p-SCHATTEN NORM INEQUALITIES FOR WEIGHTED ČEBYŠEV'S FUNCTIONAL VIA AN ANDRICA-BADEA RESULT

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Abstract. An operator $A \in \mathcal{B}(H)$ is said to belong to the *von Neumann-Schatten class* $\mathcal{B}_p(H)$, $1 \le p < \infty$ if the *p-Schatten norm* is finite

$$\|A\|_p := [\operatorname{tr}(|A|^p)]^{1/p} < \infty.$$

For a given probability density distribution $\mathbf{p} = (p_1, ..., p_n)$ we say that the set of indices $S \subset \{1, ..., n\}$ is *optimal*, if S minimizes the positive quantity

$$\left| \sum_{k \in S} p_k - \frac{1}{2} \right|.$$

For $\mathbf{A} = (A_1, ..., A_n)$, $\mathbf{B} = (B_1, ..., B_n) \in B(H)$ we define the functional

$$D_{\mathbf{p}}(\mathbf{A}, \mathbf{B}) := \sum_{k=1}^{n} p_k A_k B_k - \sum_{k=1}^{n} p_k A_k \sum_{k=1}^{n} p_k B_k.$$

If $p,\ q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, assume that $\{A_k\}_{k=0}^n \subset B_p(H)$, $\{B_k\}_{k=0}^n \subset B_q(H)$ are sequences of operators and let $\mathbf{p} = (p_1, ..., p_n)$ be a probability density distribution with $S \subset \{1, ..., n\}$ its optimal set of indices, then we have the inequality:

$$\begin{aligned} \|D_{\mathbf{p}}(\mathbf{A}, \mathbf{B})\|_1 &\le D_{\mathbf{p}}(\mathbf{V}_{\mathbf{A},p}, \mathbf{V}_{\mathbf{B},q}) \\ &\le \sum_{k \in S} p_k \left(1 - \sum_{k \in S} p_k\right) \sum_{k=1}^{n-1} \|\Delta A_k\|_p \sum_{k=1}^{n-1} \|\Delta B_k\|_q \\ &\le \frac{1}{4} \sum_{k=1}^{n-1} \|\Delta A_k\|_p \sum_{k=1}^{n-1} \|\Delta B_k\|_q, \end{aligned}$$

where

$$\mathbf{V}_{\mathbf{A},p} := \left(0, \|\Delta A_1\|_p, ..., \sum_{k=1}^{n-1} \|\Delta A_k\|_p\right), \mathbf{V}_{\mathbf{B},q} := \left(0, \|\Delta B_1\|_q, ..., \sum_{k=1}^{n-1} \|\Delta B_k\|_q\right)$$

and $\Delta B_i := B_{i+1} - B_i$ $(i = 1, ..., n-1)$ are the usual forward differences.

1. Introduction

Let $(\Omega, \mathcal{A}, \mu)$ be a measurable space consisting of a set Ω, a σ-algebra $\mathcal{A}$ of parts of Ω and a countably additive and positive measure μ on $\mathcal{A}$ with values in $\mathbb{R} \cup \{\infty\}$.

For a μ-measurable function $w : \Omega \to \mathbb{R}$, with $w(x) \ge 0$ for μ-a.e. $x \in \Omega$, consider the Lebesgue space $L_w(\Omega, \mu) := \{h : \Omega \to \mathbb{R},\ h$ is μ-measurable and $\int_\Omega w(x) |h(x)| d\mu(x) < \infty\}$. Assume $\int_\Omega w(x) d\mu(x) = 1$. In order to simplify

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the notation for the integrals, we do not write the variable, namely, instead of $\int_\Omega w(x)\,d\mu(x)$ we simply write $\int_\Omega w d\mu$.

If $h,\ k:\Omega\to\mathbb{R}$ are μ-measurable functions and $h,\ k,\ hk\in L_w(\Omega,\mu)$, then we may consider the weighted Čebyšev functional in the following form

$$D_w(h,k):=\int_\Omega whkd\mu-\int_\Omega whd\mu\int_\Omega wkd\mu.$$

The following result is known in the literature as the Grüss inequality, see for instance [8]:

$$|D_w(h,k)|\le\frac{1}{4}(\Gamma-\gamma)(\Delta-\delta), \tag{1.1}$$

provided

$$-\infty<\gamma\le h\le\Gamma<\infty,\quad -\infty<\delta\le k\le\Delta<\infty \tag{1.2}$$

μ-a.e. on Ω. The constant $\frac{1}{4}$ is sharp in the sense that it cannot be replaced by a smaller constant.

For classical and more recent upper bounds related to the Čebyšev functional see [4]-[8] and [11]-[19].

Assume that $\mathbf{a}=(a_1,...,a_n)$ and $\mathbf{b}=(b_1,...,b_n)$ are n-tuples of real numbers and $\mathbf{p}=(p_1,...,p_n)$ a probability density distribution, namely $p_k\ge0$ with $P_n:=\sum_{k=1}^n p_k=1$. We consider the *weighted Čebyšev's functional*

$$D_{\mathbf{p}}(\mathbf{a},\mathbf{b}):=\sum_{k=1}^n p_k a_k b_k-\sum_{k=1}^n p_k a_k\sum_{k=1}^n p_k b_k. \tag{1.3}$$

In the case when $\mathbf{p}=\left(\frac{1}{n},...,\frac{1}{n}\right)$ is the *uniform distribution* we also consider the *Čebyšev's functional*

$$D_n(\mathbf{a},\mathbf{b}):=\frac{1}{n}\sum_{k=1}^n a_k b_k-\frac{1}{n}\sum_{k=1}^n a_k\frac{1}{n}\sum_{k=1}^n b_k.$$

For a given probability density distribution $\mathbf{p}=(p_1,...,p_n)$ we say that the set of indices $S\subset\{1,...,n\}$ is *optimal*, if S minimizes the positive quantity

$$\left|\sum_{k\in S}p_k-\frac{1}{2}\right|.$$

We observe that for the uniform distribution $\mathbf{p}=\left(\frac{1}{n},...,\frac{1}{n}\right)$ and $S\subset\{1,...,n\}$ has m elements, then

$$\left|\sum_{k\in S}p_k-\frac{1}{2}\right|=\left|\frac{m}{n}-\frac{1}{2}\right|=\frac{1}{n}\left|m-\frac{1}{2}n\right|,$$

which is minimal if $m=\left\lfloor\frac{1}{2}n\right\rfloor$ where $\lfloor\cdot\rfloor$ is the *integer part function.*

In 1988, Andrica and Badea [1, Theorem 2] established the following weighted version of the Grüss inequality:

Theorem 1. *Assume that* $\mathbf{a}=(a_1,...,a_n)$ *and* $\mathbf{b}=(b_1,...,b_n)$ *are* n*-tuples of real numbers and so that* $-\infty<\gamma\le a_k\le\Gamma<\infty,\ -\infty<\delta\le b_k\le\Delta<\infty$

for all $k \in \{1,...,n\}$. *Let* $\mathbf{p} = (p_1,...,p_n)$ *be a probability density distribution and* $S \subset \{1,...,n\}$ *its optimal set of indices, then*

$$|D_{\mathbf{p}}(\mathbf{a},\mathbf{b})| \leq \sum_{k\in S} p_k \left(1 - \sum_{k\in S} p_k\right)(\Gamma-\gamma)(\Delta-\delta) \leq \frac{1}{4}(\Gamma-\gamma)(\Delta-\delta). \tag{1.4}$$

