second and the third identities are obvious and we omit the details. $\square$

Before we prove the second result, we need the following lemma providing an identity that is interesting in itself as well.

Lemma 1. *Let* $\mathbf{p} = (p_1, ..., p_n)$ *and* $A = (A_1, ..., A_n)$ *be as in Theorem 3. Then we have the equality*

$$\det \begin{pmatrix} P_i & P_n \\ S_i(\mathbf{p}) & S_n(\mathbf{p}) \end{pmatrix} = \sum_{j=1}^{n-1} P_{\min\{i,j\}} \bar{P}_{\max\{i,j\}} \Delta A_j, \tag{2.3}$$

for each $i \in \{1, ..., n-1\}$.

Proof. Define, for $i \in \{1, ..., n-1\}$,

$$K(i) := \sum_{j=1}^{n-1} P_{\min\{i,j\}} \bar{P}_{\max\{i,j\}} \Delta A_j.$$

We have

$$\begin{aligned}
K(i) &= \sum_{j=1}^{i} P_{\min\{i,j\}} \bar{P}_{\max\{i,j\}} \Delta A_j + \sum_{j=i+1}^{n-1} P_{\min\{i,j\}} \bar{P}_{\max\{i,j\}} \Delta A_j \\
&= \sum_{j=1}^{i} P_j \bar{P}_i \Delta A_j + \sum_{j=i+1}^{n-1} P_i \bar{P}_j \Delta A_j \\
&= \bar{P}_i \sum_{j=1}^{i} P_j \Delta A_j + P_i \sum_{j=i+1}^{n-1} \bar{P}_j \Delta A_j.
\end{aligned} \tag{2.4}$$

Using the summation by parts formula, we have

$$\sum_{j=1}^{i} P_j \Delta A_j = P_j A_j\big|_1^{i+1} - \sum_{j=1}^{i} (P_{j+1} - P_j) A_{j+1} \tag{2.5}$$
$$= P_{i+1}A_{i+1} - p_1 A_1 - \sum_{j=1}^{i} p_{j+1} A_{j+1}$$
$$= P_{i+1}A_{i+1} - \sum_{j=1}^{i+1} p_j A_j$$

and

$$\sum_{j=i+1}^{n-1} \bar{P}_j \Delta A_j = \bar{P}_j A_j\big|_{i+1}^{n} - \sum_{j=i+1}^{n-1} \left(\bar{P}_{j+1} - \bar{P}_j\right) A_{j+1} \tag{2.6}$$
$$= \bar{P}_n A_n - \bar{P}_{i+1} A_{i+1} - \sum_{j=i+1}^{n-1} (P_n - P_{j+1} - P_n + P_j) A_{j+1}$$
$$= -\bar{P}_{i+1} A_{i+1} + \sum_{j=i+1}^{n-1} p_{j+1} A_{j+1}.$$

Using (2.5) and (2.6) we have

$$K(i) = \bar{P}_i \left(P_{i+1}A_{i+1} - \sum_{j=1}^{i+1} p_j A_j \right) + P_i \left(\sum_{j=i+1}^{n-1} p_{j+1} A_{j+1} - \bar{P}_{i+1} A_{i+1} \right)$$
$$= \bar{P}_i P_{i+1} A_{i+1} - P_i \bar{P}_{i+1} A_{i+1} - \bar{P}_i \sum_{j=1}^{i+1} p_j A_j + P_i \sum_{j=i+1}^{n-1} p_{j+1} A_{j+1}$$
$$= [(P_n - P_i) P_{i+1} - P_i (P_n - P_{i+1})] A_{i+1}$$
$$+ P_i \sum_{j=i+1}^{n-1} p_{j+1} A_{j+1} - \bar{P}_i \sum_{j=1}^{i+1} p_j A_j$$

$$= P_n p_{i+1} A_{i+1} + P_i \sum_{j=i+1}^{n-1} p_{j+1} A_{j+1} - \bar{P}_i \sum_{j=1}^{i+1} p_j A_j$$
$$= \left(P_i + \bar{P}_i\right) p_{i+1} A_{i+1} + P_i \sum_{j=i+1}^{n-1} p_{j+1} A_{j+1} - \bar{P}_i \sum_{j=1}^{i+1} p_j A_j$$
$$= P_i \sum_{j=i+1}^{n-1} p_j A_j - \bar{P}_i \sum_{j=1}^{i} p_j A_j = P_i \bar{S}_i(\mathbf{p}) - \bar{P}_i S_i(\mathbf{p})$$
$$= \det \begin{pmatrix} P_i & P_n \\ S_i(\mathbf{p}) & S_n(\mathbf{p}) \end{pmatrix};$$

and the identity is proved. □

We are able now to state and prove the second identity for the Čebyšev functional

Theorem 4. *With the assumptions of Theorem 3, we have the equality*

$$T_n(\mathbf{p};\mathbf{A},\mathbf{B}) = \sum_{i=1}^{n-1}\sum_{j=1}^{n-1} P_{\min\{i,j\}}\bar{P}_{\max\{i,j\}}\Delta A_j \Delta B_i. \tag{2.7}$$

The proof is obvious by Theorem 3 and Lemma 1.

