p-SCHATTEN NORM INEQUALITIES FOR ČEBYŠEV'S FUNCTIONAL VIA A BIERNACKI, PIDEK AND RYLL-NARDZEWSKI RESULT

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Abstract. An operator $A \in \mathcal{B}(H)$ is said to belong to the *von Neumann-Schatten class* $\mathcal{B}_p(H)$, $1 \leq p < \infty$ if the *p-Schatten norm* is finite

$$\|A\|_p := [\operatorname{tr}(|A|^p)]^{1/p} < \infty.$$

For $\mathbf{A} = (A_1, ..., A_n)$, $\mathbf{B} = (B_1, ..., B_n) \in B(H)$ we define the functional

$$D_n(\mathbf{A}, \mathbf{B}) := \frac{1}{n}\sum_{i=1}^{n} A_i B_i - \frac{1}{n}\sum_{i=1}^{n} A_i \frac{1}{n}\sum_{i=1}^{n} \alpha_i B_i.$$

In this paper we show among others that, if $\mathbf{A} = (A_1, ..., A_n) \in (B_p(H))^n$ and $\mathbf{B} = (B_1, ..., B_n) \in (B_q(H))^n$ for $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, then

$$\|D_n(\mathbf{A}, \mathbf{B})\|_1 \leq D_n(\mathbf{V}_{\mathbf{A},p}, \mathbf{V}_{\mathbf{B},q})$$
$$\leq \frac{1}{n}\left[\frac{n}{2}\right]\left(1 - \frac{1}{n}\left[\frac{n}{2}\right]\right)\sum_{k=1}^{n-1}\|\Delta A_k\|_p \sum_{k=1}^{n-1}\|\Delta B_k\|_q$$
$$\leq \frac{1}{4}\sum_{k=1}^{n-1}\|\Delta A_k\|_p \sum_{k=1}^{n-1}\|\Delta B_k\|_q,$$

where $\mathbf{V}_{\mathbf{A},p} := \left(0, \|\Delta A_1\|_p, ..., \sum_{k=1}^{n-1}\|\Delta A_k\|_p\right)$, $\mathbf{V}_{\mathbf{B},q} := \left(0, \|\Delta B_1\|_q, ..., \sum_{k=1}^{n-1}\|\Delta B_k\|_q\right)$ and $\Delta B_i := B_{i+1} - B_i$ $(i = 1, ..., n-1)$ are the usual forward differences.

1. Introduction

In 1935, G. Grüss [15] proved the following integral inequality which gives an approximation of the integral mean of the product in terms of the product of the integral means integrals as follows:

$$\left|\frac{1}{b-a}\int_a^b f(x)g(x)\,dx - \frac{1}{b-a}\int_a^b f(x)\,dx \cdot \frac{1}{b-a}\int_a^b g(x)\,dx\right| \leq \frac{1}{4}(\Phi - \varphi)(\Gamma - \gamma) \tag{1.1}$$

where $f, g : [a, b] \to \mathbb{R}$ are integrable on $[a, b]$ and satisfying the assumption

$$\varphi \leq f(x) \leq \Phi, \ \gamma \leq g(x) \leq \Gamma$$

for each $x \in [a, b]$ where $\varphi, \Phi, \gamma, \Gamma$ are given real constants.

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Moreover the constant $\frac{1}{4}$ is sharp in the sense that it can not be replaced by a smaller one.

For a simple proof of (1.1) as well as for some other integral inequalities of Grüss' type see the Chapter X of the recent book [16] by Mitrinović, Pečarić and Fink.

Let $\mathbf{a} = (a_1, ..., a_n)$, $\mathbf{b} = (b_1, ..., b_n)$ be two n-tuples of real numbers. We consider the Čebyšev's functional

$$D_n(\mathbf{a}, \mathbf{b}) := \frac{1}{n}\sum_{i=1}^n a_i b_i - \frac{1}{n}\sum_{i=1}^n a_i \frac{1}{n}\sum_{i=1}^n b_i.$$

In 1950, M. Biernacki, H. Pidek and C. Ryll-Nardzewski [3] established the following discrete version of Grüss' inequality, see also [16, Ch. X]:

Theorem 1. *Let $\mathbf{a} = (a_1, ..., a_n)$, $\mathbf{b} = (b_1, ..., b_n)$ be two n-tuples of real numbers such that $r \le a_i \le R$ and $s \le b_i \le S$ for $i = 1, ..., n$. Then one has the inequality:*

$$\begin{aligned} |D_n(\mathbf{a}, \mathbf{b})| &\le \frac{1}{n}\left[\frac{n}{2}\right]\left(1 - \frac{1}{n}\left[\frac{n}{2}\right]\right)(R - r)(S - s) \\ &\le \frac{1}{4}(R - r)(S - s), \end{aligned} \tag{1.2}$$

when $[x]$ is the integer part of x, $x \in \mathbb{R}$.

In 1981, A. Lupaş [16, Ch. X] proved some similar results for the first difference of $\mathbf{a}$ as follows :

Theorem 2. *Let $\mathbf{a}$, $\mathbf{b}$ be two monotonic n-tuples in the same sense and $\mathbf{p}$ a positive n-tuple. Then*

$$\begin{aligned} 0 &\le \min_{1\le i\le n-1} |a_{i+1} - a_i| \min_{1\le i\le n-1} |b_{i+1} - b_i| \left[\frac{1}{P_n}\sum_{i=1}^n i^2 p_i - \left(\frac{1}{P_n}\sum_{i=1}^n i p_i\right)^2\right] \\ &\le \frac{1}{P_n}\sum_{i=1}^n p_i a_i b_i - \frac{1}{P_n}\sum_{i=1}^n p_i a_i \cdot \frac{1}{P_n}\sum_{i=1}^n p_i b_i \\ &\le \max_{1\le i\le n-1} |a_{i+1} - a_i| \max_{1\le i\le n-1} |b_{i+1} - b_i| \left[\frac{1}{P_n}\sum_{i=1}^n i^2 p_i - \left(\frac{1}{P_n}\sum_{i=1}^n i p_i\right)^2\right]. \end{aligned} \tag{1.3}$$

If there exists the numbers $\bar{a}$, $\bar{a}_1$, r, r_1, $(rr_1 > 0)$ such that $a_k = \bar{a} + kr$ and $b_k = \bar{a}_1 + kr_1$, then in (1.3) the equality holds.

In particular, if $p_i = \frac{1}{n}$, then by (1.3) we derive

$$\begin{aligned} 0 &\le \frac{1}{12}\left(n^2 - 1\right) \min_{1\le i\le n-1} |a_{i+1} - a_i| \min_{1\le i\le n-1} |b_{i+1} - b_i| \\ &\le D_n(\mathbf{a}, \mathbf{b}) \\ &\le \frac{1}{12}\left(n^2 - 1\right) \max_{1\le i\le n-1} |a_{i+1} - a_i| \max_{1\le i\le n-1} |b_{i+1} - b_i|, \end{aligned} \tag{1.4}$$

for $\mathbf{a}$, $\mathbf{b}$ two monotonic n-tuples in the same sense.