The case of uniform distribution is as follows and was obtained in 1950 by M. Biernacki, H. Pidek and C. Ryll-Nardzewski, [4]:

Corollary 1. *Let* $a = (a_1,...,a_n)$, $b = (b_1,...,b_n)$ *be two* n*-tuples of real numbers such that* $-\infty < \gamma \leq a_k \leq \Gamma < \infty$, $-\infty < \delta \leq b_k \leq \Delta < \infty$ *for all* $k \in \{1,...,n\}$. *Then one has the inequality:*

$$\begin{aligned} |D_n(\mathbf{a},\mathbf{b})| &\leq \frac{1}{n}\left[\frac{n}{2}\right]\left(1-\frac{1}{n}\left[\frac{n}{2}\right]\right)(\Gamma-\gamma)(\Delta-\delta) \\ &\leq \frac{1}{4}(\Gamma-\gamma)(\Delta-\delta). \end{aligned} \tag{1.5}$$

In order to extend these results for p-Schatten norms we need the following preparations.

Let $(H;\langle .,. \rangle)$ be a complex Hilbert space and $\mathcal{B}(H)$ the Banach algebra of all bounded linear operators on H. If $\{e_i\}_{i\in I}$ an orthonormal basis of H, we say that $A \in \mathcal{B}(H)$ is of *trace class* if

$$\|A\|_1 := \sum_{i\in I} \langle |A| e_i, e_i\rangle < \infty. \tag{1.6}$$

The definition of $\|A\|_1$ does not depend on the choice of the orthonormal basis $\{e_i\}_{i\in I}$. We denote by $\mathcal{B}_1(H)$ the set of trace class operators in $\mathcal{B}(H)$.

We define the *trace* of a trace class operator $A \in \mathcal{B}_1(H)$ to be

$$\operatorname{tr}(A) := \sum_{i\in I} \langle Ae_i, e_i\rangle, \tag{1.7}$$

where $\{e_i\}_{i\in I}$ an orthonormal basis of H. Note that this coincides with the usual definition of the trace if H is finite-dimensional. We observe that the series (1.7) converges absolutely and it is independent from the choice of basis.

The following result collects some properties of the trace:

Theorem 2. *We have:*

(i) If $A \in \mathcal{B}_1(H)$ *then* $A^* \in \mathcal{B}_1(H)$ *and*

$$\operatorname{tr}(A^*) = \overline{\operatorname{tr}(A)}; \tag{1.8}$$

(ii) If $A \in \mathcal{B}_1(H)$ *and* $T \in \mathcal{B}(H)$, *then* $AT, TA \in \mathcal{B}_1(H)$ *and*

$$\operatorname{tr}(AT) = \operatorname{tr}(TA) \text{ and } |\operatorname{tr}(AT)| \leq \|A\|_1 \|T\|; \tag{1.9}$$

(iii) $\operatorname{tr}(\cdot)$ *is a bounded linear functional on* $\mathcal{B}_1(H)$ *with* $\|\operatorname{tr}\| = 1$;

(iv) If $A, B \in \mathcal{B}_2(H)$ *then* $AB, BA \in \mathcal{B}_1(H)$ *and* $\operatorname{tr}(AB) = \operatorname{tr}(BA)$;

(v) $\mathcal{B}_{fin}(H)$, *the space of operators of finite rank, is a dense subspace of* $\mathcal{B}_1(H)$.

An operator $A \in \mathcal{B}(H)$ is said to belong to the *von Neumann-Schatten class* $\mathcal{B}_p(H)$, $1 \leq p < \infty$ if the *p-Schatten norm* is finite [22, p. 60-64]

$$\|A\|_p := [\operatorname{tr}(|A|^p)]^{1/p} < \infty.$$

For $1<p<q<\infty$ we have that

$$\mathcal{B}_1(H)\subset\mathcal{B}_p(H)\subset\mathcal{B}_q(H)\subset\mathcal{B}(H) \tag{1.10}$$

and

$$\|A\|_1\geq\|A\|_p\geq\|A\|_q\geq\|A\|. \tag{1.11}$$

For $p\geq 1$ the functional $\|\cdot\|_p$ is a *norm* on the $*$-ideal $\mathcal{B}_p(H)$ and $\left(\mathcal{B}_p(H),\|\cdot\|_p\right)$ is a Banach space.

Also, see for instance [22, p. 60-64],

$$\|A\|_p=\|A^*\|_p,\ A\in\mathcal{B}_p(H) \tag{1.12}$$

$$\|AB\|_p\leq\|A\|_p\|B\|_p,\ A,B\in\mathcal{B}_p(H) \tag{1.13}$$

and

$$\|AB\|_p\leq\|A\|_p\|B\|,\ \|BA\|_p\leq\|B\|\|A\|_p,\ A\in\mathcal{B}_p(H),\ B\in\mathcal{B}(H). \tag{1.14}$$

This implies that

$$\|CAB\|_p\leq\|C\|\|A\|_p\|B\|,\ A\in\mathcal{B}_p(H),\ B,\ C\in\mathcal{B}(H). \tag{1.15}$$

In terms of *p-Schatten norm* we have the *Hölder inequality* for $p,\ q>1$ with $\frac{1}{p}+\frac{1}{q}=1$

$$(|\operatorname{tr}(AB)|\leq)\,\|AB\|_1\leq\|A\|_p\|B\|_q,\ A\in\mathcal{B}_p(H),\ B\in\mathcal{B}_q(H). \tag{1.16}$$

For the theory of trace functionals and their applications the reader is referred to [21] and [22].

For some classical trace inequalities see [9], [10], and [20], which are continuations of the work of Bellman [2].

2. Sum-Bounds

Let $\mathbf{p}=(p_1,...,p_n)$ be a probability density distribution. For $\mathbf{A}=(A_1,...,A_n)$, $\mathbf{B}=(B_1,...,B_n)\in B(H)$ we define the functional

$$D_{\mathbf{p}}(\mathbf{A},\mathbf{B}):=\sum_{k=1}^{n}p_kA_kB_k-\sum_{k=1}^{n}p_kA_k\sum_{k=1}^{n}p_kB_k. \tag{2.1}$$

The following inequality of Grüss type holds.