Remark 1. *The identity* (2.7)*, for n-tuples of real numbers, was stated without a proof in paper* [16]*. It also may be found for the same sequences in* [13, p. 281]*, again without a proof. In the second place mentioned above there is a misprint for the index of* $\bar{P}$ *which, instead of* $\max\{i,j\}+1$*, should be* $\max\{i,j\}$*.*

3. Some 1-Schatten Norm Inequalities

The following result holds:

Theorem 5. *For* $p,\ q>1$ *with* $\frac{1}{p}+\frac{1}{q}=1$ *and* $\mathbf{p}=(p_1,...,p_n)\in\mathbb{R}^n$, $A=(A_1,...,A_n)\in(\mathcal{B}_p(H))^n$ *and* $B=(B_1,...,B_n)\in(\mathcal{B}_q(H))^n$*, we have the inequalities*

$$\|T_n(\mathbf{p};\mathbf{A},\mathbf{B})\|_1 \le \begin{cases} \max_{1\le i\le n-1}\left\|\det\begin{pmatrix} P_i & P_n \\ S_i(\mathbf{p}) & S_n(\mathbf{p})\end{pmatrix}\right\|_p \sum_{j=1}^{n-1}\|\Delta B_j\|_q; \\ \left(\sum_{i=1}^{n-1}\left\|\det\begin{pmatrix} P_i & P_n \\ S_i(\mathbf{p}) & S_n(\mathbf{p})\end{pmatrix}\right\|_p^\beta\right)^{1/\beta}\left(\sum_{j=1}^{n-1}\|\Delta B_j\|_q^\alpha\right)^{1/\alpha} \\ \text{for } \alpha>1,\ \frac{1}{\alpha}+\frac{1}{\beta}=1; \\ \sum_{i=1}^{n-1}\left\|\det\begin{pmatrix} P_i & P_n \\ S_i(\mathbf{p}) & S_n(\mathbf{p})\end{pmatrix}\right\|_p \max_{1\le j\le n-1}\|\Delta B_j\|_q. \end{cases} \tag{3.1}$$

In particular, if $A,\ B\in(\mathcal{B}_2(H))^n$*, then*

$$\|T_n(\mathbf{p};\mathbf{A},\mathbf{B})\|_1 \le \begin{cases} \max_{1\le i\le n-1}\left\|\det\begin{pmatrix} P_i & P_n \\ S_i(\mathbf{p}) & S_n(\mathbf{p})\end{pmatrix}\right\|_2 \sum_{j=1}^{n-1}\|\Delta B_j\|_2; \\ \left(\sum_{i=1}^{n-1}\left\|\det\begin{pmatrix} P_i & P_n \\ S_i(\mathbf{p}) & S_n(\mathbf{p})\end{pmatrix}\right\|_2^\beta\right)^{1/\beta}\left(\sum_{j=1}^{n-1}\|\Delta B_j\|_2^\alpha\right)^{1/\alpha} \\ \text{for } \alpha>1,\ \frac{1}{\alpha}+\frac{1}{\beta}=1; \\ \sum_{i=1}^{n-1}\left\|\det\begin{pmatrix} P_i & P_n \\ S_i(\mathbf{p}) & S_n(\mathbf{p})\end{pmatrix}\right\|_2 \max_{1\le j\le n-1}\|\Delta B_j\|_2. \end{cases} \tag{3.2}$$

Proof. Using the first identity in (2.1), we have by the triangle inequality for 1-Schatten norm

$$\begin{aligned} \left\|T_n(\mathbf{p};\mathbf{A},\mathbf{B})\right\|_1 &= \left\|\sum_{i=1}^{n-1}\det\begin{pmatrix} P_i & P_n \\ S_i(\mathbf{p}) & S_n(\mathbf{p})\end{pmatrix}\Delta B_i\right\|_1 \\ &\le \sum_{i=1}^{n}\left\|\det\begin{pmatrix} P_i & P_n \\ S_i(\mathbf{p}) & S_n(\mathbf{p})\end{pmatrix}\Delta B_i\right\|_1 . \end{aligned}$$

By the property (1.14) we have

$$\left\|\det\begin{pmatrix} P_i & P_n \\ S_i(\mathbf{p}) & S_n(\mathbf{p})\end{pmatrix}\Delta B_i\right\|_1 \le \left\|\det\begin{pmatrix} P_i & P_n \\ S_i(\mathbf{p}) & S_n(\mathbf{p})\end{pmatrix}\right\|_p \left\|\Delta B_i\right\|_q$$

for all $i\in\{1,...,n-1\}$. This implies by summing that

$$\begin{aligned} &\sum_{i=1}^{n-1}\left\|\det\begin{pmatrix} P_i & P_n \\ S_i(\mathbf{p}) & S_n(\mathbf{p})\end{pmatrix}\Delta B_i\right\|_1 \\ &\le \sum_{i=1}^{n-1}\left\|\det\begin{pmatrix} P_i & P_n \\ S_i(\mathbf{p}) & S_n(\mathbf{p})\end{pmatrix}\right\|_p \left\|\Delta B_i\right\|_q . \end{aligned}$$

Using Hölder's inequality for $\alpha>1$, $\frac{1}{\alpha}+\frac{1}{\beta}=1$, we deduce the desired result (3.1). The details are omitted. □

