p-SCHATTEN NORM INEQUALITIES FOR ČEBYŠEV'S FUNCTIONAL VIA A CERONE-DRAGOMIR RESULT

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Abstract. An operator $A \in \mathcal{B}(H)$ is said to belong to the *von Neumann-Schatten class* $\mathcal{B}_p(H)$, $1 \leq p < \infty$ if the *p-Schatten norm* is finite

$$\|A\|_p := [\operatorname{tr}(|A|^p)]^{1/p} < \infty.$$

For $\mathbf{A} = (A_1, ..., A_n)$, $\mathbf{B} = (B_1, ..., B_n) \in B(H)$ we define the functional

$$D_n(\mathbf{A}, \mathbf{B}) := \frac{1}{n}\sum_{i=1}^{n} A_i B_i - \frac{1}{n}\sum_{i=1}^{n} A_i \frac{1}{n}\sum_{i=1}^{n} \alpha_i B_i.$$

In this paper we show among others that, if $\mathbf{A} = (A_1, ..., A_n) \in (B_p(H))^n$ and $\mathbf{B} = (B_1, ..., B_n) \in (B_q(H))^n$ for p, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, then

$$\begin{aligned}
&\|D_n(\mathbf{A}, \mathbf{B})\|_1 \\
&\leq \max_{k\in\{1,\ldots,n-1\}} \|\Delta A_k\|_p D_n(\mathbf{e}, \mathbf{V}_{\mathbf{B},q}) \\
&\leq \frac{1}{2}\left\lfloor\frac{n+1}{2}\right\rfloor\left(1 - \frac{1}{n}\left\lfloor\frac{n+1}{2}\right\rfloor\right) \max_{k\in\{1,\ldots,n-1\}} \|\Delta A_k\|_p \sum_{k=1}^{n-1} \|\Delta B_k\|_q \\
&\leq \frac{1}{8} n \max_{k\in\{1,\ldots,n-1\}} \|\Delta A_k\|_p \sum_{k=1}^{n-1} \|\Delta B_k\|_q,
\end{aligned}$$

where $\mathbf{V}_{\mathbf{B},q} := \left(0, \|\Delta B_1\|_q, ..., \sum_{k=1}^{n-1} \|\Delta B_k\|_q\right)$ and $\Delta B_i := B_{i+1} - B_i$ $(i = 1, ..., n-1)$ are the usual forward differences.

1. Introduction

Let $(\Omega, \mathcal{A}, \mu)$ be a measurable space consisting of a set Ω, a σ-algebra $\mathcal{A}$ of parts of Ω and a countably additive and positive measure μ on $\mathcal{A}$ with values in $\mathbb{R} \cup \{\infty\}$.

For a μ-measurable function $w : \Omega \to \mathbb{R}$, with $w(x) \geq 0$ for μ-a.e. $x \in \Omega$, consider the Lebesgue space $L_w(\Omega, \mu) := \{h : \Omega \to \mathbb{R},\ h$ is μ-measurable and $\int_\Omega w(x)|h(x)| d\mu(x) < \infty\}$. Assume $\int_\Omega w(x) d\mu(x) = 1$. In order to simplify the notation for the integrals, we do not write the variable, namely, instead of $\int_\Omega w(x) d\mu(x)$ we simply write $\int_\Omega w d\mu$.

If h, $k : \Omega \to \mathbb{R}$ are μ-measurable functions and h, k, $hk \in L_w(\Omega, \mu)$, then we may consider the weighted Čebyšev functional in the following form

$$D_w(h, k) := \int_\Omega whk d\mu - \int_\Omega wh d\mu \int_\Omega wk d\mu.$$

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The following result is known in the literature as the Grüss inequality, see for instance [7]:

$$|D_w(h,k)| \leq \frac{1}{4}(\Gamma-\gamma)(\Delta-\delta), \tag{1.1}$$

provided

$$-\infty<\gamma\leq h\leq\Gamma<\infty, \quad -\infty<\delta\leq k\leq\Delta<\infty \tag{1.2}$$

μ-a.e. on Ω. The constant $\frac{1}{4}$ is sharp in the sense that it cannot be replaced by a smaller constant.

For classical and more recent upper bounds related to the Čebyšev functional see [3]-[7] and [10]-[18].

Assume that $\mathbf{a}=(a_1,...,a_n)$ and $\mathbf{b}=(b_1,...,b_n)$ are n-tuples of real numbers and $\mathbf{p}=(p_1,...,p_n)$ a density distribution, namely $p_k\geq 0$ with $P_n:=\sum_{k=1}^n p_k=1$. We consider the *weighted Čebyšev's functional*

$$D_{\mathbf{p}}(\mathbf{a},\mathbf{b}):=\sum_{k=1}^n p_k a_k b_k-\sum_{k=1}^n p_k a_k \sum_{k=1}^n p_k b_k.$$

In the case when $\mathbf{p}=\left(\frac{1}{n},...,\frac{1}{n}\right)$ is the *uniform distribution* we also consider the *Čebyšev's functional*

$$D(\mathbf{a},\mathbf{b}):=\frac{1}{n}\sum_{k=1}^n a_k b_k-\frac{1}{n}\sum_{k=1}^n a_k \frac{1}{n}\sum_{k=1}^n b_k.$$

In [7] Cerone and Dragomir proved among others the following *refinement of Grüss' discrete inequality:*