Theorem 3. *For $p,\ q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, assume that $\{A_k\}_{k=0}^{n}\subset B_p(H)$, $\{B_k\}_{k=0}^{n}\subset B_q(H)$ are sequences of operators and let $\mathbf{p}=(p_1,...,p_n)$ be a probability density distribution with $S\subset\{1,...,n\}$ its optimal set of indices, then we have the inequality:*

$$\begin{aligned}\|D_{\mathbf{p}}(\mathbf{A},\mathbf{B})\|_1 &\leq D_{\mathbf{p}}(\mathbf{V}_{\mathbf{A},p},\mathbf{V}_{\mathbf{B},q})\\ &\leq\sum_{k\in S}p_k\left(1-\sum_{k\in S}p_k\right)\sum_{k=1}^{n-1}\|\Delta A_k\|_p\sum_{k=1}^{n-1}\|\Delta B_k\|_q\\ &\leq\frac{1}{4}\sum_{k=1}^{n-1}\|\Delta A_k\|_p\sum_{k=1}^{n-1}\|\Delta B_k\|_q,\end{aligned} \tag{2.2}$$

where

$$\mathbf{V}_{\mathbf{A},p}:=\left(0,\left\|\Delta A_1\right\|_p,...,\sum_{k=1}^{n-1}\left\|\Delta A_k\right\|_p\right),\ \mathbf{V}_{\mathbf{B},q}:=\left(0,\left\|\Delta B_1\right\|_q,...,\sum_{k=1}^{n-1}\left\|\Delta B_k\right\|_q\right)$$

and $\Delta A_k := A_{k+1} - A_k$ $(k=1,...,n-1)$ *are the usual forward differences.*

In particular, if $\{A_k\}_{k=0}^n, \{B_k\}_{k=0}^n \subset B_2(H)$, *then for* $p=q=2$ *we have*

$$\begin{aligned}\left\|D_{\mathbf{p}}(\mathbf{A},\mathbf{B})\right\|_1 &\le D_{\mathbf{p}}(\mathbf{V}_{\mathbf{A},2},\mathbf{V}_{\mathbf{B},2}) \\ &\le \sum_{k\in S} p_k\left(1-\sum_{k\in S}p_k\right)\sum_{k=1}^{n-1}\left\|\Delta A_k\right\|_2\sum_{k=1}^{n-1}\left\|\Delta B_k\right\|_2 \\ &\le \frac{1}{4}\sum_{k=1}^{n-1}\left\|\Delta A_k\right\|_2\sum_{k=1}^{n-1}\left\|\Delta B_k\right\|_2 .\end{aligned} \tag{2.3}$$

Proof. Let us start with the following identity in Banach algebra $B(H)$ which can be proved by direct computation

$$\begin{aligned}\sum_{k=1}^n p_k A_k B_k - \sum_{k=1}^n p_k A_k \sum_{k=1}^n p_k B_k &= \frac{1}{2}\sum_{i,j=1}^n p_i p_j (A_j - A_i)(B_j - B_i) \\ &= \sum_{1\le i<j\le n} p_i p_j (A_j - A_i)(B_j - B_i).\end{aligned}$$

As $i<j$, we can write $A_j - A_i = \sum_{k=i}^{j-1}\Delta A_k$ and $B_j - B_i = \sum_{l=i}^{j-1}\Delta B_l$. Using the generalized triangle inequality and the property (1.16), we have successively:

$$\begin{aligned}\left\|D_{\mathbf{p}}(\mathbf{A},\mathbf{B})\right\|_1 &= \left\|\sum_{1\le i<j\le n} p_i p_j \sum_{k=i}^{j-1}\Delta A_k \sum_{l=k}^{j-1}\Delta B_l\right\|_1 \\ &\le \sum_{1\le i<j\le n} p_i p_j \left\|\sum_{k=i}^{j-1}\Delta A_k \sum_{l=i}^{j-1}\Delta B_l\right\|_1 \\ &\le \sum_{1\le i<j\le n} p_i p_j \left\|\sum_{k=i}^{j-1}\Delta A_k\right\|_p \left\|\sum_{l=i}^{j-1}\Delta B_l\right\|_q \\ &\le \sum_{1\le i<j\le n} p_i p_j \sum_{k=i}^{j-1}\left\|\Delta A_k\right\|_p \sum_{l=i}^{j-1}\left\|\Delta B_l\right\|_q .\end{aligned} \tag{2.4}$$

Now, observe that

$$\begin{aligned}&\sum_{1\le i<j\le n} p_i p_j \sum_{k=i}^{j-1}\left\|\Delta A_k\right\|_p \sum_{l=i}^{j-1}\left\|\Delta B_l\right\|_q \\ &= \sum_{1\le i<j\le n} p_i p_j \left(\sum_{k=1}^{j-1}\left\|\Delta A_k\right\|_p - \sum_{k=1}^{i-1}\left\|\Delta A_k\right\|_p\right)\left(\sum_{l=1}^{j-1}\left\|\Delta B_l\right\|_q - \sum_{l=1}^{i-1}\left\|\Delta B_l\right\|_q\right) \\ &= \frac{1}{2}\sum_{i,j=1}^n p_i p_j \left(\sum_{k=1}^{j-1}\left\|\Delta A_k\right\|_p - \sum_{k=1}^{i-1}\left\|\Delta A_k\right\|_p\right)\left(\sum_{l=1}^{j-1}\left\|\Delta B_l\right\|_q - \sum_{l=1}^{i-1}\left\|\Delta B_l\right\|_q\right) \\ &= D_{\mathbf{p}}(\mathbf{V}_{\mathbf{A},p},\mathbf{V}_{\mathbf{B},q}),\end{aligned}$$

which prove the first inequality in (2.2).

Now, if we use the Andrica-Badea inequality (1.4) for $(\mathbf{V}_{\mathbf{A},p},\mathbf{V}_{\mathbf{B},q})$ we have

$$\begin{aligned}D_{\mathbf{p}}\left(\mathbf{V}_{\mathbf{A},p},\mathbf{V}_{\mathbf{B},q}\right) &\leq \sum_{k\in S}p_k\left(1-\sum_{k\in S}p_k\right)\sum_{k=1}^{n-1}\left\|\Delta A_k\right\|_p\sum_{k=1}^{n-1}\left\|\Delta B_k\right\|_q \\ &\leq \frac{1}{4}\sum_{k=1}^{n-1}\left\|\Delta A_k\right\|_p\sum_{k=1}^{n-1}\left\|\Delta B_k\right\|_q,\end{aligned}$$

which proves the second part of (2.2). □

For $\mathbf{A}=(A_1,...,A_n)$, $\mathbf{B}=(B_1,...,B_n)\in B(H)$ we define the functional

$$D_n(\mathbf{A},\mathbf{B}) := \frac{1}{n}\sum_{k=1}^{n}A_kB_k-\frac{1}{n}\sum_{k=1}^{n}A_k\frac{1}{n}\sum_{k=1}^{n}B_k.$$

Corollary 2. *For p, $q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, assume that $\{A_k\}_{k=0}^n\subset B_p(H)$, $\{B_k\}_{k=0}^n\subset B_q(H)$ are sequences of operators, then*

$$\begin{aligned}\left\|D_n(\mathbf{A},\mathbf{B})\right\|_1 &\leq D_n\left(\mathbf{V}_{\mathbf{A},p},\mathbf{V}_{\mathbf{B},q}\right) \\ &\leq \frac{1}{n}\left[\frac{n}{2}\right]\left(1-\frac{1}{n}\left[\frac{n}{2}\right]\right)\sum_{k=1}^{n-1}\left\|\Delta A_k\right\|_p\sum_{k=1}^{n-1}\left\|\Delta B_k\right\|_q \\ &\leq \frac{1}{4}\sum_{k=1}^{n-1}\left\|\Delta A_k\right\|_p\sum_{k=1}^{n-1}\left\|\Delta B_k\right\|_q.\end{aligned}\tag{2.5}$$