In order to extend these results for p-Schatten norms we need the following preparations.

Let $(H;\langle .,.\rangle)$ be a complex Hilbert space and $\mathcal{B}(H)$ the Banach algebra of all bounded linear operators on H. If $\{e_i\}_{i\in I}$ an orthonormal basis of H, we say that $A\in\mathcal{B}(H)$ is of *trace class* if

$$\|A\|_1 := \sum_{i\in I}\langle |A| e_i, e_i\rangle < \infty. \tag{1.5}$$

The definition of $\|A\|_1$ does not depend on the choice of the orthonormal basis $\{e_i\}_{i\in I}$. We denote by $\mathcal{B}_1(H)$ the set of trace class operators in $\mathcal{B}(H)$.

We define the *trace* of a trace class operator $A\in\mathcal{B}_1(H)$ to be

$$\operatorname{tr}(A) := \sum_{i\in I}\langle Ae_i, e_i\rangle, \tag{1.6}$$

where $\{e_i\}_{i\in I}$ an orthonormal basis of H. Note that this coincides with the usual definition of the trace if H is finite-dimensional. We observe that the series (1.6) converges absolutely and it is independent from the choice of basis.

The following result collects some properties of the trace:

Theorem 3. *We have:*

(i) If $A\in\mathcal{B}_1(H)$ then $A^*\in\mathcal{B}_1(H)$ and

$$\operatorname{tr}(A^*) = \overline{\operatorname{tr}(A)}; \tag{1.7}$$

(ii) If $A\in\mathcal{B}_1(H)$ and $T\in\mathcal{B}(H)$, then AT, $TA\in\mathcal{B}_1(H)$ and

$$\operatorname{tr}(AT) = \operatorname{tr}(TA) \text{ and } |\operatorname{tr}(AT)| \le \|A\|_1\|T\|; \tag{1.8}$$

(iii) $\operatorname{tr}(\cdot)$ is a bounded linear functional on $\mathcal{B}_1(H)$ with $\|\operatorname{tr}\| = 1$;

(iv) If A, $B\in\mathcal{B}_2(H)$ then AB, $BA\in\mathcal{B}_1(H)$ and $\operatorname{tr}(AB) = \operatorname{tr}(BA)$;

(v) $\mathcal{B}_{fin}(H)$, the space of operators of finite rank, is a dense subspace of $\mathcal{B}_1(H)$.

An operator $A\in\mathcal{B}(H)$ is said to belong to the *von Neumann-Schatten class* $\mathcal{B}_p(H)$, $1\le p<\infty$ if the *p-Schatten norm* is finite [20, p. 60-64]

$$\|A\|_p := [\operatorname{tr}(|A|^p)]^{1/p} < \infty.$$

For $1<p<q<\infty$ we have that

$$\mathcal{B}_1(H)\subset\mathcal{B}_p(H)\subset\mathcal{B}_q(H)\subset\mathcal{B}(H) \tag{1.9}$$

and

$$\|A\|_1 \ge \|A\|_p \ge \|A\|_q \ge \|A\|. \tag{1.10}$$

For $p\ge 1$ the functional $\|\cdot\|_p$ is a *norm* on the $*$-ideal $\mathcal{B}_p(H)$ and $\left(\mathcal{B}_p(H), \|\cdot\|_p\right)$ is a Banach space.

Also, see for instance [20, p. 60-64],

$$\|A\|_p = \|A^*\|_p,\ A\in\mathcal{B}_p(H) \tag{1.11}$$

$$\|AB\|_p \le \|A\|_p\|B\|_p,\ A,B\in\mathcal{B}_p(H) \tag{1.12}$$

and

$$\|AB\|_p \le \|A\|_p\|B\|,\ \|BA\|_p \le \|B\|\|A\|_p,\ A\in\mathcal{B}_p(H),\ B\in\mathcal{B}(H). \tag{1.13}$$

This implies that

$$\|CAB\|_p \le \|C\|\|A\|_p\|B\|,\ A\in\mathcal{B}_p(H),\ B,\ C\in\mathcal{B}(H). \tag{1.14}$$

In terms of *p-Schatten norm* we have the *Hölder inequality* for $p,\ q>1$ with $\frac{1}{p}+\frac{1}{q}=1$

$$(|\operatorname{tr}(AB)|\le)\ \|AB\|_1\le\|A\|_p\|B\|_q\,,\ A\in\mathcal{B}_p(H)\,,\ B\in\mathcal{B}_q(H)\,. \tag{1.15}$$

For the theory of trace functionals and their applications the reader is referred to [19] and [20].

For some classical trace inequalities see [7], [8], and [18], which are continuations of the work of Bellman [1].

2. 1-Schatten Norm Inequalities

For $\mathbf{A}=(A_1,...,A_n)\,,\ \mathbf{B}=(B_1,...,B_n)\in B(H)$ we define the functional

$$D_n(\mathbf{A},\mathbf{B}):=\frac{1}{n}\sum_{i=1}^{n}A_iB_i-\frac{1}{n}\sum_{i=1}^{n}A_i\frac{1}{n}\sum_{i=1}^{n}\alpha_iB_i.$$

We can state the following result for the 1-Schatten norm:

Theorem 4. *If* $\mathbf{A}=(A_1,...,A_n)\in(B_p(H))^n$ *and* $\mathbf{B}=(B_1,...,B_n)\in(B_q(H))^n$ *for* $p,\ q>1$ *with* $\frac{1}{p}+\frac{1}{q}=1$, *then*

$$\begin{aligned}\|D_n(\mathbf{A},\mathbf{B})\|_1&\le D_n(\mathbf{V}_{\mathbf{A},p},\mathbf{V}_{\mathbf{B},q})\\&\le\frac{1}{n}\left[\frac{n}{2}\right]\left(1-\frac{1}{n}\left[\frac{n}{2}\right]\right)\sum_{k=1}^{n-1}\|\Delta A_k\|_p\sum_{k=1}^{n-1}\|\Delta B_k\|_q\\&\le\frac{1}{4}\sum_{k=1}^{n-1}\|\Delta A_k\|_p\sum_{k=1}^{n-1}\|\Delta B_k\|_q\,,\end{aligned} \tag{2.1}$$

where $\mathbf{V}_{\mathbf{A},p}:=\left(0,\|\Delta A_1\|_p\,,...,\sum_{k=1}^{n-1}\|\Delta A_k\|_p\right)$ *and* $\mathbf{V}_{\mathbf{B},q}:=\left(0,\|\Delta B_1\|_q\,,...,\sum_{k=1}^{n-1}\|\Delta B_k\|_q\right)$.