Theorem 1. *Assume that* $\mathbf{a}=(a_1,...,a_n)$ *and* $\mathbf{b}=(b_1,...,b_n)$ *are n-tuples of real numbers and so that* $-\infty<\gamma\leq a_k\leq\Gamma<\infty$, $-\infty<\delta\leq b_k\leq\Delta<\infty$ *for all* $k\in\{1,...,n\}$, *then*

$$\begin{aligned}|D_{\mathbf{p}}(\mathbf{a},\mathbf{b})| &\leq \frac{1}{2}(\Delta-\delta)\sum_{k=1}^n p_k\left|b_k-\sum_{j=1}^n p_j b_j\right| \\ &\leq \frac{1}{2}(\Delta-\delta)\left[\sum_{k=1}^n p_k b_k^2-\left(\sum_{k=1}^n p_k b_k\right)^2\right]^{\frac{1}{2}} \\ &\leq \frac{1}{4}(\Delta-\delta)(\Gamma-\gamma).\end{aligned} \tag{1.3}$$

The constants $\frac{1}{2}$ and $\frac{1}{4}$ are best possible.

In particular, we have

$$\begin{aligned}|D(\mathbf{a},\mathbf{b})| &\leq \frac{1}{2}(\Delta-\delta)\frac{1}{n}\sum_{k=1}^n\left|b_k-\frac{1}{n}\sum_{j=1}^n b_j\right| \\ &\leq \frac{1}{2}(\Delta-\delta)\left[\frac{1}{n}\sum_{k=1}^n b_k^2-\left(\frac{1}{n}\sum_{k=1}^n b_k\right)^2\right]^{\frac{1}{2}} \\ &\leq \frac{1}{4}(\Delta-\delta)(\Gamma-\gamma).\end{aligned} \tag{1.4}$$

In order to extend these results for p-Schatten norms we need the following preparations.

Let $(H;\langle .,.\rangle)$ be a complex Hilbert space and $\mathcal{B}(H)$ the Banach algebra of all bounded linear operators on H. If $\{e_i\}_{i\in I}$ an orthonormal basis of H, we say that $A\in\mathcal{B}(H)$ is of *trace class* if

$$\|A\|_1:=\sum_{i\in I}\langle |A|\,e_i,e_i\rangle<\infty. \tag{1.5}$$

The definition of $\|A\|_1$ does not depend on the choice of the orthonormal basis $\{e_i\}_{i\in I}$. We denote by $\mathcal{B}_1(H)$ the set of trace class operators in $\mathcal{B}(H)$.

We define the *trace* of a trace class operator $A\in\mathcal{B}_1(H)$ to be

$$\operatorname{tr}(A):=\sum_{i\in I}\langle Ae_i,e_i\rangle, \tag{1.6}$$

where $\{e_i\}_{i\in I}$ an orthonormal basis of H. Note that this coincides with the usual definition of the trace if H is finite-dimensional. We observe that the series (1.6) converges absolutely and it is independent from the choice of basis.

The following result collects some properties of the trace:

Theorem 2. *We have:*

(i) If $A\in\mathcal{B}_1(H)$ *then* $A^*\in\mathcal{B}_1(H)$ *and*

$$\operatorname{tr}(A^*)=\overline{\operatorname{tr}(A)}; \tag{1.7}$$

(ii) If $A\in\mathcal{B}_1(H)$ *and* $T\in\mathcal{B}(H)$, *then* $AT,\,TA\in\mathcal{B}_1(H)$ *and*

$$\operatorname{tr}(AT)=\operatorname{tr}(TA)\ \text{ and }\ |\operatorname{tr}(AT)|\le\|A\|_1\|T\|; \tag{1.8}$$

(iii) $\operatorname{tr}(\cdot)$ *is a bounded linear functional on* $\mathcal{B}_1(H)$ *with* $\|\operatorname{tr}\|=1$;

(iv) If $A,\ B\in\mathcal{B}_2(H)$ *then* $AB,\ BA\in\mathcal{B}_1(H)$ *and* $\operatorname{tr}(AB)=\operatorname{tr}(BA)$;

(v) $\mathcal{B}_{fin}(H)$, *the space of operators of finite rank, is a dense subspace of* $\mathcal{B}_1(H)$.

An operator $A\in\mathcal{B}(H)$ is said to belong to the *von Neumann-Schatten class* $\mathcal{B}_p(H)$, $1\le p<\infty$ if the *p-Schatten norm* is finite [21, p. 60-64]

$$\|A\|_p:=\left[\operatorname{tr}(|A|^p)\right]^{1/p}<\infty.$$

For $1<p<q<\infty$ we have that

$$\mathcal{B}_1(H)\subset\mathcal{B}_p(H)\subset\mathcal{B}_q(H)\subset\mathcal{B}(H) \tag{1.9}$$

and

$$\|A\|_1\ge\|A\|_p\ge\|A\|_q\ge\|A\|. \tag{1.10}$$

For $p\ge1$ the functional $\|\cdot\|_p$ is a *norm* on the $*$-ideal $\mathcal{B}_p(H)$ and $\left(\mathcal{B}_p(H),\|\cdot\|_p\right)$ is a Banach space.

Also, see for instance [21, p. 60-64],

$$\|A\|_p=\|A^*\|_p,\ A\in\mathcal{B}_p(H) \tag{1.11}$$

$$\|AB\|_p\le\|A\|_p\|B\|_p,\ A,B\in\mathcal{B}_p(H) \tag{1.12}$$

and

$$\|AB\|_p\le\|A\|_p\|B\|,\ \ \|BA\|_p\le\|B\|\,\|A\|_p,\ A\in\mathcal{B}_p(H),\ B\in\mathcal{B}(H). \tag{1.13}$$

This implies that

$$\|CAB\|_p \le \|C\| \, \|A\|_p \, \|B\| \,, \ A \in \mathcal{B}_p(H)\,, \ B, \ C \in \mathcal{B}(H)\,. \tag{1.14}$$

In terms of *p-Schatten norm* we have the *Hölder inequality* for $p,\ q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$

$$(|\mathrm{tr}(AB)| \le) \, \|AB\|_1 \le \|A\|_p \, \|B\|_q \,, \ A \in \mathcal{B}_p(H)\,, \ B \in \mathcal{B}_q(H)\,. \tag{1.15}$$

For the theory of trace functionals and their applications the reader is referred to [20] and [21].