In particular, if $\{A_k\}_{k=0}^n$, $\{B_k\}_{k=0}^n\subset B_2(H)$, then for $p=q=2$ we have

$$\begin{aligned}\left\|D_n(\mathbf{A},\mathbf{B})\right\|_1 &\leq D_n\left(\mathbf{V}_{\mathbf{A},2},\mathbf{V}_{\mathbf{B},2}\right) \\ &\leq \frac{1}{n}\left[\frac{n}{2}\right]\left(1-\frac{1}{n}\left[\frac{n}{2}\right]\right)\sum_{k=1}^{n-1}\left\|\Delta A_k\right\|_2\sum_{k=1}^{n-1}\left\|\Delta B_k\right\|_2 \\ &\leq \frac{1}{4}\sum_{k=1}^{n-1}\left\|\Delta A_k\right\|_2\sum_{k=1}^{n-1}\left\|\Delta B_k\right\|_2.\end{aligned}\tag{2.6}$$

We also have:

Theorem 4. *For $p\geq 1$, assume that $\{A_k\}_{k=0}^n$, $\{B_k\}_{k=0}^n\subset B_p(H)$ are sequences of operators and let $\mathbf{p}=(p_1,...,p_n)$ be a probability density distribution with $S\subset\{1,...,n\}$ its optimal set of indices, then we have the inequality:*

$$\begin{aligned}\left\|D_{\mathbf{p}}(\mathbf{A},\mathbf{B})\right\|_p &\leq D_{\mathbf{p}}\left(\mathbf{V}_{\mathbf{A},p},\mathbf{V}_{\mathbf{B},p}\right) \\ &\leq \sum_{k\in S}p_k\left(1-\sum_{k\in S}p_k\right)\sum_{k=1}^{n-1}\left\|\Delta A_k\right\|_p\sum_{k=1}^{n-1}\left\|\Delta B_k\right\|_p \\ &\leq \frac{1}{4}\sum_{k=1}^{n-1}\left\|\Delta A_k\right\|_p\sum_{k=1}^{n-1}\left\|\Delta B_k\right\|_p.\end{aligned}\tag{2.7}$$

Proof. As in the proof of (2.4) and by using property (1.13) we get

$$\begin{aligned}\|D_{\mathbf{p}}(\mathbf{A},\mathbf{B})\|_p &= \left\|\sum_{1\le i<j\le n} p_i p_j \sum_{k=i}^{j-1}\Delta A_k \sum_{l=k}^{j-1}\Delta B_l\right\|_p \\ &\le \sum_{1\le i<j\le n} p_i p_j \left\|\sum_{k=i}^{j-1}\Delta A_k \sum_{l=i}^{j-1}\Delta B_l\right\|_p \\ &\le \sum_{1\le i<j\le n} p_i p_j \left\|\sum_{k=i}^{j-1}\Delta A_k\right\|_p \left\|\sum_{l=i}^{j-1}\Delta B_l\right\|_p \\ &\le \sum_{1\le i<j\le n} p_i p_j \sum_{k=i}^{j-1}\|\Delta A_k\|_p \sum_{l=i}^{j-1}\|\Delta B_l\|_p .\end{aligned}$$

Now, by utilising a similar proof as the one in Theorem 3 we derive the desired result (2.7). □

Remark 1. *The case of uniform distribution gives that*

$$\begin{aligned}\|D_n(\mathbf{A},\mathbf{B})\|_p &\le D_n(\mathbf{V}_{\mathbf{A},p},\mathbf{V}_{\mathbf{B},p}) \\ &\le \frac{1}{n}\left[\frac{n}{2}\right]\left(1-\frac{1}{n}\left[\frac{n}{2}\right]\right)\sum_{k=1}^{n-1}\|\Delta A_k\|_p \sum_{k=1}^{n-1}\|\Delta B_k\|_p \\ &\le \frac{1}{4}\sum_{k=1}^{n-1}\|\Delta A_k\|_p \sum_{k=1}^{n-1}\|\Delta B_k\|_p ,\end{aligned} \tag{2.8}$$

provided that $\{A_k\}_{k=0}^n, \{B_k\}_{k=0}^n \subset B_p(H)$ *for* $p \ge 1$.

3. Max-Bounds

From a different perspective we have:

Theorem 5. *For* $p, q > 1$ *with* $\frac{1}{p}+\frac{1}{q}=1$, *assume that* $\{A_k\}_{k=0}^n \subset B_p(H)$, $\{B_k\}_{k=0}^n \subset B_q(H)$ *are sequences of operators and let* $\mathbf{p}=(p_1,...,p_n)$ *be a probability density distribution with* $S \subset \{1,...,n\}$ *its optimal set of indices, then we have the inequality:*

$$\begin{aligned}\|D_{\mathbf{p}}(\mathbf{A},\mathbf{B})\|_1 &\le \max_{k\in\{1,\dots,n-1\}}\|\Delta A_k\|_p D_{\mathbf{p}}(\mathbf{e},\mathbf{V}_{\mathbf{B},q}) \\ &\le \left[\sum_{k\in S} p_k\left(1-\sum_{k\in S}p_k\right)\right]^{1/2}[D_{\mathbf{p}}(\mathbf{e},\mathbf{e})]^{1/2} \\ &\quad\times \max_{k\in\{1,\dots,n-1\}}\|\Delta A_k\|_p\left(\sum_{k=1}^{n-1}\|\Delta B_k\|_q\right) \\ &\le \frac{1}{2}[D_{\mathbf{p}}(\mathbf{e},\mathbf{e})]^{1/2}\max_{k\in\{1,\dots,n-1\}}\|\Delta A_k\|_p\left(\sum_{k=1}^{n-1}\|\Delta B_k\|_q\right),\end{aligned} \tag{3.1}$$

where

$$D_{\mathbf{p}}(\mathbf{e},\mathbf{e}) = \sum_{k=1}^n k^2 p_k - \left(\sum_{k=1}^n k p_k\right)^2 .$$

In particular, if $\{A_k\}_{k=0}^n, \{B_k\}_{k=0}^n \subset B_2(H)$, *then for* $p = q = 2$ *we have*

$$\begin{aligned}
\|D_{\mathbf{p}}(\mathbf{A},\mathbf{B})\|_1 &\le \max_{k\in\{1,\dots,n-1\}} \|\Delta A_k\|_2 \, D_{\mathbf{p}}(\mathbf{e},\mathbf{V}_{\mathbf{B},2}) \\
&\le \left[\sum_{k\in S} p_k \left(1-\sum_{k\in S} p_k\right)\right]^{1/2} [D_{\mathbf{p}}(\mathbf{e},\mathbf{e})]^{1/2} \\
&\quad\times \max_{k\in\{1,\dots,n-1\}} \|\Delta A_k\|_2 \left(\sum_{k=1}^{n-1} \|\Delta B_k\|_2\right) \\
&\le \frac{1}{2} [D_{\mathbf{p}}(\mathbf{e},\mathbf{e})]^{1/2} \max_{k\in\{1,\dots,n-1\}} \|\Delta A_k\|_2 \left(\sum_{k=1}^{n-1} \|\Delta B_k\|_2\right).
\end{aligned} \tag{3.2}$$