Corollary 2. *With the assumptions of Theorem 5 for A and B, we have the inequalities*

$$\begin{aligned} &\left\|T_n(\mathbf{A},\mathbf{B})\right\|_1 \\ &\le \frac{1}{n^2}\times\begin{cases} \max_{1\le i\le n-1}\left\|\det\begin{pmatrix} i & n \\ \sum_{k=1}^{i}A_k & \sum_{k=1}^{n}A_k\end{pmatrix}\right\|_p \\ \times\sum_{j=1}^{n-1}\left\|\Delta B_j\right\|_q ; \\ \\ \left(\sum_{i=1}^{n-1}\left\|\det\begin{pmatrix} i & n \\ \sum_{k=1}^{i}A_k & \sum_{k=1}^{n}A_k\end{pmatrix}\right\|_p^\beta\right)^{1/\beta} \\ \times\left(\sum_{j=1}^{n-1}\left\|\Delta B_j\right\|_q^\alpha\right)^{1/\alpha}, \text{ for } \alpha>1,\ \frac{1}{\alpha}+\frac{1}{\beta}=1; \\ \\ \sum_{i=1}^{n-1}\left\|\det\begin{pmatrix} i & n \\ \sum_{k=1}^{i}A_k & \sum_{k=1}^{n}A_k\end{pmatrix}\right\|_p \\ \times\max_{1\le j\le n-1}\left\|\Delta B_j\right\|_q . \end{cases} \end{aligned} \tag{3.3}$$

The following result may be stated as well.

Theorem 6. *With the assumptions of Theorem 5 and if* $P_i \neq 0$ $(i = 1, ..., n)$, *then we have the inequalities*

$$\|T_n(\mathbf{p}; \mathbf{A}, \mathbf{B})\|_1 \tag{3.4}$$

$$\leq |P_n| \times \begin{cases} \max_{1\leq i\leq n-1} \left\| \frac{S_n(\mathbf{p})}{P_n} - \frac{S_i(\mathbf{p})}{Pi} \right\|_p \sum_{i=1}^{n-1} |P_i| \|\Delta B_i\|_q ; \\ \left(\sum_{i=1}^{n-1} |P_i| \left\| \frac{S_n(\mathbf{p})}{P_n} - \frac{S_i(\mathbf{p})}{Pi} \right\|_p^\beta \right)^{1/\beta} \left(\sum_{i=1}^{n-1} |P_i| \|\Delta B_i\|_q \right)^{1/p} \\ \text{for } p > 1, \frac{1}{p} + \frac{1}{\beta} = 1; \\ \sum_{i=1}^{n-1} |P_i| \left\| \frac{S_n(\mathbf{p})}{P_n} - \frac{S_i(\mathbf{p})}{Pi} \right\|_p \max_{1\leq i\leq n-1} \|\Delta B_i\|_q . \end{cases}$$

In particular, for $p = q = 2$

$$\|T_n(\mathbf{p}; \mathbf{A}, \mathbf{B})\|_1 \tag{3.5}$$

$$\leq |P_n| \times \begin{cases} \max_{1\leq i\leq n-1} \left\| \frac{S_n(\mathbf{p})}{P_n} - \frac{S_i(\mathbf{p})}{Pi} \right\|_2 \sum_{i=1}^{n-1} |P_i| \|\Delta B_i\|_2 ; \\ \left(\sum_{i=1}^{n-1} |P_i| \left\| \frac{S_n(\mathbf{p})}{P_n} - \frac{S_i(\mathbf{p})}{Pi} \right\|_2^\beta \right)^{1/\beta} \left(\sum_{i=1}^{n-1} |P_i| \|\Delta B_i\|_2 \right)^{1/p} \\ \text{for } \alpha > 1, \frac{1}{\alpha} + \frac{1}{\beta} = 1; \\ \sum_{i=1}^{n-1} |P_i| \left\| \frac{S_n(\mathbf{p})}{P_n} - \frac{S_i(\mathbf{p})}{Pi} \right\|_2 \max_{1\leq i\leq n-1} \|\Delta B_i\|_2 . \end{cases}$$

Proof. Follows by the second identity in (2.1) and taking into account that

$$\|T_n(\mathbf{p}; \mathbf{A}, \mathbf{B})\|_1 \leq |P_n| \sum_{i=1}^{n-1} |P_i| \left\| \frac{S_n(\mathbf{p})}{P_n} - \frac{S_i(\mathbf{p})}{P_i} \right\|_p \|\Delta B_i\|_q .$$

Using Hölder's weighted inequality, we easily deduce (3.4). □

The following corollary containing the unweighted inequalities holds.