For some classical trace inequalities see [8], [9], and [19], which are continuations of the work of Bellman [1].

2. Main Results

We have for $\mathbf{e} = (1, 2, ..., n)$ and $\mathbf{x} = (x_1, ..., x_n)$ that

$$D_n(\mathbf{x}, \mathbf{e}) := \frac{1}{n} \sum_{i=1}^{n} i x_i - \frac{n+1}{2} \frac{1}{n} \sum_{i=1}^{n} x_i. \tag{2.1}$$

Now from (1.4) for $\mathbf{a} = \mathbf{e}$ and $\mathbf{b} = \mathbf{x}$ we have

$$|D(\mathbf{e}, \mathbf{x})| \le \frac{1}{2} \left(1 - \frac{1}{n}\right) \sum_{k=1}^{n} \left| x_k - \frac{1}{n} \sum_{j=1}^{n} x_j \right|. \tag{2.2}$$

Also, if $\gamma \le x_k \le \Gamma$ for $k \in \{1, ..., n\}$ then by (1.4) for $\mathbf{a} = \mathbf{x}$ and $\mathbf{b} = \mathbf{e}$ we have

$$|D_{\mathbf{p}}(\mathbf{x}, \mathbf{e})| \le \frac{1}{2} (\Gamma - \gamma) \frac{1}{n} \sum_{k=1}^{n} \left| k - \frac{n+1}{2} \right|. \tag{2.3}$$

Now, observe that, if $\lfloor \cdot \rfloor$ is the integer part function, then

$$\begin{aligned}
&\sum_{k=1}^{n} \left| k - \frac{n+1}{2} \right| \\
&= \sum_{k=1}^{\lfloor \frac{n+1}{2} \rfloor} \left| k - \frac{n+1}{2} \right| + \sum_{k=\lfloor \frac{n+1}{2} \rfloor + 1}^{n} \left| k - \frac{n+1}{2} \right| \\
&= \sum_{k=1}^{\lfloor \frac{n+1}{2} \rfloor} \left(\frac{n+1}{2} - k \right) + \sum_{k=\lfloor \frac{n+1}{2} \rfloor + 1}^{n} \left(k - \frac{n+1}{2} \right) \\
&= \frac{n+1}{2} \left\lfloor \frac{n+1}{2} \right\rfloor - \sum_{k=1}^{\lfloor \frac{n+1}{2} \rfloor} k + \sum_{k=\lfloor \frac{n+1}{2} \rfloor + 1}^{n} k - \frac{n+1}{2} \left(n - \left\lfloor \frac{n+1}{2} \right\rfloor \right)
\end{aligned}$$

$$= 2\frac{n+1}{2}\left\lfloor\frac{n+1}{2}\right\rfloor - \frac{n+1}{2}n - \sum_{k=1}^{\left\lfloor\frac{n+1}{2}\right\rfloor} k + \sum_{k=1}^{n} k - \sum_{k=1}^{\left\lfloor\frac{n+1}{2}\right\rfloor} k$$
$$= 2\frac{n+1}{2}\left\lfloor\frac{n+1}{2}\right\rfloor - \frac{n+1}{2}n + \frac{n+1}{2}n - 2\sum_{k=1}^{\left\lfloor\frac{n+1}{2}\right\rfloor} k$$
$$= 2\frac{n+1}{2}\left\lfloor\frac{n+1}{2}\right\rfloor - 2\frac{\left\lfloor\frac{n+1}{2}\right\rfloor\left(\left\lfloor\frac{n+1}{2}\right\rfloor+1\right)}{2} =$$
$$= (n+1)\left\lfloor\frac{n+1}{2}\right\rfloor - \left\lfloor\frac{n+1}{2}\right\rfloor^2 - \left\lfloor\frac{n+1}{2}\right\rfloor$$
$$= \left\lfloor\frac{n+1}{2}\right\rfloor\left[(n+1) - \left\lfloor\frac{n+1}{2}\right\rfloor - 1\right] = \left\lfloor\frac{n+1}{2}\right\rfloor\left(n - \left\lfloor\frac{n+1}{2}\right\rfloor\right).$$

So, the inequality (2.3) is equivalent to

$$\begin{aligned}|D_{\mathbf{p}}(\mathbf{x},\mathbf{e})| &\le \frac{1}{2}(\Gamma-\gamma)\left\lfloor\frac{n+1}{2}\right\rfloor\left(1-\frac{1}{n}\left\lfloor\frac{n+1}{2}\right\rfloor\right)\\ &\le \frac{1}{8}(\Gamma-\gamma)\,n\end{aligned} \tag{2.4}$$

provided that $\gamma \le x_k \le \Gamma$ for $k \in \{1, ..., n\}$.

The last inequality in (2.4) follows by the fact that $ab \le \frac{1}{4}(a+b)^2$, where $a = \frac{1}{n}\left\lfloor\frac{n+1}{2}\right\rfloor$ and $b = 1 - \frac{1}{n}\left\lfloor\frac{n+1}{2}\right\rfloor$.