Proof. From (2.4) we also derive

$$\begin{aligned}
&\|D_{\mathbf{p}}(\mathbf{A},\mathbf{B})\|_1 \\
&\le \sum_{1\le i<j\le n} p_i p_j \sum_{k=i}^{j-1} \|\Delta A_k\|_p \sum_{l=i}^{j-1} \|\Delta B_l\|_q \\
&\le \sum_{1\le i<j\le n} p_i p_j (j-i) \max_{k\in\{i,\dots,j-1\}} \|\Delta A_k\|_p \sum_{l=i}^{j-1} \|\Delta B_l\|_q \\
&\le \max_{k\in\{1,\dots,n-1\}} \|\Delta A_k\|_p \sum_{1\le i<j\le n} p_i p_j (j-i) \sum_{l=i}^{j-1} \|\Delta B_l\|_q \\
&= \max_{k\in\{1,\dots,n-1\}} \|\Delta A_k\|_p \sum_{1\le i<j\le n} p_i p_j (j-i) \left(\sum_{l=1}^{j-1} \|\Delta B_l\|_q - \sum_{l=1}^{i-1} \|\Delta B_l\|_q\right) \\
&= \max_{k\in\{1,\dots,n-1\}} \|\Delta A_k\|_p \frac{1}{2} \sum_{i,j=1}^{n} p_i p_j (j-i) \left(\sum_{l=1}^{j-1} \|\Delta B_l\|_q - \sum_{l=1}^{i-1} \|\Delta B_l\|_q\right) \\
&= \max_{k\in\{1,\dots,n-1\}} \|\Delta A_k\|_p \, D_{\mathbf{p}}(\mathbf{e},\mathbf{V}_{\mathbf{B},q}),
\end{aligned} \tag{3.3}$$

where $\mathbf{e} = (1, 2, ..., n)$. This proves the first inequality in (3.1).

We use now the *pre-Grüss inequality*

$$|D_{\mathbf{p}}(\mathbf{a},\mathbf{b})| \le [D_{\mathbf{p}}(\mathbf{a},\mathbf{a})]^{1/2} [D_{\mathbf{p}}(\mathbf{b},\mathbf{b})]^{1/2} \tag{3.4}$$

to get

$$D_{\mathbf{p}}(\mathbf{e},\mathbf{V}_{\mathbf{B},q}) \le [D_{\mathbf{p}}(\mathbf{e},\mathbf{e})]^{1/2} [D_{\mathbf{p}}(\mathbf{V}_{\mathbf{B},q},\mathbf{V}_{\mathbf{B},q})]^{1/2}.$$

Now by Andrica-Badea inequality we have

$$\begin{aligned}
D_{\mathbf{p}}(\mathbf{V}_{\mathbf{B},q},\mathbf{V}_{\mathbf{B},q}) &\le \sum_{k\in S} p_k \left(1-\sum_{k\in S} p_k\right) \left(\sum_{k=1}^{n-1} \|\Delta B_k\|_q\right)^2 \\
&\le \frac{1}{4} \left(\sum_{k=1}^{n-1} \|\Delta B_k\|_q\right)^2,
\end{aligned}$$

which proves the last part of (3.1). □

We have for $\mathbf{e} = (1, 2, ..., n)$ that

$$D_n(\mathbf{e}, \mathbf{e}) := \frac{1}{n}\sum_{i=1}^{n} i^2 - \frac{n+1}{2}\frac{1}{n}\sum_{i=1}^{n} i$$
$$= \frac{(n+1)(2n+1)}{6} - \frac{(n+1)^2}{4} = \frac{1}{12}\left(n^2 - 1\right).$$

This implies that

$$[D_n(\mathbf{e}, \mathbf{e})]^{1/2} = \frac{\sqrt{3}}{6}\left(n^2 - 1\right)^{1/2}.$$

Corollary 3. *For p, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, assume that $\{A_k\}_{k=0}^{n} \subset B_p(H)$, $\{B_k\}_{k=0}^{n} \subset B_q(H)$ are sequences of operators, then*

$$\|D_n(\mathbf{A}, \mathbf{B})\|_1 \le \max_{k\in\{1,\dots,n-1\}} \|\Delta A_k\|_p D_n(\mathbf{e}, \mathbf{V}_{\mathbf{B},q}) \tag{3.5}$$
$$\le \frac{\sqrt{3}}{6}\left(n^2 - 1\right)^{1/2}\left[\frac{1}{n}\left[\frac{n}{2}\right]\left(1 - \frac{1}{n}\left[\frac{n}{2}\right]\right)\right]^{1/2}$$
$$\times \max_{k\in\{1,\dots,n-1\}} \|\Delta A_k\|_p \sum_{k=1}^{n-1}\|\Delta B_k\|_q$$
$$\le \frac{\sqrt{3}}{12}\left(n^2 - 1\right)^{1/2} \max_{k\in\{1,\dots,n-1\}} \|\Delta A_k\|_p \sum_{k=1}^{n-1}\|\Delta B_k\|_q.$$

In particular, if $\{A_k\}_{k=0}^{n}$, $\{B_k\}_{k=0}^{n} \subset B_2(H)$, then for $p = q = 2$ we have

$$\|D_n(\mathbf{A}, \mathbf{B})\|_1 \le \max_{k\in\{1,\dots,n-1\}} \|\Delta A_k\|_2 D_n(\mathbf{e}, \mathbf{V}_{\mathbf{B},2}) \tag{3.6}$$
$$\le \frac{\sqrt{3}}{6}\left(n^2 - 1\right)^{1/2}\left[\frac{1}{n}\left[\frac{n}{2}\right]\left(1 - \frac{1}{n}\left[\frac{n}{2}\right]\right)\right]^{1/2}$$
$$\times \max_{k\in\{1,\dots,n-1\}} \|\Delta A_k\|_2 \sum_{k=1}^{n-1}\|\Delta B_k\|_2$$
$$\le \frac{\sqrt{3}}{12}\left(n^2 - 1\right)^{1/2} \max_{k\in\{1,\dots,n-1\}} \|\Delta A_k\|_2 \sum_{k=1}^{n-1}\|\Delta B_k\|_2.$$

We also have:

Theorem 6. *For $p \ge 1$, assume that $\{A_k\}_{k=0}^{n}$, $\{B_k\}_{k=0}^{n} \subset B_p(H)$ are sequences of operators and let $\mathbf{p} = (p_1, ..., p_n)$ be a probability density distribution with $S \subset$*

$\{1,...,n\}$ *its optimal set of indices, then we have the inequality:*

$$\begin{aligned}(3.7)\qquad \|D_{\mathbf{p}}(\mathbf{A},\mathbf{B})\|_p &\leq \max_{k\in\{1,...,n-1\}}\|\Delta A_k\|_p\, D_{\mathbf{p}}(\mathbf{e},\mathbf{V}_{\mathbf{B},p})\\ &\leq \left[\sum_{k\in S}p_k\left(1-\sum_{k\in S}p_k\right)\right]^{1/2}[D_{\mathbf{p}}(\mathbf{e},\mathbf{e})]^{1/2}\\ &\times \max_{k\in\{1,...,n-1\}}\|\Delta A_k\|_p\left(\sum_{k=1}^{n-1}\|\Delta B_k\|_p\right)\\ &\leq \frac{1}{2}[D_{\mathbf{p}}(\mathbf{e},\mathbf{e})]^{1/2}\max_{k\in\{1,...,n-1\}}\|\Delta A_k\|_p\left(\sum_{k=1}^{n-1}\|\Delta B_k\|_p\right).\end{aligned}$$

Remark 2. *The case of uniform distribution gives that*

$$\begin{aligned}(3.8)\qquad \|D_n(\mathbf{A},\mathbf{B})\|_p &\leq \max_{k\in\{1,...,n-1\}}\|\Delta A_k\|_p\, D_n(\mathbf{e},\mathbf{V}_{\mathbf{B},p})\\ &\leq \frac{\sqrt{3}}{6}\left(n^2-1\right)^{1/2}\left[\frac{1}{n}\left[\frac{n}{2}\right]\left(1-\frac{1}{n}\left[\frac{n}{2}\right]\right)\right]^{1/2}\\ &\times \max_{k\in\{1,...,n-1\}}\|\Delta A_k\|_p\sum_{k=1}^{n-1}\|\Delta B_k\|_p\\ &\leq \frac{\sqrt{3}}{12}\left(n^2-1\right)^{1/2}\max_{k\in\{1,...,n-1\}}\|\Delta A_k\|_p\sum_{k=1}^{n-1}\|\Delta B_k\|_p,\end{aligned}$$

provided that $\{A_k\}_{k=0}^n, \{B_k\}_{k=0}^n \subset B_p(H)$.