In particular, if $\mathbf{A},\ \mathbf{B}\in(B_2(H))^n$, *then*

$$\begin{aligned}\|D_n(\mathbf{A},\mathbf{B})\|_1&\le D_n(\mathbf{V}_{\mathbf{A},2},\mathbf{V}_{\mathbf{B},2})\\&\le\frac{1}{n}\left[\frac{n}{2}\right]\left(1-\frac{1}{n}\left[\frac{n}{2}\right]\right)\sum_{k=1}^{n-1}\|\Delta A_k\|_2\sum_{k=1}^{n-1}\|\Delta B_k\|_2\\&\le\frac{1}{4}\sum_{k=1}^{n-1}\|\Delta A_k\|_2\sum_{k=1}^{n-1}\|\Delta B_k\|_2\,.\end{aligned} \tag{2.2}$$

Proof. Let us start with the following identity which can be proved by direct computation for $\mathbf{A}=(A_1,...,A_n)\in(B_p(H))^n$ and $\mathbf{B}=(B_1,...,B_n)\in(B_q(H))^n$

$$\begin{aligned}D_n(\mathbf{A},\mathbf{B})&=\frac{1}{2n^2}\sum_{i,j=1}^{n}(A_j-A_i)(B_j-B_i)\\&=\frac{1}{n^2}\sum_{1\le i<j\le n}(A_j-A_i)(B_j-B_i)\,.\end{aligned} \tag{2.3}$$

As $i<j$ we can write that $A_j-A_i=\sum_{k=i}^{j-1}\Delta A_k$ and $B_j-B_i=\sum_{k=i}^{j-1}\Delta B_k$. Using the generalized triangle inequality and the property (1.15) we have successively from

(2.3) that:

$$
\begin{aligned}
(2.4)\quad & \left\| D_n(\mathbf{A},\mathbf{B}) \right\|_1 \\
&= \frac{1}{n^2}\left\| \sum_{1\le i<j\le n} \sum_{k=i}^{j-1} \Delta A_k \sum_{k=i}^{j-1} \Delta B_k \right\|_1 \le \frac{1}{n^2} \sum_{1\le i<j\le n} \left\| \sum_{k=i}^{j-1} \Delta A_k \sum_{k=i}^{j-1} \Delta B_k \right\|_1 \\
&\le \frac{1}{n^2} \sum_{1\le i<j\le n} \left\| \sum_{k=i}^{j-1} \Delta A_k \right\|_p \left\| \sum_{k=i}^{j-1} \Delta B_k \right\|_q \\
&\le \frac{1}{n^2} \sum_{1\le i<j\le n} \sum_{k=i}^{j-1} \left\| \Delta A_k \right\|_p \sum_{k=i}^{j-1} \left\| \Delta B_k \right\|_q \\
&= \frac{1}{n^2} \sum_{1\le i<j\le n} \left(\sum_{k=1}^{j-1} \left\| \Delta A_k \right\|_p - \sum_{k=1}^{i-1} \left\| \Delta A_k \right\|_p \right) \left(\sum_{k=1}^{j-1} \left\| \Delta B_k \right\|_q - \sum_{k=1}^{i-1} \left\| \Delta B_k \right\|_q \right) \\
&= \frac{1}{2n^2} \sum_{i,j=1}^{n} \left(\sum_{k=1}^{j-1} \left\| \Delta A_k \right\|_p - \sum_{k=1}^{i-1} \left\| \Delta A_k \right\|_p \right) \left(\sum_{k=1}^{j-1} \left\| \Delta B_k \right\|_q - \sum_{k=1}^{i-1} \left\| \Delta B_k \right\|_q \right) \\
&= D_n\left(\mathbf{V}_{\mathbf{A},p}, \mathbf{V}_{\mathbf{B},q}\right).
\end{aligned}
$$

Observe that

$$0 \le \sum_{k=1}^{j-1} \left\| \Delta A_k \right\|_p \le \sum_{k=1}^{n-1} \left\| \Delta A_k \right\|_p$$

and

$$0 \le \sum_{k=1}^{j-1} \left\| \Delta B_k \right\|_q \le \sum_{k=1}^{n-1} \left\| \Delta B_k \right\|_q$$

for all $j \in \{1, ..., n\}$.

Utilising the inequality (1.2) for $\mathbf{V_A}$ and $\mathbf{V_B}$ we have

$$
\begin{aligned}
(0 \le)\, D_n\left(\mathbf{V}_{\mathbf{A},p}, \mathbf{V}_{\mathbf{B},q}\right) &\le \frac{1}{n}\left[\frac{n}{2}\right]\left(1 - \frac{1}{n}\left[\frac{n}{2}\right]\right) \sum_{k=1}^{n-1} \left\| \Delta A_k \right\|_p \sum_{k=1}^{n-1} \left\| \Delta B_k \right\|_q \\
&\le \frac{1}{4} \sum_{k=1}^{n-1} \left\| \Delta A_k \right\|_p \sum_{k=1}^{n-1} \left\| \Delta B_k \right\|_q
\end{aligned}
$$

and by (2.4) we get (2.1). □

We have for $\mathbf{e} = (1, 2, ..., n)$ that

$$D_n(\mathbf{a},\mathbf{e}) := \frac{1}{n}\sum_{i=1}^{n} i a_i - \frac{n+1}{2}\frac{1}{n}\sum_{i=1}^{n} a_i$$

and

$$
\begin{aligned}
D_n(\mathbf{e},\mathbf{e}) &:= \frac{1}{n}\sum_{i=1}^{n} i^2 - \frac{n+1}{2}\frac{1}{n}\sum_{i=1}^{n} i \\
&= \frac{(n+1)(2n+1)}{6} - \frac{(n+1)^2}{4} = \frac{1}{12}\left(n^2 - 1\right).
\end{aligned}
$$

If we use the *pre-Grüss inequality*

$$|D_n(\mathbf{a},\mathbf{b})|^2 \le D_n(\mathbf{a},\mathbf{a})\, D_n(\mathbf{b},\mathbf{b}),$$

we get

$$|D_n(\mathbf{a},\mathbf{e})|^2 \le \frac{1}{12}\left(n^2-1\right) D_n(\mathbf{a},\mathbf{a}). \tag{2.5}$$

If $r \le a_i \le R$ for $i = 1, ..., n$, then by (1.2) we have

$$D_n(\mathbf{a},\mathbf{a}) \le \frac{1}{n}\left[\frac{n}{2}\right]\left(1-\frac{1}{n}\left[\frac{n}{2}\right]\right)(R-r)^2 \le \frac{1}{4}(R-r)^2. \tag{2.6}$$