For $\mathbf{A} = (A_1, ..., A_n)$, $\mathbf{B} = (B_1, ..., B_n) \in B(H)$ we define the functional

$$D_n(\mathbf{A},\mathbf{B}) := \frac{1}{n}\sum_{i=1}^{n} A_iB_i - \frac{1}{n}\sum_{i=1}^{n} A_i \frac{1}{n}\sum_{i=1}^{n} B_i. \tag{2.5}$$

Theorem 3. *If* $\mathbf{A} = (A_1, ..., A_n) \in (B_p(H))^n$ *and* $\mathbf{B} = (B_1, ..., B_n) \in (B_q(H))^n$ *for* $p, q > 1$ *with* $\frac{1}{p} + \frac{1}{q} = 1$, *then*

$$\begin{aligned}&\|D_n(\mathbf{A},\mathbf{B})\|_1\\ &\le \max_{k\in\{1,...,n-1\}} \|\Delta A_k\|_p\, D_n(\mathbf{e},\mathbf{V}_{\mathbf{B},q})\\ &\le \frac{1}{2}\left\lfloor\frac{n+1}{2}\right\rfloor\left(1-\frac{1}{n}\left\lfloor\frac{n+1}{2}\right\rfloor\right)\max_{k\in\{1,...,n-1\}}\|\Delta A_k\|_p\sum_{k=1}^{n-1}\|\Delta B_k\|_q\\ &\le \frac{1}{8}n\max_{k\in\{1,...,n-1\}}\|\Delta A_k\|_p\sum_{k=1}^{n-1}\|\Delta B_k\|_q,\end{aligned} \tag{2.6}$$

where $\mathbf{V}_{\mathbf{B},q} := \left(0, \|\Delta B_1\|_q, ..., \sum_{k=1}^{n-1}\|\Delta B_k\|_q\right)$.

In particular, if $\mathbf{A}$, $\mathbf{B}\in(B_2(H))^n$, *then*

$$\begin{aligned}\|D_n(\mathbf{A},\mathbf{B})\|_1 & \\ \leq & \max_{k\in\{1,...,n-1\}}\|\Delta A_k\|_p D_n(\mathbf{e},\mathbf{V}_{\mathbf{B},2}) \\ \leq & \frac{1}{2}\left\lfloor\frac{n+1}{2}\right\rfloor\left(1-\frac{1}{n}\left\lfloor\frac{n+1}{2}\right\rfloor\right)\max_{k\in\{1,...,n-1\}}\|\Delta A_k\|_2\sum_{k=1}^{n-1}\|\Delta B_k\|_2 \\ \leq & \frac{1}{8}n\max_{k\in\{1,...,n-1\}}\|\Delta A_k\|_2\sum_{k=1}^{n-1}\|\Delta B_k\|_2 ,\end{aligned}\tag{2.7}$$

where $\mathbf{V}_{\mathbf{B},2}:=\left(0,\|\Delta B_1\|_2,...,\sum_{k=1}^{n-1}\|\Delta B_k\|_2\right)$.

Proof. Let us start with the following identity which can be proved by direct computation for $\mathbf{A}=(A_1,...,A_n)\in(B_p(H))^n$ and $\mathbf{B}=(B_1,...,B_n)\in(B_q(H))^n$

$$\begin{aligned}D_n(\mathbf{A},\mathbf{B}) &= \frac{1}{2n^2}\sum_{i,j=1}^{n}(A_j-A_i)(B_j-B_i) \\ &= \frac{1}{n^2}\sum_{1\leq i<j\leq n}(A_j-A_i)(B_j-B_i).\end{aligned}\tag{2.8}$$

As $i<j$ we can write that $A_j-A_i=\sum_{k=i}^{j-1}\Delta A_k$ and $B_j-B_i=\sum_{k=i}^{j-1}\Delta B_k$. Using the generalized triangle inequality and the property (1.15) we have successively from (2.8) that:

$$\begin{aligned}&\|D_n(\mathbf{A},\mathbf{B})\|_1 \\ &= \frac{1}{n^2}\left\|\sum_{1\leq i<j\leq n}\sum_{k=i}^{j-1}\Delta A_k\sum_{k=i}^{j-1}\Delta B_k\right\|_1 \leq \frac{1}{n^2}\sum_{1\leq i<j\leq n}\left\|\sum_{k=i}^{j-1}\Delta A_k\sum_{k=i}^{j-1}\Delta B_k\right\|_1 \\ &\leq \frac{1}{n^2}\sum_{1\leq i<j\leq n}\left\|\sum_{k=i}^{j-1}\Delta A_k\right\|_p\left\|\sum_{k=i}^{j-1}\Delta B_k\right\|_q \\ &\leq \frac{1}{n^2}\sum_{1\leq i<j\leq n}\sum_{k=i}^{j-1}\|\Delta A_k\|_p\sum_{k=i}^{j-1}\|\Delta B_k\|_q .\end{aligned}\tag{2.9}$$