4. Hölder Type Bounds

Further, we have:

Theorem 7. *For* $p,\ q>1$ *with* $\frac{1}{p}+\frac{1}{q}=1$, *assume that* $\{A_k\}_{k=0}^n\subset B_p(H)$, $\{B_k\}_{k=0}^n\subset B_q(H)$ *are sequences of operators and let* $\mathbf{p}=(p_1,...,p_n)$ *be a probability density distribution with* $S\subset\{1,...,n\}$ *its optimal set of indices. If* $\alpha,\ \beta>1$ *with* $\frac{1}{\alpha}+\frac{1}{\beta}=1$, *then we have the inequality:*

$$\begin{aligned}(4.1)\qquad \|D_{\mathbf{p}}(\mathbf{A},\mathbf{B})\|_1 &\leq [D_{\mathbf{p}}(\mathbf{e},\mathbf{V}_{\mathbf{A},p,\alpha})]^{1/\alpha}[D_{\mathbf{p}}(\mathbf{e},\mathbf{V}_{\mathbf{B},q,\beta})]^{1/\beta}\\ &\leq [D_{\mathbf{p}}(\mathbf{e},\mathbf{e})]^{1/2}\left[\sum_{k\in S}p_k\left(1-\sum_{k\in S}p_k\right)\right]^{1/2}\\ &\times\left(\sum_{k=1}^{n-1}\|\Delta A_k\|_p^{\alpha}\right)^{1/\alpha}\left(\sum_{k=1}^{n-1}\|\Delta B_k\|_q^{\beta}\right)^{1/\beta}\\ &\leq\frac{1}{2}[D_{\mathbf{p}}(\mathbf{e},\mathbf{e})]^{1/2}\left(\sum_{k=1}^{n-1}\|\Delta A_k\|_p^{\alpha}\right)^{1/\alpha}\left(\sum_{k=1}^{n-1}\|\Delta B_k\|_q^{\beta}\right)^{1/\beta},\end{aligned}$$

where

$$\mathbf{V}_{\mathbf{A},p,\alpha} \quad : \quad = \left(0, \|\Delta A_1\|_p^\alpha, ..., \sum_{k=1}^{n-1} \|\Delta A_k\|_p^\alpha\right),$$
$$\mathbf{V}_{\mathbf{B},q,\beta} \quad : \quad = \left(0, \|\Delta B_1\|_q^\beta, ..., \sum_{k=1}^{n-1} \|\Delta B_k\|_q^\beta\right).$$

In particular, if $\{A_k\}_{k=0}^n, \{B_k\}_{k=0}^n \subset B_2(H)$, *then*

$$\begin{aligned} \|D_{\mathbf{p}}(\mathbf{A},\mathbf{B})\|_1 &\le [D_{\mathbf{p}}(\mathbf{e},\mathbf{V}_{\mathbf{A},2,\alpha})]^{1/\alpha}[D_{\mathbf{p}}(\mathbf{e},\mathbf{V}_{\mathbf{B},2,\beta})]^{1/\beta} \\ &\le [D_{\mathbf{p}}(\mathbf{e},\mathbf{e})]^{1/2}\left[\sum_{k\in S} p_k\left(1-\sum_{k\in S} p_k\right)\right]^{1/2} \\ &\times \left(\sum_{k=1}^{n-1}\|\Delta A_k\|_2^\alpha\right)^{1/\alpha}\left(\sum_{k=1}^{n-1}\|\Delta B_k\|_2^\beta\right)^{1/\beta} \\ &\le \frac{1}{2}[D_{\mathbf{p}}(\mathbf{e},\mathbf{e})]^{1/2}\left(\sum_{k=1}^{n-1}\|\Delta A_k\|_2^\alpha\right)^{1/\alpha}\left(\sum_{k=1}^{n-1}\|\Delta B_k\|_2^\beta\right)^{1/\beta}. \end{aligned} \tag{4.2}$$

Proof. As above we have

$$\|D_{\mathbf{p}}(\mathbf{A},\mathbf{B})\|_1 \le \sum_{1\le i<j\le n} p_i p_j \sum_{k=i}^{j-1}\|\Delta A_k\|_p \sum_{l=i}^{j-1}\|\Delta B_l\|_q. \tag{4.3}$$

By Hölder's inequality for α, $\beta > 1$ with $\frac{1}{\alpha}+\frac{1}{\beta}=1$ we have for $j > i$ that

$$\sum_{k=i}^{j-1}\|\Delta A_k\|_p \le (j-i)^{1/\beta}\left(\sum_{k=i}^{j-1}\|\Delta A_k\|_p^\alpha\right)^{1/\alpha}$$

and

$$\sum_{k=i}^{j-1}\|\Delta B_k\|_q \le (j-i)^{1/\alpha}\left(\sum_{k=i}^{j-1}\|\Delta B_k\|_q^\beta\right)^{1/\beta},$$

which gives, by multiplication, that

$$\sum_{k=i}^{j-1}\|\Delta A_k\|_p \sum_{k=i}^{j-1}\|\Delta B_k\|_q \le (j-i)\left(\sum_{k=i}^{j-1}\|\Delta A_k\|_p^\alpha\right)^{1/\alpha}\left(\sum_{k=i}^{j-1}\|\Delta B_k\|_q^\beta\right)^{1/\beta}.$$

Therefore, by weighted Hölder's inequality we can state

$$\sum_{1\le i<j\le n} p_i p_j \sum_{k=i}^{j-1} \|\Delta A_k\|_p \sum_{l=i}^{j-1} \|\Delta B_l\|_q \tag{4.4}$$
$$\le \sum_{1\le i<j\le n} p_i p_j (j-i) \left(\sum_{k=i}^{j-1} \|\Delta A_k\|_p^\alpha\right)^{1/\alpha} \left(\sum_{k=i}^{j-1} \|\Delta B_k\|_q^\beta\right)^{1/\beta}$$
$$\le \left[\sum_{1\le i<j\le n} p_i p_j (j-i) \left[\left(\sum_{k=i}^{j-1} \|\Delta A_k\|_p^\alpha\right)^{1/\alpha}\right]^\alpha\right]^{1/\alpha}$$
$$\times \left[\sum_{1\le i<j\le n} p_i p_j (j-i) \left[\left(\sum_{k=i}^{j-1} \|\Delta B_k\|_q^\beta\right)^{1/\beta}\right]^\beta\right]^{1/\beta}$$
$$= \left[\sum_{1\le i<j\le n} p_i p_j (j-i) \sum_{k=i}^{j-1} \|\Delta A_k\|_p^\alpha\right]^{1/\alpha}$$
$$\times \left[\sum_{1\le i<j\le n} p_i p_j (j-i) \sum_{k=i}^{j-1} \|\Delta B_k\|_q^\beta\right]^{1/\beta}.$$