Corollary 3. *With the above assumptions for* A *and* B *one has for* p, $q > 1$ *with* $\frac{1}{p} + \frac{1}{q} = 1$ *that*

$$\|T_n(\mathbf{A}, \mathbf{B})\|_1 \tag{3.6}$$

$$\leq \frac{1}{n} \times \begin{cases} \max_{1\leq i\leq n-1} \left\| \frac{1}{n} \sum_{k=1}^n A_k - \frac{1}{i} \sum_{k=1}^i A_k \right\|_p \sum_{i=1}^{n-1} i \|\Delta B_i\|_q ; \\ \left(\sum_{i=1}^{n-1} i \left\| \frac{1}{n} \sum_{k=1}^n A_k - \frac{1}{i} \sum_{k=1}^i A_k \right\|_p^\beta \right)^{1/\beta} \\ \times \left(\sum_{i=1}^{n-1} i \|\Delta B_i\|_q^\alpha \right)^{1/\alpha}, \text{ for } \alpha > 1, \frac{1}{\alpha} + \frac{1}{\beta} = 1; \\ \sum_{i=1}^{n-1} i \left\| \frac{1}{n} \sum_{k=1}^n A_k - \frac{1}{i} \sum_{k=1}^i A_k \right\|_p \max_{1\leq i\leq n-1} \|\Delta B_i\|_q . \end{cases}$$

Another type of inequalities may be stated if one uses the third identity in (2.1).

Theorem 7. *With the assumptions in Theorem 5 and if* $P_i, \overline{P}_i \neq 0, i \in \{1, ..., n-1\}$, *then we have the inequalities*

$$\|T_n(\mathbf{p};\mathbf{A},\mathbf{B})\|_1 \tag{3.7}$$
$$\leq \begin{cases} \max_{1\leq i\leq n-1}\left\|\frac{\overline{S}_i(\mathbf{p})}{\overline{P}_i}-\frac{S_i(\mathbf{p})}{Pi}\right\|_p \sum_{i=1}^{n-1}|P_i|\left|\overline{P}_i\right|\|\Delta B_i\|_q; \\ \left(\sum_{i=1}^{n-1}|P_i|\left|\overline{P}_i\right|\left\|\frac{\overline{S}_i(\mathbf{p})}{\overline{P}_i}-\frac{S_i(\mathbf{p})}{Pi}\right\|_p^\beta\right)^{1/\beta}\left(\sum_{i=1}^{n-1}|P_i|\left|\overline{P}_i\right|\|\Delta B_i\|_q^\alpha\right)^{1/\alpha} \\ \text{for } \alpha>1, \frac{1}{\alpha}+\frac{1}{\beta}=1; \\ \sum_{i=1}^{n-1}|P_i|\left|\overline{P}_i\right|\left\|\frac{\overline{S}_i(\mathbf{p})}{\overline{P}_i}-\frac{S_i(\mathbf{p})}{Pi}\right\|_p \max_{1\leq i\leq n-1}\|\Delta B_i\|_q. \end{cases}$$

In particular, if $p_i = \frac{1}{n}, i \in \{1, ..., n\}$, *then we have*

$$\|T_n(\mathbf{A},\mathbf{B})\|_1 \tag{3.8}$$
$$\leq \frac{1}{n^2}\begin{cases} \max_{1\leq i\leq n-1}\left\|\frac{1}{n-i}\sum_{k=i+1}^n A_k-\frac{1}{i}\sum_{k=1}^i A_k\right\|_p \\ \times\sum_{i=1}^{n-1} i(n-i)\|\Delta B_i\|_q; \\ \left(\sum_{i=1}^{n-1} i(n-i)\left\|\frac{1}{n-i}\sum_{k=i+1}^n A_k-\frac{1}{i}\sum_{k=1}^i A_k\right\|_p^\beta\right)^{1/\beta} \\ \times\left(\sum_{i=1}^{n-1} i(n-i)\|\Delta B_i\|_p^\alpha\right)^{1/\alpha} \text{ for } \alpha>1, \frac{1}{\alpha}+\frac{1}{\beta}=1; \\ \sum_{i=1}^{n-1} i(n-i)\left\|\frac{1}{n-i}\sum_{k=i+1}^n A_k-\frac{1}{i}\sum_{k=1}^i A_k\right\|_p \\ \times\max_{1\leq i\leq n-1}\|\Delta B_i\|_q. \end{cases}$$

A different approach may be considered if one uses the representation in terms of double sums for the Čebyšev functional provided by the Theorem 4.

The following result from a different perspective also holds.

Theorem 8. *With the assumptions in Theorem 5, we have the inequalities*

$$\|T_n(\mathbf{p};\mathbf{A},\mathbf{B})\|_1 \tag{3.9}$$
$$\leq \begin{cases} \max_{1\leq i,j\leq n-1}\left\{\left|P_{\min\{i,j\}}\right|\left|\bar{P}_{\max\{i,j\}}\right|\right\}\sum_{i=1}^{n-1}\|\Delta A_i\|_p\sum_{i=1}^{n-1}\|\Delta B_i\|_q; \\ \left(\sum_{i=1}^{n-1}\sum_{j=1}^{n-1}\left|P_{\min\{i,j\}}\right|^\beta\left|\bar{P}_{\max\{i,j\}}\right|^\beta\right)^{1/\beta} \\ \times\left(\sum_{i=1}^{n-1}\|\Delta A_i\|_p^\alpha\right)^{1/\alpha}\left(\sum_{i=1}^{n-1}\|\Delta B_i\|_q^\alpha\right)^{1/\alpha} \text{ for } \alpha>1, \frac{1}{\alpha}+\frac{1}{\beta}=1; \\ \sum_{i=1}^{n-1}\sum_{j=1}^{n-1}\left|P_{\min\{i,j\}}\right|\left|\bar{P}_{\max\{i,j\}}\right| \\ \times\max_{1\leq i\leq n-1}\|\Delta A_i\|_p\max_{1\leq i\leq n-1}\|\Delta B_i\|_q. \end{cases}$$