Since $\sum_{k=i}^{j-1}\|\Delta A_k\|_p \le (j-i)\max_{k\in\{i,\dots,j-1\}}\|\Delta A_k\|_p$ for $j>i$, hence by (2.9)

$$\begin{aligned}
\|D_n(\mathbf{A},\mathbf{B})\|_1 &\le \frac{1}{n^2}\sum_{1\le i<j\le n}(j-i)\max_{k\in\{i,\dots,j-1\}}\|\Delta A_k\|_p\sum_{k=i}^{j-1}\|\Delta B_k\|_q \\
&\le \frac{1}{n^2}\max_{k\in\{1,\dots,n-1\}}\|\Delta A_k\|_p\sum_{1\le i<j\le n}(j-i)\sum_{k=i}^{j-1}\|\Delta B_k\|_q \\
&= \frac{1}{n^2}\max_{k\in\{1,\dots,n-1\}}\|\Delta A_k\|_p\frac{1}{2}\sum_{i,j=1}^{n}(j-i)\sum_{k=i}^{j-1}\|\Delta B_k\|_q \\
&= \max_{k\in\{1,\dots,n-1\}}\|\Delta A_k\|_p\frac{1}{2}\frac{1}{n^2}\sum_{i,j=1}^{n}(j-i)\left(\sum_{k=1}^{j-1}\|\Delta B_k\|_q-\sum_{k=1}^{i-1}\|\Delta B_k\|_q\right) \\
&= \max_{k\in\{1,\dots,n-1\}}\|\Delta A_k\|_p D_n(\mathbf{e},\mathbf{V}_{\mathbf{B},q}),
\end{aligned}$$

which proves the first inequality in (2.6).

By utilising (2.4) we derive

$$\begin{aligned}
D_n(\mathbf{e},\mathbf{V}_{\mathbf{B},q}) &\le \frac{1}{2}\left\lfloor\frac{n+1}{2}\right\rfloor\left(1-\frac{1}{n}\left\lfloor\frac{n+1}{2}\right\rfloor\right)\sum_{k=1}^{n-1}\|\Delta B_k\|_q \\
&\le \frac{1}{8}n\sum_{k=1}^{n-1}\|\Delta B_k\|_q,
\end{aligned}$$

which proves the last part of 2.6 □

Further, we can state the following result as well:

Theorem 4. *If* $\mathbf{A}=(A_1,\dots,A_n)\in(B_p(H))^n$ *and* $\mathbf{B}=(B_1,\dots,B_n)\in(B_q(H))^n$ *for* $p,\ q>1$ *with* $\frac{1}{p}+\frac{1}{q}=1$, *then for* $\alpha,\ \beta>1$ *with* $\frac{1}{\alpha}+\frac{1}{\beta}=1$ *we have*

$$\begin{aligned}
\|D_n(\mathbf{A},\mathbf{B})\|_1 &\le [D_n(\mathbf{V}_{\mathbf{A},p,\alpha},\mathbf{e})]^{1/\alpha}[D_n(\mathbf{V}_{\mathbf{B},q,\beta},\mathbf{e})]^{1/\beta} \\
&\le \frac{1}{2}\left\lfloor\frac{n+1}{2}\right\rfloor\left(1-\frac{1}{n}\left\lfloor\frac{n+1}{2}\right\rfloor\right) \\
&\quad\times\left(\sum_{k=1}^{n-1}\|\Delta A_k\|_p^\alpha\right)^{1/\alpha}\left(\sum_{k=1}^{n-1}\|\Delta B_k\|_q^\beta\right)^{1/\beta} \\
&\le \frac{1}{8}n\left(\sum_{k=1}^{n-1}\|\Delta A_k\|_p^\alpha\right)^{1/\alpha}\left(\sum_{k=1}^{n-1}\|\Delta B_k\|_q^\beta\right)^{1/\beta},
\end{aligned} \tag{2.10}$$

where

$$\mathbf{V}_{\mathbf{A},p,\alpha}:=\left(0,\|\Delta A_1\|_p^\alpha,\dots,\sum_{k=1}^{n-1}\|\Delta A_k\|_p^\alpha\right)$$

and

$$\mathbf{V}_{\mathbf{B},q,\beta}:=\left(0,\|\Delta B_1\|_q^\beta,\dots,\sum_{k=1}^{n-1}\|\Delta B_k\|_q^\beta\right).$$

Proof. We have by (2.9) that

$$\left\| D_n(\mathbf{A},\mathbf{B})\right\|_1 \le \frac{1}{n^2}\sum_{1\le i<j\le n}\sum_{k=i}^{j-1}\left\|\Delta A_k\right\|_p\sum_{k=i}^{j-1}\left\|\Delta B_k\right\|_q . \tag{2.11}$$

By Hölder's inequality for α, $\beta>1$ with $\frac{1}{\alpha}+\frac{1}{\beta}=1$ we have that

$$\sum_{k=i}^{j-1}\left\|\Delta A_k\right\|_p \le (j-i)^{1/\beta}\left(\sum_{k=i}^{j-1}\left\|\Delta A_k\right\|_p^{\alpha}\right)^{1/\alpha}$$

and

$$\sum_{k=i}^{j-1}\left\|\Delta B_k\right\|_q \le (j-i)^{1/\alpha}\left(\sum_{k=i}^{j-1}\left\|\Delta B_k\right\|_q^{\beta}\right)^{1/\beta},$$

which gives, by multiplication, that

$$\sum_{k=i}^{j-1}\left\|\Delta A_k\right\|_p\sum_{k=i}^{j-1}\left\|\Delta B_k\right\|_q \le (j-i)\left(\sum_{k=i}^{j-1}\left\|\Delta A_k\right\|_p^{\alpha}\right)^{1/\alpha}\left(\sum_{k=i}^{j-1}\left\|\Delta B_k\right\|_q^{\beta}\right)^{1/\beta}.$$

Then by (2.11) we get

$$\left\| D_n(\mathbf{A},\mathbf{B})\right\|_1 \le \frac{1}{n^2}\sum_{1\le i<j\le n}(j-i)\left(\sum_{k=i}^{j-1}\left\|\Delta A_k\right\|_p^{\alpha}\right)^{1/\alpha}\left(\sum_{k=i}^{j-1}\left\|\Delta B_k\right\|_q^{\beta}\right)^{1/\beta}. \tag{2.12}$$