By (2.5) and (2.6) we also have

$$\begin{aligned} |D_n(\mathbf{a},\mathbf{e})|^2 &\le \frac{1}{12}\left(n^2-1\right) D_n(\mathbf{a},\mathbf{a}) \\ &\le \frac{1}{12}\left(n^2-1\right)\frac{1}{n}\left[\frac{n}{2}\right]\left(1-\frac{1}{n}\left[\frac{n}{2}\right]\right)(R-r)^2 \\ &\le \frac{1}{48}\left(n^2-1\right)(R-r)^2, \end{aligned} \tag{2.7}$$

provided that $r \le a_i \le R$ for $i = 1, ..., n$. This implies that

$$\begin{aligned} |D_n(\mathbf{a},\mathbf{e})| &\le \frac{\sqrt{3}}{6}\left(n^2-1\right)^{1/2}\left[D_n(\mathbf{a},\mathbf{a})\right]^{1/2} \\ &\le \frac{\sqrt{3}}{6}\left(n^2-1\right)^{1/2}\left[\frac{1}{n}\left[\frac{n}{2}\right]\left(1-\frac{1}{n}\left[\frac{n}{2}\right]\right)\right]^{1/2}(R-r) \\ &\le \frac{\sqrt{3}}{12}\left(n^2-1\right)^{1/2}(R-r). \end{aligned} \tag{2.8}$$

Theorem 5. *If* $\mathbf{A} = (A_1, ..., A_n) \in (B_p(H))^n$ *and* $\mathbf{B} = (B_1, ..., B_n) \in (B_q(H))^n$ *for* $p,\ q > 1$ *with* $\frac{1}{p} + \frac{1}{q} = 1$, *then*

$$\begin{aligned} \|D_n(\mathbf{A},\mathbf{B})\|_1 &\le \max_{k\in\{1,\dots,n-1\}} \|\Delta A_k\|_p\, D_n(\mathbf{e},\mathbf{V}_{\mathbf{B},q}) \\ &\le \max_{k\in\{1,\dots,n-1\}} \|\Delta A_k\|_p \frac{\sqrt{3}}{6}\left(n^2-1\right)^{1/2} \\ &\times \left[D_n(\mathbf{V}_{\mathbf{B},q},\mathbf{V}_{\mathbf{B},q})\right]^{1/2} \\ &\le \frac{\sqrt{3}}{6}\left(n^2-1\right)^{1/2}\left[\frac{1}{n}\left[\frac{n}{2}\right]\left(1-\frac{1}{n}\left[\frac{n}{2}\right]\right)\right]^{1/2} \\ &\times \max_{k\in\{1,\dots,n-1\}} \|\Delta A_k\|_p \sum_{k=1}^{n-1}\|\Delta B_k\|_q \\ &\le \frac{\sqrt{3}}{12}\left(n^2-1\right)^{1/2}\max_{k\in\{1,\dots,n-1\}} \|\Delta A_k\|_p \sum_{k=1}^{n-1}\|\Delta B_k\|_q, \end{aligned} \tag{2.9}$$

where $\mathbf{V}_{\mathbf{B},q} := \left(0, \|\Delta B_1\|_q, ..., \sum_{k=1}^{n-1}\|\Delta B_k\|_q\right)$.

In particular, if $\mathbf{A}$, $\mathbf{B} \in (B_2(H))^n$, *then*

$$\begin{aligned} \|D_n(\mathbf{A},\mathbf{B})\|_1 &\leq \max_{k\in\{1,\dots,n-1\}} \|\Delta A_k\|_2 \, D_n(\mathbf{e},\mathbf{V}_{\mathbf{B},2}) \\ &\leq \max_{k\in\{1,\dots,n-1\}} \|\Delta A_k\|_2 \frac{\sqrt{3}}{6}\left(n^2-1\right)^{1/2} \\ &\times \left[D_n(\mathbf{V}_{\mathbf{B},2},\mathbf{V}_{\mathbf{B},2})\right]^{1/2} \\ &\leq \frac{\sqrt{3}}{6}\left(n^2-1\right)^{1/2}\left[\frac{1}{n}\left[\frac{n}{2}\right]\left(1-\frac{1}{n}\left[\frac{n}{2}\right]\right)\right]^{1/2} \\ &\times \max_{k\in\{1,\dots,n-1\}} \|\Delta A_k\|_2 \sum_{k=1}^{n-1}\|\Delta B_k\|_2 \\ &\leq \frac{\sqrt{3}}{12}\left(n^2-1\right)^{1/2} \max_{k\in\{1,\dots,n-1\}} \|\Delta A_k\|_2 \sum_{k=1}^{n-1}\|\Delta B_k\|_2 . \end{aligned} \tag{2.10}$$

Proof. We have by (2.4) that,

$$\begin{aligned} \|D_n(\mathbf{A},\mathbf{B})\|_1 &\leq \frac{1}{n^2}\sum_{1\leq i<j\leq n}\sum_{k=i}^{j-1}\|\Delta A_k\|_p \sum_{k=i}^{j-1}\|\Delta B_k\|_q \\ &\leq \frac{1}{n^2}\sum_{1\leq i<j\leq n}(j-i)\max_{k\in\{i,\dots,j-1\}}\|\Delta A_k\|_p\sum_{k=i}^{j-1}\|\Delta B_k\|_q \end{aligned} \tag{2.11}$$

$$\begin{aligned} &\leq \frac{1}{n^2}\max_{k\in\{1,\dots,n-1\}}\|\Delta A_k\|_p\sum_{1\leq i<j\leq n}(j-i)\sum_{k=i}^{j-1}\|\Delta B_k\|_q \\ &= \frac{1}{n^2}\max_{k\in\{1,\dots,n-1\}}\|\Delta A_k\|_p\frac{1}{2}\sum_{i,j=1}^{n}(j-i)\sum_{k=i}^{j-1}\|\Delta B_k\|_q \\ &= \frac{1}{n^2}\max_{k\in\{1,\dots,n-1\}}\|\Delta A_k\|_p\frac{1}{2}\sum_{i,j=1}^{n}(j-i)\left(\sum_{k=1}^{j-1}\|\Delta B_k\|_q-\sum_{k=1}^{i-1}\|\Delta B_k\|_q\right) \\ &= \max_{k\in\{1,\dots,n-1\}}\|\Delta A_k\|_p \, D_n(\mathbf{e},\mathbf{V}_{\mathbf{B},q}). \end{aligned}$$