In particular, for $p = q = 2$ we get

$$\left\Vert T_n(\mathbf{p};\mathbf{A},\mathbf{B})\right\Vert_1 \leq \begin{cases} \max_{1\leq i,j\leq n-1}\left\{\left\vert P_{\min\{i,j\}}\right\vert \left\vert \bar{P}_{\max\{i,j\}}\right\vert\right\} \\ \times \sum_{i=1}^{n-1}\left\Vert \Delta A_i\right\Vert_2 \sum_{i=1}^{n-1}\left\Vert \Delta B_i\right\Vert_2 ; \\ \\ \left(\sum_{i=1}^{n-1}\sum_{j=1}^{n-1}\left\vert P_{\min\{i,j\}}\right\vert^{\beta}\left\vert \bar{P}_{\max\{i,j\}}\right\vert^{\beta}\right)^{1/\beta} \\ \times \left(\sum_{i=1}^{n-1}\left\Vert \Delta A_i\right\Vert_2^{\alpha}\right)^{1/\alpha}\left(\sum_{i=1}^{n-1}\left\Vert \Delta B_i\right\Vert_2^{\alpha}\right)^{1/\alpha} \\ \text{for } \alpha>1, \frac{1}{\alpha}+\frac{1}{\beta}=1; \\ \\ \sum_{i=1}^{n-1}\sum_{j=1}^{n-1}\left\vert P_{\min\{i,j\}}\right\vert \left\vert \bar{P}_{\max\{i,j\}}\right\vert \\ \times \max_{1\leq i\leq n-1}\left\Vert \Delta A_i\right\Vert_2 \max_{1\leq i\leq n-1}\left\Vert \Delta B_i\right\Vert_2 . \end{cases} \tag{3.10}$$

Proof. Taking the 1-Schatten norm in the identity (2.7) we get

$$\begin{aligned} \left\Vert T_n(\mathbf{p};\mathbf{A},\mathbf{B})\right\Vert_1 &= \left\Vert \sum_{i=1}^{n-1}\sum_{j=1}^{n-1} P_{\min\{i,j\}}\bar{P}_{\max\{i,j\}}\Delta A_j \Delta B_i\right\Vert_1 \\ &\leq \sum_{i=1}^{n-1}\sum_{j=1}^{n-1}\left\Vert P_{\min\{i,j\}}\bar{P}_{\max\{i,j\}}\Delta A_j \Delta B_i\right\Vert_1 \\ &= \sum_{i=1}^{n-1}\sum_{j=1}^{n-1}\left\vert P_{\min\{i,j\}}\right\vert \left\vert \bar{P}_{\max\{i,j\}}\right\vert \left\Vert \Delta A_j \Delta B_i\right\Vert_1 \\ &\leq \sum_{i=1}^{n-1}\sum_{j=1}^{n-1}\left\vert P_{\min\{i,j\}}\right\vert \left\vert \bar{P}_{\max\{i,j\}}\right\vert \left\Vert \Delta A_j\right\Vert_p \left\Vert \Delta B_i\right\Vert_q , \end{aligned} \tag{3.11}$$

where for the last inequality we used the property (1.14).

We have

$$\begin{aligned} &\sum_{i=1}^{n-1}\sum_{j=1}^{n-1}\left\vert P_{\min\{i,j\}}\right\vert \left\vert \bar{P}_{\max\{i,j\}}\right\vert \left\Vert \Delta A_j\right\Vert_p \left\Vert \Delta B_i\right\Vert_q \\ &\leq \max_{i,j\in\{1,\dots,n-1\}}\left\{\left\Vert \Delta A_j\right\Vert_p \left\Vert \Delta B_i\right\Vert_q\right\}\sum_{i=1}^{n-1}\sum_{j=1}^{n-1}\left\vert P_{\min\{i,j\}}\right\vert \left\vert \bar{P}_{\max\{i,j\}}\right\vert \\ &= \max_{1\leq i\leq n-1}\left\Vert \Delta A_i\right\Vert_p \max_{1\leq i\leq n-1}\left\Vert \Delta B_i\right\Vert_q \sum_{i=1}^{n-1}\sum_{j=1}^{n-1}\left\vert P_{\min\{i,j\}}\right\vert \left\vert \bar{P}_{\max\{i,j\}}\right\vert \end{aligned}$$

and

$$\begin{aligned} &\sum_{i=1}^{n-1}\sum_{j=1}^{n-1}\left\vert P_{\min\{i,j\}}\right\vert \left\vert \bar{P}_{\max\{i,j\}}\right\vert \left\Vert \Delta A_j\right\Vert_p \left\Vert \Delta B_i\right\Vert_q \\ &\leq \max_{1\leq i,j\leq n-1}\left\{\left\vert P_{\min\{i,j\}}\right\vert \left\vert \bar{P}_{\max\{i,j\}}\right\vert\right\}\sum_{i=1}^{n-1}\sum_{j=1}^{n-1}\left\Vert \Delta A_j\right\Vert_p \left\Vert \Delta B_i\right\Vert_q \\ &= \max_{1\leq i,j\leq n-1}\left\{\left\vert P_{\min\{i,j\}}\right\vert \left\vert \bar{P}_{\max\{i,j\}}\right\vert\right\}\sum_{i=1}^{n-1}\left\Vert \Delta A_i\right\Vert_p \sum_{i=1}^{n-1}\left\Vert \Delta B_i\right\Vert_q . \end{aligned}$$