Now, observe that

$$\sum_{1\le i<j\le n} p_i p_j (j-i) \sum_{k=i}^{j-1} \|\Delta A_k\|_p^\alpha$$
$$= \frac{1}{2}\sum_{i,j=} p_i p_j (j-i) \sum_{k=i}^{j-1} \|\Delta A_k\|_p^\alpha$$
$$= \frac{1}{2}\sum_{i,j=} p_i p_j (j-i) \left(\sum_{k=1}^{j-1} \|\Delta A_k\|_p^\alpha - \sum_{k=1}^{i-1} \|\Delta A_k\|_p^\alpha\right)$$
$$= D_{\mathbf{p}}(\mathbf{e}, \mathbf{V}_{\mathbf{A},p,\alpha})$$

and, similarly,

$$\sum_{1\le i<j\le n} p_i p_j (j-i) \sum_{k=i}^{j-1} \|\Delta B_k\|_q^\beta = D_{\mathbf{p}}(\mathbf{e}, \mathbf{V}_{\mathbf{B},q,\beta}).$$

Therefore, by (4.4)

$$\sum_{1\le i<j\le n} p_i p_j \sum_{k=i}^{j-1} \|\Delta A_k\|_p \sum_{l=i}^{j-1} \|\Delta B_l\|_q \le [D_{\mathbf{p}}(\mathbf{e}, \mathbf{V}_{\mathbf{A},p,\alpha})]^{1/\alpha} [D_{\mathbf{p}}(\mathbf{e}, \mathbf{V}_{\mathbf{B},q,\beta})]^{1/\beta}. \tag{4.5}$$

Utilising (4.3) and (4.5) we derive the first inequality in (4.1).

By (3.4) we derive

$$D_{\mathbf{p}}(\mathbf{e}, \mathbf{V}_{\mathbf{A},p,\alpha}) \le [D_{\mathbf{p}}(\mathbf{e}, \mathbf{e})]^{1/2} [D_{\mathbf{p}}(\mathbf{V}_{\mathbf{A},p,\alpha}, \mathbf{V}_{\mathbf{A},p,\alpha})]^{1/2}$$

and

$$D_{\mathbf{p}}\left(\mathbf{e},\mathbf{V}_{\mathbf{B},q,\beta}\right)\leq\left[D_{\mathbf{p}}\left(\mathbf{e},\mathbf{e}\right)\right]^{1/2}\left[D_{\mathbf{p}}\left(\mathbf{V}_{\mathbf{B},q,\beta},\mathbf{V}_{\mathbf{B},q,\beta}\right)\right]^{1/2}.$$

Now by Andrica-Badea inequality we have

$$\begin{aligned}D_{\mathbf{p}}\left(\mathbf{V}_{\mathbf{A},p,\alpha},\mathbf{V}_{\mathbf{A},p,\alpha}\right)&\leq\sum_{k\in S}p_k\left(1-\sum_{k\in S}p_k\right)\left(\sum_{k=1}^{n-1}\left\|\Delta A_k\right\|_p^{\alpha}\right)^2\\&\leq\frac{1}{4}\left(\sum_{k=1}^{n-1}\left\|\Delta A_k\right\|_p^{\alpha}\right)^2,\end{aligned}$$

and

$$\begin{aligned}D_{\mathbf{p}}\left(\mathbf{V}_{\mathbf{B},q,\beta},\mathbf{V}_{\mathbf{B},q,\beta}\right)&\leq\sum_{k\in S}p_k\left(1-\sum_{k\in S}p_k\right)\left(\sum_{k=1}^{n-1}\left\|\Delta B_k\right\|_q^{\beta}\right)^2\\&\leq\frac{1}{4}\left(\sum_{k=1}^{n-1}\left\|\Delta B_k\right\|_q^{\beta}\right)^2.\end{aligned}$$

Therefore 4.1

$$\begin{aligned}&\left[D_{\mathbf{p}}\left(\mathbf{e},\mathbf{V}_{\mathbf{A},p,\alpha}\right)\right]^{1/\alpha}\left[D_{\mathbf{p}}\left(\mathbf{e},\mathbf{V}_{\mathbf{B},q,\beta}\right)\right]^{1/\beta}\\&\leq\left[D_{\mathbf{p}}\left(\mathbf{e},\mathbf{e}\right)\right]^{1/(2\alpha)}\left[\sum_{k\in S}p_k\left(1-\sum_{k\in S}p_k\right)\left(\sum_{k=1}^{n-1}\left\|\Delta A_k\right\|_p^{\alpha}\right)^2\right]^{1/(2\alpha)}\\&\times\left[D_{\mathbf{p}}\left(\mathbf{e},\mathbf{e}\right)\right]^{1/(2\beta)}\left[\sum_{k\in S}p_k\left(1-\sum_{k\in S}p_k\right)\left(\sum_{k=1}^{n-1}\left\|\Delta B_k\right\|_q^{\beta}\right)^2\right]^{1/(2\beta)}\\&=\left[D_{\mathbf{p}}\left(\mathbf{e},\mathbf{e}\right)\right]^{1/2}\left[\sum_{k\in S}p_k\left(1-\sum_{k\in S}p_k\right)\right]^{1/2}\\&\times\left(\sum_{k=1}^{n-1}\left\|\Delta A_k\right\|_p^{\alpha}\right)^{1/\alpha}\left(\sum_{k=1}^{n-1}\left\|\Delta B_k\right\|_q^{\beta}\right)^{1/\beta},\end{aligned}$$

which proves the second inequality in (4.1).

The last part is obvious. □

Remark 3. *For $p,\ q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, assume that $\{A_k\}_{k=0}^{n}\subset B_p(H)$, $\{B_k\}_{k=0}^{n}\subset B_q(H)$ are sequences of operators and let $\mathbf{p}=(p_1,...,p_n)$ be a probability* *density distribution with* $S \subset \{1, ..., n\}$ *its optimal set of indices, then*

$$\begin{aligned}(4.6)\qquad \|D_{\mathbf{p}}(\mathbf{A},\mathbf{B})\|_1 &\leq [D_{\mathbf{p}}(\mathbf{e},\mathbf{V}_{\mathbf{A},2,p})]^{1/p}[D_{\mathbf{p}}(\mathbf{e},\mathbf{V}_{\mathbf{B},2,q})]^{1/q} \\ &\leq [D_{\mathbf{p}}(\mathbf{e},\mathbf{e})]^{1/2}\left[\sum_{k\in S} p_k\left(1-\sum_{k\in S} p_k\right)\right]^{1/2} \\ &\times \left(\sum_{k=1}^{n-1}\|\Delta A_k\|_p^p\right)^{1/p}\left(\sum_{k=1}^{n-1}\|\Delta B_k\|_q^q\right)^{1/q} \\ &\leq \frac{1}{2}[D_{\mathbf{p}}(\mathbf{e},\mathbf{e})]^{1/2}\left(\sum_{k=1}^{n-1}\|\Delta A_k\|_p^p\right)^{1/p}\left(\sum_{k=1}^{n-1}\|\Delta B_k\|_q^q\right)^{1/q}.\end{aligned}$$