By (2.8) we derive

$$\begin{aligned} &(0\leq)\, D_n(\mathbf{V}_{\mathbf{B},q},\mathbf{e}) \\ &\leq \frac{\sqrt{3}}{6}\left(n^2-1\right)^{1/2}\left[D_n(\mathbf{V}_{\mathbf{B},q},\mathbf{V}_{\mathbf{B},q})\right]^{1/2} \\ &\leq \frac{\sqrt{3}}{6}\left(n^2-1\right)^{1/2}\left[\frac{1}{n}\left[\frac{n}{2}\right]\left(1-\frac{1}{n}\left[\frac{n}{2}\right]\right)\right]^{1/2}\sum_{k=1}^{n-1}\|\Delta B_k\|_q \\ &\leq \frac{\sqrt{3}}{12}\left(n^2-1\right)^{1/2}\sum_{k=1}^{n-1}\|\Delta B_k\|_q . \end{aligned} \tag{2.12}$$

By making use of (2.11) and (2.12) we derive (2.9). □

Theorem 6. *If* $\mathbf{A}=(A_1,...,A_n)\in(B_p(H))^n$ *and* $\mathbf{B}=(B_1,...,B_n)\in(B_q(H))^n$ *for* $p,\ q>1$ *with* $\frac{1}{p}+\frac{1}{q}=1$, *then for* $\alpha,\ \beta>1$ *with* $\frac{1}{\alpha}+\frac{1}{\beta}=1$ *we have*

$$\begin{aligned}\|D_n(\mathbf{A},\mathbf{B})\|_1 &\le [D_n(\mathbf{V}_{\mathbf{A},p,\alpha},\mathbf{e})]^{1/\alpha}[D_n(\mathbf{V}_{\mathbf{B},q,\beta},\mathbf{e})]^{1/\beta} \\ &\le \frac{\sqrt{3}}{6}\left(n^2-1\right)^{1/2} \\ &\times [D_n(\mathbf{V}_{\mathbf{A},p,\alpha},\mathbf{V}_{\mathbf{A},p,\alpha})]^{1/(2\alpha)}[D_n(\mathbf{V}_{\mathbf{B},q,\beta},\mathbf{V}_{\mathbf{B},q,\beta})]^{1/(2\beta)} \\ &\le \frac{\sqrt{3}}{6}\left(n^2-1\right)^{1/2}\left[\frac{1}{n}\left[\frac{n}{2}\right]\left(1-\frac{1}{n}\left[\frac{n}{2}\right]\right)\right]^{1/2} \\ &\times\left(\sum_{k=1}^{n-1}\|\Delta A_k\|_p^\alpha\right)^{1/\alpha}\left(\sum_{k=1}^{n-1}\|\Delta B_k\|_q^\beta\right)^{1/\beta} \\ &\le \frac{\sqrt{3}}{12}\left(n^2-1\right)^{1/2}\left(\sum_{k=1}^{n-1}\|\Delta A_k\|_p^\alpha\right)^{1/\alpha}\left(\sum_{k=1}^{n-1}\|\Delta B_k\|_q^\beta\right)^{1/\beta},\end{aligned}\tag{2.13}$$

where

$$\mathbf{V}_{\mathbf{A},p,\alpha}:=\left(0,\|\Delta A_1\|_p^\alpha,...,\sum_{k=1}^{n-1}\|\Delta A_k\|_p^\alpha\right)$$

and

$$\mathbf{V}_{\mathbf{B},q,\beta}:=\left(0,\|\Delta B_1\|_q^\beta,...,\sum_{k=1}^{n-1}\|\Delta B_k\|_q^\beta\right).$$

Proof. We have by (2.4) that

$$\|D_n(\mathbf{A},\mathbf{B})\|_1\le\frac{1}{n^2}\sum_{1\le i<j\le n}\sum_{k=i}^{j-1}\|\Delta A_k\|_p\sum_{k=i}^{j-1}\|\Delta B_k\|_q.\tag{2.14}$$

By Hölder's inequality for $\alpha,\ \beta>1$ with $\frac{1}{\alpha}+\frac{1}{\beta}=1$ we have that

$$\sum_{k=i}^{j-1}\|\Delta A_k\|_p\le(j-i)^{1/\beta}\left(\sum_{k=i}^{j-1}\|\Delta A_k\|_p^\alpha\right)^{1/\alpha}$$

and

$$\sum_{k=i}^{j-1}\|\Delta B_k\|_q\le(j-i)^{1/\alpha}\left(\sum_{k=i}^{j-1}\|\Delta B_k\|_q^\beta\right)^{1/\beta},$$

which gives, by multiplication, that

$$\sum_{k=i}^{j-1}\|\Delta A_k\|_p\sum_{k=i}^{j-1}\|\Delta B_k\|_q\le(j-i)\left(\sum_{k=i}^{j-1}\|\Delta A_k\|_p^\alpha\right)^{1/\alpha}\left(\sum_{k=i}^{j-1}\|\Delta B_k\|_q^\beta\right)^{1/\beta}.$$

Then by (2.14) we get

$$\|D_n(\mathbf{A},\mathbf{B})\|_1\le\frac{1}{n^2}\sum_{1\le i<j\le n}(j-i)\left(\sum_{k=i}^{j-1}\|\Delta A_k\|_p^\alpha\right)^{1/\alpha}\left(\sum_{k=i}^{j-1}\|\Delta B_k\|_q^\beta\right)^{1/\beta}.\tag{2.15}$$

By using Hölder's weighted inequality for α, $\beta > 1$ with $\frac{1}{\alpha} + \frac{1}{\beta} = 1$ we get

$$\begin{aligned}(2.16)\quad &\frac{1}{n^2} \sum_{1\leq i<j\leq n} (j-i) \left(\sum_{k=i}^{j-1} \|\Delta A_k\|_p^{\alpha}\right)^{1/\alpha} \left(\sum_{k=i}^{j-1} \|\Delta B_k\|_q^{\beta}\right)^{1/\beta} \\ &\leq \frac{1}{n^2} \left\{ \sum_{1\leq i<j\leq n} (j-i) \left[\left(\sum_{k=i}^{j-1} \|\Delta A_k\|_p^{\alpha}\right)^{1/\alpha}\right]^{\alpha} \right\}^{1/\alpha} \\ &\times \left\{ \sum_{1\leq i<j\leq n} (j-i) \left[\left(\sum_{k=i}^{j-1} \|\Delta B_k\|_q^{\beta}\right)^{1/\beta}\right]^{\beta} \right\}^{1/\beta}\end{aligned}$$

$$\begin{aligned}&= \frac{1}{n^2} \left\{ \sum_{1\leq i<j\leq n} (j-i) \left(\sum_{k=i}^{j-1} \|\Delta A_k\|_p^{\alpha}\right) \right\}^{1/\alpha} \left\{ \sum_{1\leq i<j\leq n} (j-i) \left(\sum_{k=i}^{j-1} \|\Delta B_k\|_q^{\beta}\right) \right\}^{1/\beta} \\ &= \left\{ \frac{1}{2}\frac{1}{n^2} \sum_{i,j=1}^{n} (j-i) \sum_{k=i}^{j-1} \|\Delta A_k\|_p^{\alpha} \right\}^{1/\alpha} \left\{ \frac{1}{2}\frac{1}{n^2} \sum_{i,j=1}^{n} (j-i) \sum_{k=i}^{j-1} \|\Delta B_k\|_q^{\beta} \right\}^{1/\beta} \\ &= \left[D_n\left(\mathbf{V}_{\mathbf{A},p,\alpha}, \mathbf{e}\right)\right]^{1/\alpha} \left[D_n\left(\mathbf{V}_{\mathbf{B},q,\beta}, \mathbf{e}\right)\right]^{1/\beta}.\end{aligned}$$

By making use of (2.14)-(2.16) we derive the first inequality in (2.13).