By using Hölder's weighted inequality for α, $\beta>1$ with $\frac{1}{\alpha}+\frac{1}{\beta}=1$ we get

$$\begin{aligned}
&\frac{1}{n^2}\sum_{1\le i<j\le n}(j-i)\left(\sum_{k=i}^{j-1}\left\|\Delta A_k\right\|_p^{\alpha}\right)^{1/\alpha}\left(\sum_{k=i}^{j-1}\left\|\Delta B_k\right\|_q^{\beta}\right)^{1/\beta} \\
&\le \frac{1}{n^2}\left\{\sum_{1\le i<j\le n}(j-i)\left[\left(\sum_{k=i}^{j-1}\left\|\Delta A_k\right\|_p^{\alpha}\right)^{1/\alpha}\right]^{\alpha}\right\}^{1/\alpha} \\
&\times\left\{\sum_{1\le i<j\le n}(j-i)\left[\left(\sum_{k=i}^{j-1}\left\|\Delta B_k\right\|_q^{\beta}\right)^{1/\beta}\right]^{\beta}\right\}^{1/\beta}
\end{aligned} \tag{2.13}$$

$$\begin{aligned}
&= \frac{1}{n^2}\left\{\sum_{1\le i<j\le n}(j-i)\left(\sum_{k=i}^{j-1}\left\|\Delta A_k\right\|_p^{\alpha}\right)\right\}^{1/\alpha}\left\{\sum_{1\le i<j\le n}(j-i)\left(\sum_{k=i}^{j-1}\left\|\Delta B_k\right\|_q^{\beta}\right)\right\}^{1/\beta} \\
&= \left\{\frac{1}{2}\frac{1}{n^2}\sum_{i,j=1}^{n}(j-i)\sum_{k=i}^{j-1}\left\|\Delta A_k\right\|_p^{\alpha}\right\}^{1/\alpha}\left\{\frac{1}{2}\frac{1}{n^2}\sum_{i,j=1}^{n}(j-i)\sum_{k=i}^{j-1}\left\|\Delta B_k\right\|_q^{\beta}\right\}^{1/\beta} \\
&= \left[D_n\left(\mathbf{V}_{\mathbf{A},p,\alpha},\mathbf{e}\right)\right]^{1/\alpha}\left[D_n\left(\mathbf{V}_{\mathbf{B},q,\beta},\mathbf{e}\right)\right]^{1/\beta}.
\end{aligned}$$

This proves the first inequality in (2.10).

By the inequality (2.4) we have

$$\begin{aligned}\left|D_n\left(\mathbf{V}_{\mathbf{A},p,\alpha},\mathbf{e}\right)\right| &\leq \frac{1}{2}\left\lfloor\frac{n+1}{2}\right\rfloor\left(1-\frac{1}{n}\left\lfloor\frac{n+1}{2}\right\rfloor\right)\sum_{k=1}^{n-1}\left\|\Delta A_k\right\|_p^{\alpha} \\ &\leq \frac{1}{8}n\sum_{k=1}^{n-1}\left\|\Delta A_k\right\|_p^{\alpha}\end{aligned}$$

and

$$\begin{aligned}D_n\left(\mathbf{V}_{\mathbf{B},q,\beta},\mathbf{e}\right) &\leq \frac{1}{2}\left\lfloor\frac{n+1}{2}\right\rfloor\left(1-\frac{1}{n}\left\lfloor\frac{n+1}{2}\right\rfloor\right)\sum_{k=1}^{n-1}\left\|\Delta B_k\right\|_q^{\beta} \\ &\leq \frac{1}{8}n\sum_{k=1}^{n-1}\left\|\Delta B_k\right\|_q^{\beta}.\end{aligned}$$

These two inequalities imply

$$\begin{aligned}&\left[D_n\left(\mathbf{V}_{\mathbf{A},p,\alpha},\mathbf{e}\right)\right]^{1/\alpha}\left[D_n\left(\mathbf{V}_{\mathbf{B},q,\beta},\mathbf{e}\right)\right]^{1/\beta} \\ &\leq \frac{1}{2}\left\lfloor\frac{n+1}{2}\right\rfloor\left(1-\frac{1}{n}\left\lfloor\frac{n+1}{2}\right\rfloor\right)\left(\sum_{k=1}^{n-1}\left\|\Delta A_k\right\|_p^{\alpha}\right)^{1/\alpha}\left(\sum_{k=1}^{n-1}\left\|\Delta B_k\right\|_q^{\beta}\right)^{1/\beta} \\ &\leq \frac{1}{8}n\left(\sum_{k=1}^{n-1}\left\|\Delta A_k\right\|_p^{\alpha}\right)^{1/\alpha}\left(\sum_{k=1}^{n-1}\left\|\Delta B_k\right\|_q^{\beta}\right)^{1/\beta},\end{aligned}$$

which proves the last part of (2.10). □

Remark 1. *If*

$$\mathbf{V}_{\mathbf{A},2,\alpha}:=\left(0,\left\|\Delta A_1\right\|_2^{\alpha},...,\sum_{k=1}^{n-1}\left\|\Delta A_k\right\|_2^{\alpha}\right)$$

and

$$\mathbf{V}_{\mathbf{B},2,\beta}:=\left(0,\left\|\Delta B_1\right\|_2^{\beta},...,\sum_{k=1}^{n-1}\left\|\Delta B_k\right\|_2^{\beta}\right),$$