If we use Hölder's inequality for $\alpha > 1$, $\frac{1}{\alpha} + \frac{1}{\beta} = 1$ we get

$$\begin{aligned}
&\sum_{i=1}^{n-1}\sum_{j=1}^{n-1} \left|P_{\min\{i,j\}}\right| \left|\bar{P}_{\max\{i,j\}}\right| \left\|\Delta A_j\right\|_p \left\|\Delta B_i\right\|_q \\
&\leq \left(\sum_{i=1}^{n-1}\sum_{j=1}^{n-1} \left|P_{\min\{i,j\}}\right|^\beta \left|\bar{P}_{\max\{i,j\}}\right|^\beta\right)^{1/\beta} \left(\sum_{i=1}^{n-1}\sum_{j=1}^{n-1} \left\|\Delta A_j\right\|_p^\alpha \left\|\Delta B_i\right\|_q^\alpha\right)^{1/\alpha} \\
&= \left(\sum_{i=1}^{n-1}\sum_{j=1}^{n-1} \left|P_{\min\{i,j\}}\right|^\beta \left|\bar{P}_{\max\{i,j\}}\right|^\beta\right)^{1/\beta} \\
&\times \left(\sum_{i=1}^{n-1} \left\|\Delta A_i\right\|_p^\alpha\right)^{1/\alpha} \left(\sum_{i=1}^{n-1} \left\|\Delta B_i\right\|_q^\alpha\right)^{1/\alpha}
\end{aligned}$$

and the theorem is thus proved. □

Now, define

$$k_\infty := \max_{1\leq i,j\leq n-1} \left\{ \frac{\min\{i,j\}}{n} \left(1 - \frac{\max\{i,j\}}{n}\right)\right\},\ n \geq 2.$$

Using the elementary inequality

$$ab \leq \frac{1}{4}(a+b)^2, \quad a,b \in \mathbb{R};$$

we deduce

$$\begin{aligned}
\min\{i,j\}(n - \max\{i,j\}) &\leq \frac{1}{4}(n + \min\{i,j\} - \max\{i,j\})^2 \\
&= \frac{1}{4}(n - |i-j|)^2, 1 \leq i,j \leq n-1.
\end{aligned}$$

Consequently, we observe that

$$k_\infty \leq \frac{1}{4n^2} \max_{1\leq i,j\leq n-1} \left\{(n - |i-j|)^2\right\} = \frac{1}{4}.$$

We may state now the following corollary of Theorem 8.

Corollary 4. *For $p,\ q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ and $\mathbf{p} = (p_1, ..., p_n) \in \mathbb{R}^n$, $A = (A_1, ..., A_n) \in (\mathcal{B}_p(H))^n$ and $B = (B_1, ..., B_n) \in (\mathcal{B}_q(H))^n$. Then we have the inequality*

$$\left\|T_n(\mathbf{A},\mathbf{B})\right\|_1 \leq k_\infty \sum_{i=1}^{n-1} \left\|\Delta A_i\right\|_p \sum_{i=1}^{n-1} \left\|\Delta B_i\right\|_q \leq \frac{1}{4} \sum_{i=1}^{n-1} \left\|\Delta A_i\right\|_p \sum_{i=1}^{n-1} \left\|\Delta B_i\right\|_q. \tag{3.12}$$

In particular, if $A,\ B \in (\mathcal{B}_2(H))^n$, then

$$\left\|T_n(\mathbf{A},\mathbf{B})\right\|_1 \leq k_\infty \sum_{i=1}^{n-1} \left\|\Delta A_i\right\|_2 \sum_{i=1}^{n-1} \left\|\Delta B_i\right\|_2 \leq \frac{1}{4} \sum_{i=1}^{n-1} \left\|\Delta A_i\right\|_2 \sum_{i=1}^{n-1} \left\|\Delta B_i\right\|_2. \tag{3.13}$$

Consider now, for $\beta > 1$, the number

$$k_\beta := \frac{1}{n^2} \left(\sum_{i=1}^{n-1}\sum_{j=1}^{n-1} \left[\min\{i,j\}(n - \max\{i,j\})\right]^\beta\right)^{1/\beta}.$$

We observe, by the symmetry of the terms under the sums symbol, we have that

$$k_{\beta}=\frac{1}{n^{2}}\left(2\sum_{1\leq i<j\leq n-1}i^{\beta}\left(n-j\right)^{\beta}+\sum_{i=1}^{n-1}i^{\beta}\left(n-i\right)^{\beta}\right)^{1/\beta},$$

that may be computed exactly if $\beta=2$ or another natural number.