By (2.8) we get

$$\begin{aligned}(2.17)\quad D_n\left(\mathbf{V}_{\mathbf{A},p,\alpha}, \mathbf{e}\right) &\leq \frac{\sqrt{3}}{6}\left(n^2-1\right)^{1/2} \left[D_n\left(\mathbf{V}_{\mathbf{A},p,\alpha}, \mathbf{V}_{\mathbf{A},p,\alpha}\right)\right]^{1/2} \\ &\leq \frac{\sqrt{3}}{6}\left(n^2-1\right)^{1/2} \left[\frac{1}{n}\left[\frac{n}{2}\right]\left(1-\frac{1}{n}\left[\frac{n}{2}\right]\right)\right]^{1/2} \sum_{k=1}^{n-1} \|\Delta A_k\|_p^{\alpha} \\ &\leq \frac{\sqrt{3}}{12}\left(n^2-1\right)^{1/2} \sum_{k=1}^{n-1} \|\Delta A_k\|_p^{\alpha},\end{aligned}$$

and

$$\begin{aligned}(2.18)\quad D_n\left(\mathbf{V}_{\mathbf{B},q,\beta}, \mathbf{e}\right) &\leq \frac{\sqrt{3}}{6}\left(n^2-1\right)^{1/2} \left[D_n\left(\mathbf{V}_{\mathbf{B},q,\beta}, \mathbf{V}_{\mathbf{B},q,\beta}\right)\right]^{1/2} \\ &\leq \frac{\sqrt{3}}{6}\left(n^2-1\right)^{1/2} \left[\frac{1}{n}\left[\frac{n}{2}\right]\left(1-\frac{1}{n}\left[\frac{n}{2}\right]\right)\right]^{1/2} \sum_{k=1}^{n-1} \|\Delta B_k\|_q^{\beta} \\ &\leq \frac{\sqrt{3}}{12}\left(n^2-1\right)^{1/2} \sum_{k=1}^{n-1} \|\Delta B_k\|_q^{\beta}.\end{aligned}$$

Therefore

$$
\begin{aligned}
&\left[D_n\left(\mathbf{V}_{\mathbf{A},p,\alpha},\mathbf{e}\right)\right]^{1/\alpha}\left[D_n\left(\mathbf{V}_{\mathbf{B},q,\beta},\mathbf{e}\right)\right]^{1/\beta} \\
&\leq \frac{\sqrt{3}}{6}\left(n^2-1\right)^{1/2}\left[D_n\left(\mathbf{V}_{\mathbf{A},p,\alpha},\mathbf{V}_{\mathbf{A},p,\alpha}\right)\right]^{1/(2\alpha)}\left[D_n\left(\mathbf{V}_{\mathbf{B},q,\beta},\mathbf{V}_{\mathbf{B},q,\beta}\right)\right]^{1/(2\beta)} \\
&\leq \frac{\sqrt{3}}{6}\left(n^2-1\right)^{1/2}\left[\frac{1}{n}\left[\frac{n}{2}\right]\left(1-\frac{1}{n}\left[\frac{n}{2}\right]\right)\right]^{1/2} \\
&\times\left(\sum_{k=1}^{n-1}\left\|\Delta A_k\right\|_p^{\alpha}\right)^{1/\alpha}\left(\sum_{k=1}^{n-1}\left\|\Delta B_k\right\|_q^{\beta}\right)^{1/\beta} \\
&\leq \frac{\sqrt{3}}{12}\left(n^2-1\right)^{1/2}\left(\sum_{k=1}^{n-1}\left\|\Delta A_k\right\|_p^{\alpha}\right)^{1/\alpha}\left(\sum_{k=1}^{n-1}\left\|\Delta B_k\right\|_q^{\beta}\right)^{1/\beta},
\end{aligned}
$$

which proves the second part of (2.13). □

Remark 1. *If*

$$
\mathbf{V}_{\mathbf{A},2,\alpha}:=\left(0,\left\|\Delta A_1\right\|_2^{\alpha},...,\sum_{k=1}^{n-1}\left\|\Delta A_k\right\|_2^{\alpha}\right)
$$

and

$$
\mathbf{V}_{\mathbf{B},2,\beta}:=\left(0,\left\|\Delta B_1\right\|_2^{\beta},...,\sum_{k=1}^{n-1}\left\|\Delta B_k\right\|_2^{\beta}\right),
$$

then for $\alpha,\ \beta>1$ *with* $\frac{1}{\alpha}+\frac{1}{\beta}=1$ *we have*

$$
\begin{aligned}
\left\|D_n\left(\mathbf{A},\mathbf{B}\right)\right\|_1 &\leq \left[D_n\left(\mathbf{V}_{\mathbf{A},2,\alpha},\mathbf{e}\right)\right]^{1/\alpha}\left[D_n\left(\mathbf{V}_{\mathbf{B},2,\beta},\mathbf{e}\right)\right]^{1/\beta} \\
&\leq \frac{\sqrt{3}}{6}\left(n^2-1\right)^{1/2} \\
&\times\left[D_n\left(\mathbf{V}_{\mathbf{A},2,\alpha},\mathbf{V}_{\mathbf{A},2,\alpha}\right)\right]^{1/(2\alpha)}\left[D_n\left(\mathbf{V}_{\mathbf{B},2,\beta},\mathbf{V}_{\mathbf{B},2,\beta}\right)\right]^{1/(2\beta)} \\
&\leq \frac{\sqrt{3}}{6}\left(n^2-1\right)^{1/2}\left[\frac{1}{n}\left[\frac{n}{2}\right]\left(1-\frac{1}{n}\left[\frac{n}{2}\right]\right)\right]^{1/2} \\
&\times\left(\sum_{k=1}^{n-1}\left\|\Delta A_k\right\|_2^{\alpha}\right)^{1/\alpha}\left(\sum_{k=1}^{n-1}\left\|\Delta B_k\right\|_2^{\beta}\right)^{1/\beta} \\
&\leq \frac{\sqrt{3}}{12}\left(n^2-1\right)^{1/2}\left(\sum_{k=1}^{n-1}\left\|\Delta A_k\right\|_2^{\alpha}\right)^{1/\alpha}\left(\sum_{k=1}^{n-1}\left\|\Delta B_k\right\|_2^{\beta}\right)^{1/\beta}.
\end{aligned}
\tag{2.19}
$$