Using the properties of the trace, we get from (4.6) that

$$\begin{aligned}(4.7)\qquad &\left|\operatorname{tr}\left(\sum_{k=1}^{n} p_k A_k B_k - \sum_{k=1}^{n} p_k A_k \sum_{k=1}^{n} p_k B_k\right)\right| \\ &\leq [D_{\mathbf{p}}(\mathbf{e},\mathbf{e})]^{1/2}\left[\sum_{k\in S} p_k\left(1-\sum_{k\in S} p_k\right)\right]^{1/2} \\ &\times \left[\operatorname{tr}\left(\sum_{k=1}^{n-1}|\Delta A_k|^p\right)\right]^{1/p}\left[\operatorname{tr}\left(\sum_{k=1}^{n-1}|\Delta B_k|^q\right)^{1/q}\right] \\ &\leq \frac{1}{2}[D_{\mathbf{p}}(\mathbf{e},\mathbf{e})]^{1/2}\left[\operatorname{tr}\left(\sum_{k=1}^{n-1}|\Delta A_k|^p\right)\right]^{1/p}\left[\operatorname{tr}\left(\sum_{k=1}^{n-1}|\Delta B_k|^q\right)^{1/q}\right]\end{aligned}$$

for $p, q > 1$ *with* $\frac{1}{p}+\frac{1}{q}=1$ *and* $\{A_k\}_{k=0}^n \subset B_p(H)$, $\{B_k\}_{k=0}^n \subset B_q(H)$.

For $p = q = 2$ *we derive*

$$\begin{aligned}(4.8)\qquad &\left|\operatorname{tr}\left(\sum_{k=1}^{n} p_k A_k B_k - \sum_{k=1}^{n} p_k A_k \sum_{k=1}^{n} p_k B_k\right)\right|^2 \\ &\leq D_{\mathbf{p}}(\mathbf{e},\mathbf{e})\sum_{k\in S} p_k\left(1-\sum_{k\in S} p_k\right)\operatorname{tr}\left(\sum_{k=1}^{n-1}|\Delta A_k|^2\right)\operatorname{tr}\left(\sum_{k=1}^{n-1}|\Delta B_k|^2\right) \\ &\leq \frac{1}{4}D_{\mathbf{p}}(\mathbf{e},\mathbf{e})\operatorname{tr}\left(\sum_{k=1}^{n-1}|\Delta A_k|^2\right)\operatorname{tr}\left(\sum_{k=1}^{n-1}|\Delta B_k|^q\right).\end{aligned}$$

In the case of uniform distribution we get for $p,\ q>1$ with $\frac{1}{p}+\frac{1}{q}=1$ and $\{A_k\}_{k=0}^n \subset B_p(H)$, $\{B_k\}_{k=0}^n \subset B_q(H)$ that

$$\begin{aligned}\|D_n(\mathbf{A},\mathbf{B})\|_1 &\le [D_n(\mathbf{e},\mathbf{V}_{\mathbf{A},2,p})]^{1/p}[D_n(\mathbf{e},\mathbf{V}_{\mathbf{B},2,q})]^{1/q} \\ &\le \frac{\sqrt{3}}{6}\left(n^2-1\right)^{1/2}\left[\frac{1}{n}\left[\frac{n}{2}\right]\left(1-\frac{1}{n}\left[\frac{n}{2}\right]\right)\right]^{1/2} \\ &\times\left(\sum_{k=1}^{n-1}\|\Delta A_k\|_p^p\right)^{1/p}\left(\sum_{k=1}^{n-1}\|\Delta B_k\|_q^q\right)^{1/q} \\ &\le \frac{\sqrt{3}}{12}\left(n^2-1\right)^{1/2}\left(\sum_{k=1}^{n-1}\|\Delta A_k\|_p^p\right)^{1/p}\left(\sum_{k=1}^{n-1}\|\Delta B_k\|_q^q\right)^{1/q}.\end{aligned}\tag{4.9}$$

In particular, if $\{A_k\}_{k=0}^n$, $\{B_k\}_{k=0}^n \subset B_2(H)$, then

$$\begin{aligned}\|D_n(\mathbf{A},\mathbf{B})\|_1 &\le [D_n(\mathbf{e},\mathbf{V}_{\mathbf{A},2,2})]^{1/2}[D_n(\mathbf{e},\mathbf{V}_{\mathbf{B},2,2})]^{1/2} \\ &\le \frac{\sqrt{3}}{6}\left(n^2-1\right)^{1/2}\left[\frac{1}{n}\left[\frac{n}{2}\right]\left(1-\frac{1}{n}\left[\frac{n}{2}\right]\right)\right]^{1/2} \\ &\times\left(\sum_{k=1}^{n-1}\|\Delta A_k\|_2^2\right)^{1/2}\left(\sum_{k=1}^{n-1}\|\Delta B_k\|_2^2\right)^{1/2} \\ &\le \frac{\sqrt{3}}{12}\left(n^2-1\right)^{1/2}\left(\sum_{k=1}^{n-1}\|\Delta A_k\|_2^2\right)^{1/p}\left(\sum_{k=1}^{n-1}\|\Delta B_k\|_2^2\right)^{1/q}.\end{aligned}\tag{4.10}$$

We finally can state:

Theorem 8. *For $p\ge 1$, assume that $\{A_k\}_{k=0}^n$, $\{B_k\}_{k=0}^n \subset B_p(H)$ are sequences of operators and let $\mathbf{p}=(p_1,...,p_n)$ be a probability density distribution with $S\subset \{1,...,n\}$ its optimal set of indices, then we have the inequality:*

$$\begin{aligned}\|D_{\mathbf{p}}(\mathbf{A},\mathbf{B})\|_p &\le [D_{\mathbf{p}}(\mathbf{e},\mathbf{V}_{\mathbf{A},p,\alpha})]^{1/\alpha}[D_{\mathbf{p}}(\mathbf{e},\mathbf{V}_{\mathbf{B},p,\beta})]^{1/\beta} \\ &\le [D_{\mathbf{p}}(\mathbf{e},\mathbf{e})]^{1/2}\left[\sum_{k\in S}p_k\left(1-\sum_{k\in S}p_k\right)\right]^{1/2} \\ &\times\left(\sum_{k=1}^{n-1}\|\Delta A_k\|_p^\alpha\right)^{1/\alpha}\left(\sum_{k=1}^{n-1}\|\Delta B_k\|_p^\beta\right)^{1/\beta} \\ &\le \frac{1}{2}[D_{\mathbf{p}}(\mathbf{e},\mathbf{e})]^{1/2}\left(\sum_{k=1}^{n-1}\|\Delta A_k\|_p^\alpha\right)^{1/\alpha}\left(\sum_{k=1}^{n-1}\|\Delta B_k\|_p^\beta\right)^{1/\beta},\end{aligned}\tag{4.11}$$

where $\alpha,\ \beta>1$ with $\frac{1}{\alpha}+\frac{1}{\beta}=1$.

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