then for α, $\beta>1$ with $\frac{1}{\alpha}+\frac{1}{\beta}=1$ we have

$$\begin{aligned}\left\|D_n\left(\mathbf{A},\mathbf{B}\right)\right\|_1 &\leq \left[D_n\left(\mathbf{V}_{\mathbf{A},2,\alpha},\mathbf{e}\right)\right]^{1/\alpha}\left[D_n\left(\mathbf{V}_{\mathbf{B},2,\beta},\mathbf{e}\right)\right]^{1/\beta} \\ &\leq \frac{1}{2}\left\lfloor\frac{n+1}{2}\right\rfloor\left(1-\frac{1}{n}\left\lfloor\frac{n+1}{2}\right\rfloor\right) \\ &\quad\times\left(\sum_{k=1}^{n-1}\left\|\Delta A_k\right\|_2^{\alpha}\right)^{1/\alpha}\left(\sum_{k=1}^{n-1}\left\|\Delta B_k\right\|_2^{\beta}\right)^{1/\beta} \\ &\leq \frac{1}{8}n\left(\sum_{k=1}^{n-1}\left\|\Delta A_k\right\|_2^{\alpha}\right)^{1/\alpha}\left(\sum_{k=1}^{n-1}\left\|\Delta B_k\right\|_2^{\beta}\right)^{1/\beta}.\end{aligned} \tag{2.14}$$

Now, if we take $\alpha = p$ and $\beta = q$ in (2.10), then we get

$$\begin{aligned} \|D_n(\mathbf{A},\mathbf{B})\|_1 &\le [D_n(\mathbf{V}_{\mathbf{A},p,p},\mathbf{e})]^{1/p}[D_n(\mathbf{V}_{\mathbf{B},q,q},\mathbf{e})]^{1/q} \\ &\le \frac{1}{2}\left\lfloor\frac{n+1}{2}\right\rfloor\left(1-\frac{1}{n}\left\lfloor\frac{n+1}{2}\right\rfloor\right) \\ &\quad\times\left(\sum_{k=1}^{n-1}\|\Delta A_k\|_p^p\right)^{1/p}\left(\sum_{k=1}^{n-1}\|\Delta B_k\|_q^q\right)^{1/q} \\ &\le \frac{1}{8}n\left(\sum_{k=1}^{n-1}\|\Delta A_k\|_p^p\right)^{1/p}\left(\sum_{k=1}^{n-1}\|\Delta B_k\|_q^q\right)^{1/q}. \end{aligned} \tag{2.15}$$

By using the trace properties, we can state that

$$\begin{aligned} &|\mathrm{tr}(D_n(\mathbf{A},\mathbf{B}))| \\ &\le \frac{1}{2}\left\lfloor\frac{n+1}{2}\right\rfloor\left(1-\frac{1}{n}\left\lfloor\frac{n+1}{2}\right\rfloor\right) \\ &\quad\times\left[\mathrm{tr}\left(\sum_{k=1}^{n-1}|\Delta A_k|^p\right)\right]^{1/p}\left[\mathrm{tr}\left(\sum_{k=1}^{n-1}|\Delta B_k|^q\right)\right]^{1/q} \\ &\le \frac{1}{8}n\left[\mathrm{tr}\left(\sum_{k=1}^{n-1}|\Delta A_k|^p\right)\right]^{1/p}\left[\mathrm{tr}\left(\sum_{k=1}^{n-1}|\Delta B_k|^q\right)\right]^{1/q}, \end{aligned} \tag{2.16}$$

for p, $q > 1$ with $\frac{1}{p}+\frac{1}{q}=1$.

In particular, for $p = q = 2$ we get the trace inequality:

$$\begin{aligned} |\mathrm{tr}(D_n(\mathbf{A},\mathbf{B}))| &\le \frac{1}{2}\left\lfloor\frac{n+1}{2}\right\rfloor\left(1-\frac{1}{n}\left\lfloor\frac{n+1}{2}\right\rfloor\right) \\ &\quad\times\left[\mathrm{tr}\left(\sum_{k=1}^{n-1}|\Delta A_k|^2\right)\right]^{1/2}\left[\mathrm{tr}\left(\sum_{k=1}^{n-1}|\Delta B_k|^2\right)\right]^{1/2} \\ &\le \frac{1}{8}n\left[\mathrm{tr}\left(\sum_{k=1}^{n-1}|\Delta A_k|^2\right)\right]^{1/2}\left[\mathrm{tr}\left(\sum_{k=1}^{n-1}|\Delta B_k|^2\right)\right]^{1/2}. \end{aligned} \tag{2.17}$$

3. p-Schatten Norm Inequalities

We have

Theorem 5. *If $\mathbf{A} = (A_1, ..., A_n)$ and $\mathbf{B} = (B_1, ..., B_n) \in (B_p(H))^n$ for $p \ge 1$, then*

$$\begin{aligned} &\|D_n(\mathbf{A},\mathbf{B})\|_p \\ &\le \max_{k\in\{1,\dots,n-1\}}\|\Delta A_k\|_p D_n(\mathbf{e},\mathbf{V}_{\mathbf{B},p}) \\ &\le \frac{1}{2}\left\lfloor\frac{n+1}{2}\right\rfloor\left(1-\frac{1}{n}\left\lfloor\frac{n+1}{2}\right\rfloor\right)\max_{k\in\{1,\dots,n-1\}}\|\Delta A_k\|_p\sum_{k=1}^{n-1}\|\Delta B_k\|_p \\ &\le \frac{1}{8}n\max_{k\in\{1,\dots,n-1\}}\|\Delta A_k\|_p\sum_{k=1}^{n-1}\|\Delta B_k\|_p, \end{aligned} \tag{3.1}$$

where $\mathbf{V}_{\mathbf{B},p}:=\left(0,\left\|\Delta B_1\right\|_p,...,\sum_{k=1}^{n-1}\left\|\Delta B_k\right\|_p\right)$.