Now, if we take $\alpha = p$ and $\beta = q$ in (2.13), then we get

$$\begin{aligned}(2.20)\qquad \|D_n(\mathbf{A},\mathbf{B})\|_1 &\leq [D_n(\mathbf{V}_{\mathbf{A},p,p},\mathbf{e})]^{1/p}[D_n(\mathbf{V}_{\mathbf{B},q,q},\mathbf{e})]^{1/q} \\ &\leq \frac{\sqrt{3}}{6}\left(n^2-1\right)^{1/2} \\ &\times [D_n(\mathbf{V}_{\mathbf{A},p,p},\mathbf{V}_{\mathbf{A},p,p})]^{1/(2p)}[D_n(\mathbf{V}_{\mathbf{B},q,q},\mathbf{V}_{\mathbf{B},q,q})]^{1/(2q)} \\ &\leq \frac{\sqrt{3}}{6}\left(n^2-1\right)^{1/2}\left[\frac{1}{n}\left[\frac{n}{2}\right]\left(1-\frac{1}{n}\left[\frac{n}{2}\right]\right)\right]^{1/2} \\ &\times \left(\sum_{k=1}^{n-1}\|\Delta A_k\|_p^p\right)^{1/p}\left(\sum_{k=1}^{n-1}\|\Delta B_k\|_q^q\right)^{1/q} \\ &\leq \frac{\sqrt{3}}{12}\left(n^2-1\right)^{1/2}\left(\sum_{k=1}^{n-1}\|\Delta A_k\|_p^p\right)^{1/p}\left(\sum_{k=1}^{n-1}\|\Delta B_k\|_q^q\right)^{1/q}.\end{aligned}$$

By using the trace properties, we can state that

$$\begin{aligned}(2.21)\qquad &|\mathrm{tr}\,(D_n(\mathbf{A},\mathbf{B}))| \\ &\leq \frac{\sqrt{3}}{6}\left(n^2-1\right)^{1/2}\left[\frac{1}{n}\left[\frac{n}{2}\right]\left(1-\frac{1}{n}\left[\frac{n}{2}\right]\right)\right]^{1/2} \\ &\times \left[\mathrm{tr}\left(\sum_{k=1}^{n-1}|\Delta A_k|^p\right)\right]^{1/p}\left[\mathrm{tr}\left(\sum_{k=1}^{n-1}|\Delta B_k|^q\right)\right]^{1/q} \\ &\leq \frac{\sqrt{3}}{12}\left(n^2-1\right)^{1/2}\left[\mathrm{tr}\left(\sum_{k=1}^{n-1}|\Delta A_k|^p\right)\right]^{1/p}\left[\mathrm{tr}\left(\sum_{k=1}^{n-1}|\Delta B_k|^q\right)\right]^{1/q},\end{aligned}$$

for $p,\ q > 1$ with $\frac{1}{p}+\frac{1}{q}=1$.

In particular, for $p = q = 2$ we get the trace inequality:

$$\begin{aligned}(2.22)\qquad &|\mathrm{tr}\,(D_n(\mathbf{A},\mathbf{B}))| \\ &\leq \frac{\sqrt{3}}{6}\left(n^2-1\right)^{1/2}\left[\frac{1}{n}\left[\frac{n}{2}\right]\left(1-\frac{1}{n}\left[\frac{n}{2}\right]\right)\right]^{1/2} \\ &\times \left[\mathrm{tr}\left(\sum_{k=1}^{n-1}|\Delta A_k|^2\right)\right]^{1/2}\left[\mathrm{tr}\left(\sum_{k=1}^{n-1}|\Delta B_k|^2\right)\right]^{1/2} \\ &\leq \frac{\sqrt{3}}{12}\left(n^2-1\right)^{1/2}\left[\mathrm{tr}\left(\sum_{k=1}^{n-1}|\Delta A_k|^2\right)\right]^{1/2}\left[\mathrm{tr}\left(\sum_{k=1}^{n-1}|\Delta B_k|^2\right)\right]^{1/2}.\end{aligned}$$

3. p-Schatten Norm Inequalities

We also have:

Theorem 7. *If* $\mathbf{A}=(A_1,...,A_n)$ *and* $\mathbf{B}=(B_1,...,B_n)\in(B_p(H))^n$ *for* $p\geq 1$, *then*

$$\|D_n(\mathbf{A},\mathbf{B})\|_p \leq D_n(\mathbf{V}_{\mathbf{A},p},\mathbf{V}_{\mathbf{B},p}) \leq \frac{1}{n}\left[\frac{n}{2}\right]\left(1-\frac{1}{n}\left[\frac{n}{2}\right]\right)\sum_{k=1}^{n-1}\|\Delta A_k\|_p\sum_{k=1}^{n-1}\|\Delta B_k\|_p \leq \frac{1}{4}\sum_{k=1}^{n-1}\|\Delta A_k\|_p\sum_{k=1}^{n-1}\|\Delta B_k\|_p, \tag{3.1}$$

where $\mathbf{V}_{\mathbf{A},p}:=\left(0,\|\Delta A_1\|_p,...,\sum_{k=1}^{n-1}\|\Delta A_k\|_p\right)$ *and* $\mathbf{V}_{\mathbf{B},p}:=\left(0,\|\Delta B_1\|_p,...,\sum_{k=1}^{n-1}\|\Delta B_k\|_p\right)$.

Proof. As in the proof of Theorem 4 we have, by utilising property (1.12), that

$$\begin{aligned}
&\|D_n(\mathbf{A},\mathbf{B})\|_p \\
&=\frac{1}{n^2}\left\|\sum_{1\leq i<j\leq n}\sum_{k=i}^{j-1}\Delta A_k\sum_{k=i}^{j-1}\Delta B_k\right\|_p \leq \frac{1}{n^2}\sum_{1\leq i<j\leq n}\left\|\sum_{k=i}^{j-1}\Delta A_k\sum_{k=i}^{j-1}\Delta B_k\right\|_p \\
&\leq \frac{1}{n^2}\sum_{1\leq i<j\leq n}\left\|\sum_{k=i}^{j-1}\Delta A_k\right\|_p\left\|\sum_{k=i}^{j-1}\Delta B_k\right\|_p \\
&\leq \frac{1}{n^2}\sum_{1\leq i<j\leq n}\sum_{k=i}^{j-1}\|\Delta A_k\|_p\sum_{k=i}^{j-1}\|\Delta B_k\|_p \\
&=\frac{1}{n^2}\sum_{1\leq i<j\leq n}\left(\sum_{k=1}^{j-1}\|\Delta A_k\|_p-\sum_{k=1}^{i-1}\|\Delta A_k\|_p\right)\left(\sum_{k=1}^{j-1}\|\Delta B_k\|_p-\sum_{k=1}^{i-1}\|\Delta B_k\|_p\right) \\
&=\frac{1}{2n^2}\sum_{i,j=1}^{n}\left(\sum_{k=1}^{j-1}\|\Delta A_k\|_p-\sum_{k=1}^{i-1}\|\Delta A_k\|_p\right)\left(\sum_{k=1}^{j-1}\|\Delta B_k\|_p-\sum_{k=1}^{i-1}\|\Delta B_k\|_p\right) \\
&=D_n(\mathbf{V}_{\mathbf{A},p},\mathbf{V}_{\mathbf{B},p}).
\end{aligned}\tag{3.2}$$