Since, as above,

$$\left[\min\left\{i,j\right\}\left(n-\max\left\{i,j\right\}\right)\right]^{\beta}\leq\frac{1}{4^{\beta}}\left(n-\left|i-j\right|\right)^{2\beta}$$

we deduce

$$\begin{aligned}k_{\beta}&\leq\frac{1}{4n^{2}}\left(\sum_{i=1}^{n-1}\sum_{j=1}^{n-1}\left(n-\left|i-j\right|\right)^{2\beta}\right)^{1/\beta}\\&\leq\frac{1}{4n^{2}}\left[\left(n-1\right)^{2}n^{2\beta}\right]^{1/\beta}=\frac{1}{4}\left(n-1\right)^{2/\beta}.\end{aligned}$$

Consequently, we may state the following corollary as well.

Corollary 5. *With the assumption in Corollary 4, we have the inequalities*

$$\begin{aligned}\left\|T_{n}\left(\mathbf{A},\mathbf{B}\right)\right\|_{1}&\leq k_{\beta}\left(\sum_{i=1}^{n-1}\left\|\Delta A_{i}\right\|_{p}^{\alpha}\right)^{1/\alpha}\left(\sum_{i=1}^{n-1}\left\|\Delta B_{i}\right\|_{q}^{\alpha}\right)^{1/\alpha}\\&\leq\frac{1}{4}\left(n-1\right)^{2/\beta}\left(\sum_{i=1}^{n-1}\left\|\Delta A_{i}\right\|_{p}^{\alpha}\right)^{1/\alpha}\left(\sum_{i=1}^{n-1}\left\|\Delta B_{i}\right\|_{q}^{\alpha}\right)^{1/\alpha};\end{aligned}$$

provided $\alpha>1,\ \frac{1}{\alpha}+\frac{1}{\beta}=1$.

Finally, if we denote

$$k_{1}:=\frac{1}{n^{2}}\sum_{i=1}^{n-1}\sum_{j=1}^{n-1}\left[\min\left\{i,j\right\}\left(n-\max\left\{i,j\right\}\right)\right],$$

then we observe, for $\mathbf{u}=\left(\frac{1}{n},...,\frac{1}{n}\right),\mathbf{e}=\left(1,2,...,n\right)$, that

$$k_{1}=T_{n}\left(\mathbf{u};\mathbf{e},\mathbf{e}\right)=\frac{1}{n}\sum_{i=1}^{n}i^{2}-\left(\frac{1}{n}\sum_{i=1}^{n}i\right)^{2}=\frac{1}{12}\left(n^{2}-1\right),$$

and by Theorem 8, we deduce:

Corollary 6. *With the assumption in Corollary 4, we have the inequalities*

$$\left\|T_{n}\left(\mathbf{A},\mathbf{B}\right)\right\|_{1}\leq\frac{1}{12}\left(n^{2}-1\right)\max_{1\leq j\leq n-1}\left\|\Delta A_{j}\right\|_{p}\max_{1\leq j\leq n-1}\left\|\Delta B_{j}\right\|_{q}.$$

4. p-Schatten Norm Inequalities

From a different view point we also have:

Theorem 9. *For* $p \geq 1$ *and* $\mathbf{p} = (p_1, ..., p_n) \in \mathbb{R}^n$, $A = (A_1, ..., A_n)$, $\mathbf{B} = (B_1, ..., B_n) \in (\mathcal{B}_p(H))^n$ *we have the inequalities*

$$\|T_n(\mathbf{p}; \mathbf{A}, \mathbf{B})\|_p \tag{4.1}$$
$$\leq \begin{cases} \max_{1\leq i\leq n-1} \left\| \det \begin{pmatrix} P_i & P_n \\ S_i(\mathbf{p}) & S_n(\mathbf{p}) \end{pmatrix} \right\|_p \sum_{j=1}^{n-1} \|\Delta B_j\|_p ; \\ \left(\sum_{i=1}^{n-1} \left\| \det \begin{pmatrix} P_i & P_n \\ S_i(\mathbf{p}) & S_n(\mathbf{p}) \end{pmatrix} \right\|_p^\beta \right)^{1/\beta} \left(\sum_{j=1}^{n-1} \|\Delta B_j\|_p^\alpha \right)^{1/\alpha} \\ \text{for } \alpha > 1, \frac{1}{\alpha} + \frac{1}{\beta} = 1; \\ \sum_{i=1}^{n-1} \left\| \det \begin{pmatrix} P_i & P_n \\ S_i(\mathbf{p}) & S_n(\mathbf{p}) \end{pmatrix} \right\|_p \max_{1\leq j\leq n-1} \|\Delta B_j\|_p . \end{cases}$$

Proof. Using the first identity in (2.1), we have by the triangle inequality for p-Schatten norm

$$\begin{aligned} \|T_n(\mathbf{p}; \mathbf{A}, \mathbf{B})\|_p &= \left\| \sum_{i=1}^{n-1} \det \begin{pmatrix} P_i & P_n \\ S_i(\mathbf{p}) & S_n(\mathbf{p}) \end{pmatrix} \Delta B_i \right\|_p \\ &\leq \sum_{i=1}^{n} \left\| \det \begin{pmatrix} P_i & P_n \\ S_i(\mathbf{p}) & S_n(\mathbf{p}) \end{pmatrix} \Delta B_i \right\|_p . \end{aligned}$$