Proof. Using the generalized triangle inequality and the property (1.12) we have successively from (2.8) that:

$$\begin{aligned}
(3.2)\qquad & \left\|D_n\left(\mathbf{A},\mathbf{B}\right)\right\|_p \\
& = \frac{1}{n^2}\left\|\sum_{1\leq i<j\leq n}\sum_{k=i}^{j-1}\Delta A_k\sum_{k=i}^{j-1}\Delta B_k\right\|_p \leq \frac{1}{n^2}\sum_{1\leq i<j\leq n}\left\|\sum_{k=i}^{j-1}\Delta A_k\sum_{k=i}^{j-1}\Delta B_k\right\|_p \\
& \leq \frac{1}{n^2}\sum_{1\leq i<j\leq n}\left\|\sum_{k=i}^{j-1}\Delta A_k\right\|_p\left\|\sum_{k=i}^{j-1}\Delta B_k\right\|_p \\
& \leq \frac{1}{n^2}\sum_{1\leq i<j\leq n}\sum_{k=i}^{j-1}\left\|\Delta A_k\right\|_p\sum_{k=i}^{j-1}\left\|\Delta B_k\right\|_p .
\end{aligned}$$

Now, by utilising a similar argument to the one in the proof of Theorem 3 we derive the desired result (3.1). □

By utilising a similar argument to the one in Theorem 4 we can state:

Theorem 6. *If* $\mathbf{A}=(A_1,...,A_n)$ *and* $\mathbf{B}=(B_1,...,B_n)\in\left(B_p\left(H\right)\right)^n$ *for* $p\geq 1$, *then for* α, $\beta>1$ *with* $\frac{1}{\alpha}+\frac{1}{\beta}=1$ *we have*

$$\begin{aligned}
(3.3)\qquad \left\|D_n\left(\mathbf{A},\mathbf{B}\right)\right\|_p & \leq \left[D_n\left(\mathbf{V}_{\mathbf{A},p,\alpha},\mathbf{e}\right)\right]^{1/\alpha}\left[D_n\left(\mathbf{V}_{\mathbf{B},p,\beta},\mathbf{e}\right)\right]^{1/\beta} \\
& \leq \frac{1}{2}\left\lfloor\frac{n+1}{2}\right\rfloor\left(1-\frac{1}{n}\left\lfloor\frac{n+1}{2}\right\rfloor\right) \\
& \quad\times\left(\sum_{k=1}^{n-1}\left\|\Delta A_k\right\|_p^\alpha\right)^{1/\alpha}\left(\sum_{k=1}^{n-1}\left\|\Delta B_k\right\|_p^\beta\right)^{1/\beta} \\
& \leq \frac{1}{8}n\left(\sum_{k=1}^{n-1}\left\|\Delta A_k\right\|_p^\alpha\right)^{1/\alpha}\left(\sum_{k=1}^{n-1}\left\|\Delta B_k\right\|_p^\beta\right)^{1/\beta},
\end{aligned}$$

where

$$\mathbf{V}_{\mathbf{A},p,\alpha}:=\left(0,\left\|\Delta A_1\right\|_p^\alpha,...,\sum_{k=1}^{n-1}\left\|\Delta A_k\right\|_p^\alpha\right)$$

and

$$\mathbf{V}_{\mathbf{B},p,\beta}:=\left(0,\left\|\Delta B_1\right\|_p^\beta,...,\sum_{k=1}^{n-1}\left\|\Delta B_k\right\|_p^\beta\right).$$

We observe that for $\alpha=\beta=2$ we derive the inequality

$$\begin{aligned}
\left\|D_n(\mathbf{A},\mathbf{B})\right\|_p &\leq \left[D_n\left(\mathbf{V}_{\mathbf{A},p,\alpha},\mathbf{e}\right)\right]^{1/2}\left[D_n\left(\mathbf{V}_{\mathbf{B},p,2},\mathbf{e}\right)\right]^{1/2} \\
&\leq \frac{1}{2}\left\lfloor\frac{n+1}{2}\right\rfloor\left(1-\frac{1}{n}\left\lfloor\frac{n+1}{2}\right\rfloor\right) \\
&\quad\times\left(\sum_{k=1}^{n-1}\left\|\Delta A_k\right\|_p^2\right)^{1/2}\left(\sum_{k=1}^{n-1}\left\|\Delta B_k\right\|_p^2\right)^{1/2} \\
&\leq \frac{1}{8}n\left(\sum_{k=1}^{n-1}\left\|\Delta A_k\right\|_p^2\right)^{1/2}\left(\sum_{k=1}^{n-1}\left\|\Delta B_k\right\|_p^2\right)^{1/2},
\end{aligned} \tag{3.4}$$

where

$$\mathbf{V}_{\mathbf{A},p,2}:=\left(0,\left\|\Delta A_1\right\|_p^2,...,\sum_{k=1}^{n-1}\left\|\Delta A_k\right\|_p^2\right)$$

and

$$\mathbf{V}_{\mathbf{B},p,\beta}:=\left(0,\left\|\Delta B_1\right\|_p^2,...,\sum_{k=1}^{n-1}\left\|\Delta B_k\right\|_p^2\right).$$

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