The rest follows as in the proof of Theorem 4 and we omit the details. □

By utilising a similar argument to the one in the proof of Theorem 5 we can state

Theorem 8. *If* $\mathbf{A}=(A_1,...,A_n)$ *and* $\mathbf{B}=(B_1,...,B_n)\in(B_p(H))^n$ *for* $p\geq 1$, *then*

$$\begin{aligned}
\|D_n(\mathbf{A},\mathbf{B})\|_p &\leq \max_{k\in\{1,...,n-1\}}\|\Delta A_k\|_p D_n(\mathbf{e},\mathbf{V}_{\mathbf{B},p}) \\
&\leq \frac{\sqrt{3}}{6}\left(n^2-1\right)^{1/2}\max_{k\in\{1,...,n-1\}}\|\Delta A_k\|_p\left[D_n(\mathbf{V}_{\mathbf{B},p},\mathbf{V}_{\mathbf{B},p})\right]^{1/2} \\
&\leq \frac{\sqrt{3}}{6}\left(n^2-1\right)^{1/2}\left[\frac{1}{n}\left[\frac{n}{2}\right]\left(1-\frac{1}{n}\left[\frac{n}{2}\right]\right)\right]^{1/2} \\
&\quad\times\max_{k\in\{1,...,n-1\}}\|\Delta A_k\|_p\sum_{k=1}^{n-1}\|\Delta B_k\|_p \\
&\leq \frac{\sqrt{3}}{12}\left(n^2-1\right)^{1/2}\max_{k\in\{1,...,n-1\}}\|\Delta A_k\|_p\sum_{k=1}^{n-1}\|\Delta B_k\|_p.
\end{aligned}\tag{3.3}$$

Finally, we have:

Theorem 9. *If* $\mathbf{A} = (A_1, ..., A_n)$ *and* $\mathbf{B} = (B_1, ..., B_n) \in (B_p(H))^n$ *for* $p \geq 1$, *then for* α, $\beta > 1$ *with* $\frac{1}{\alpha} + \frac{1}{\beta} = 1$ *we have*

$$\begin{aligned}
\|D_n(\mathbf{A}, \mathbf{B})\|_p &\leq [D_n(\mathbf{V}_{\mathbf{A},p,\alpha}, \mathbf{e})]^{1/\alpha} [D_n(\mathbf{V}_{\mathbf{B},p,\beta}, \mathbf{e})]^{1/\beta} \\
&\leq \frac{\sqrt{3}}{6} (n^2 - 1)^{1/2} \\
&\times [D_n(\mathbf{V}_{\mathbf{A},p,\alpha}, \mathbf{V}_{\mathbf{A},p,\alpha})]^{1/(2\alpha)} [D_n(\mathbf{V}_{\mathbf{B},p,\beta}, \mathbf{V}_{\mathbf{B},p,\beta})]^{1/(2\beta)} \\
&\leq \frac{\sqrt{3}}{6} (n^2 - 1)^{1/2} \left[\frac{1}{n}\left[\frac{n}{2}\right]\left(1 - \frac{1}{n}\left[\frac{n}{2}\right]\right)\right]^{1/2} \\
&\times \left(\sum_{k=1}^{n-1} \|\Delta A_k\|_p^\alpha\right)^{1/\alpha} \left(\sum_{k=1}^{n-1} \|\Delta B_k\|_p^\beta\right)^{1/\beta} \\
&\leq \frac{\sqrt{3}}{12} (n^2 - 1)^{1/2} \left(\sum_{k=1}^{n-1} \|\Delta A_k\|_p^\alpha\right)^{1/\alpha} \left(\sum_{k=1}^{n-1} \|\Delta B_k\|_p^\beta\right)^{1/\beta},
\end{aligned} \tag{3.4}$$

where

$$\mathbf{V}_{\mathbf{A},p,\alpha} := \left(0, \|\Delta A_1\|_p^\alpha, ..., \sum_{k=1}^{n-1} \|\Delta A_k\|_p^\alpha\right), \ \mathbf{V}_{\mathbf{B},p,\beta} := \left(0, \|\Delta B_1\|_p^\beta, ..., \sum_{k=1}^{n-1} \|\Delta B_k\|_p^\beta\right).$$

We observe that for $\alpha = \beta = 2$ we derive the inequality

$$\begin{aligned}
\|D_n(\mathbf{A}, \mathbf{B})\|_p &\leq [D_n(\mathbf{V}_{\mathbf{A},p,2}, \mathbf{e})]^{1/2} [D_n(\mathbf{V}_{\mathbf{B},p,2}, \mathbf{e})]^{1/2} \\
&\leq \frac{\sqrt{3}}{6} (n^2 - 1)^{1/2} \\
&\times [D_n(\mathbf{V}_{\mathbf{A},p,2}, \mathbf{V}_{\mathbf{A},p,2})]^{1/4} [D_n(\mathbf{V}_{\mathbf{B},p,2}, \mathbf{V}_{\mathbf{B},p,2})]^{1/4} \\
&\leq \frac{\sqrt{3}}{6} (n^2 - 1)^{1/2} \left[\frac{1}{n}\left[\frac{n}{2}\right]\left(1 - \frac{1}{n}\left[\frac{n}{2}\right]\right)\right]^{1/2} \\
&\times \left(\sum_{k=1}^{n-1} \|\Delta A_k\|_p^2\right)^{1/2} \left(\sum_{k=1}^{n-1} \|\Delta B_k\|_p^2\right)^{1/2} \\
&\leq \frac{\sqrt{3}}{12} (n^2 - 1)^{1/2} \left(\sum_{k=1}^{n-1} \|\Delta A_k\|_p^2\right)^{1/2} \left(\sum_{k=1}^{n-1} \|\Delta B_k\|_p^2\right)^{1/2}.
\end{aligned} \tag{3.5}$$

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