By the property (1.11) we have

$$\left\| \det \begin{pmatrix} P_i & P_n \\ S_i(\mathbf{p}) & S_n(\mathbf{p}) \end{pmatrix} \Delta B_i \right\|_p \leq \left\| \det \begin{pmatrix} P_i & P_n \\ S_i(\mathbf{p}) & S_n(\mathbf{p}) \end{pmatrix} \right\|_p \|\Delta B_i\|_p$$

for all $i \in \{1, ..., n-1\}$. This implies by summing that

$$\sum_{i=1}^{n-1} \left\| \det \begin{pmatrix} P_i & P_n \\ S_i(\mathbf{p}) & S_n(\mathbf{p}) \end{pmatrix} \Delta B_i \right\|_p \leq \sum_{i=1}^{n-1} \left\| \det \begin{pmatrix} P_i & P_n \\ S_i(\mathbf{p}) & S_n(\mathbf{p}) \end{pmatrix} \right\|_p \|\Delta B_i\|_p .$$

Using Hölder's inequality for $\alpha > 1$, $\frac{1}{\alpha} + \frac{1}{\beta} = 1$, we deduce the desired result (4.1). The details are omitted. □

We also have

Theorem 10. *With the assumptions in Theorem 9, we have the inequalities*

$$\|T_n(\mathbf{p}; \mathbf{A}, \mathbf{B})\|_p \tag{4.2}$$
$$\leq \begin{cases} \max_{1\leq i,j\leq n-1} \left\{ \left|P_{\min\{i,j\}}\right| \left|\bar{P}_{\max\{i,j\}}\right| \right\} \sum_{i=1}^{n-1} \|\Delta A_i\|_p \sum_{i=1}^{n-1} \|\Delta B_i\|_p ; \\ \left(\sum_{i=1}^{n-1} \sum_{j=1}^{n-1} \left|P_{\min\{i,j\}}\right|^\beta \left|\bar{P}_{\max\{i,j\}}\right|^\beta \right)^{1/\beta} \\ \times \left(\sum_{i=1}^{n-1} \|\Delta A_i\|_p^\alpha \right)^{1/\alpha} \left(\sum_{i=1}^{n-1} \|\Delta B_i\|_p^\alpha \right)^{1/\alpha} \text{ for } \alpha > 1, \frac{1}{\alpha} + \frac{1}{\beta} = 1; \\ \sum_{i=1}^{n-1} \sum_{j=1}^{n-1} \left|P_{\min\{i,j\}}\right| \left|\bar{P}_{\max\{i,j\}}\right| \\ \times \max_{1\leq i\leq n-1} \|\Delta A_i\|_p \max_{1\leq i\leq n-1} \|\Delta B_i\|_p . \end{cases}$$

Proof. Taking the p-Schatten norm in the identity (2.7) we get

$$\begin{aligned}\left\|T_n(\mathbf{p};\mathbf{A},\mathbf{B})\right\|_p &= \left\|\sum_{i=1}^{n-1}\sum_{j=1}^{n-1}P_{\min\{i,j\}}\bar{P}_{\max\{i,j\}}\Delta A_j\Delta B_i\right\|_p \\ &\le \sum_{i=1}^{n-1}\sum_{j=1}^{n-1}\left\|P_{\min\{i,j\}}\bar{P}_{\max\{i,j\}}\Delta A_j\Delta B_i\right\|_p \\ &= \sum_{i=1}^{n-1}\sum_{j=1}^{n-1}\left|P_{\min\{i,j\}}\right|\left|\bar{P}_{\max\{i,j\}}\right|\left\|\Delta A_j\Delta B_i\right\|_p \\ &\le \sum_{i=1}^{n-1}\sum_{j=1}^{n-1}\left|P_{\min\{i,j\}}\right|\left|\bar{P}_{\max\{i,j\}}\right|\left\|\Delta A_j\right\|_p\left\|\Delta B_i\right\|_p,\end{aligned}$$

where for the last inequality we used the property (1.11).

By utilising a similar argument to the one in the proof of Theorem 8 and we omit the details. □

Finally, we can state:

Corollary 7. *With the assumptions of Theorem 9, we have the inequalities*

$$\left\|T_n(\mathbf{A},\mathbf{B})\right\|_p \le \frac{1}{4}\times\begin{cases}\sum_{i=1}^{n-1}\left\|\Delta A_i\right\|_p\sum_{i=1}^{n-1}\left\|\Delta B_i\right\|_p, \\ \\ (n-1)^{2/\beta}\left(\sum_{i=1}^{n-1}\left\|\Delta A_i\right\|_p^{\alpha}\right)^{1/\alpha}\left(\sum_{i=1}^{n-1}\left\|\Delta B_i\right\|_p^{\alpha}\right)^{1/\alpha} \\ \text{for } \alpha>1, \frac{1}{\alpha}+\frac{1}{\beta}=1, \\ \\ \frac{1}{3}\left(n^2-1\right)\max_{1\le j\le n-1}\left\|\Delta A_j\right\|_p\max_{1\le j\le n-1}\left\|\Delta B_j\right\|_p.\end{cases} \tag{4.3}$$

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