p-SCHATTEN NORM INEQUALITIES FOR WEIGHTED ČEBYŠEV'S FUNCTIONAL VIA A CERONE-DRAGOMIR RESULT

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Abstract. An operator $A\in\mathcal{B}(H)$ is said to belong to the *von Neumann-Schatten class* $\mathcal{B}_p(H)$, $1\le p<\infty$ if the *p-Schatten norm* is finite

$$\|A\|_p:=\left[\operatorname{tr}\left(|A|^p\right)\right]^{1/p}<\infty.$$

For $\mathbf{A}=(A_1,...,A_n)$, $\mathbf{B}=(B_1,...,B_n)\in B(H)$ and $\mathbf{p}=(p_1,...,p_n)$ a probability density distribution we define the functional

$$D_{\mathbf{p}}(\mathbf{A},\mathbf{B}):=\sum_{k=1}^n p_kA_kB_k-\sum_{k=1}^n p_kA_k\sum_{k=1}^n p_kB_k.$$

If $p,\ q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, assume that $\{A_k\}_{k=0}^n\subset B_p(H)$, $\{B_k\}_{k=0}^n\subset B_q(H)$ are sequences of operators, then we have the inequality:

$$\begin{aligned}\|D_{\mathbf{p}}(\mathbf{A},\mathbf{B})\|_1&\le\max_{k\in\{1,...,n-1\}}\|\Delta A_k\|_p\,D_{\mathbf{p}}(\mathbf{e},\mathbf{V}_{\mathbf{B},q})\\&\le\frac{1}{2}\sum_{k=1}^n p_k\left|k-\sum_{j=1}^n jp_j\right|\max_{k\in\{1,...,n-1\}}\|\Delta A_k\|_p\sum_{k=1}^{n-1}\|\Delta B_k\|_q,\end{aligned}$$

where $\mathbf{e}=(1,2,...,n)$,

$$\mathbf{V}_{\mathbf{B},q}:=\left(0,\|\Delta B_1\|_q,...,\sum_{k=1}^{n-1}\|\Delta B_k\|_q\right)$$

and $\Delta B_i:=B_{i+1}-B_i$ $(i=1,...,n-1)$ are the usual forward differences.

1. Introduction

Let $(\Omega,\mathcal{A},\mu)$ be a measurable space consisting of a set Ω, a σ-algebra $\mathcal{A}$ of parts of Ω and a countably additive and positive measure μ on $\mathcal{A}$ with values in $\mathbb{R}\cup\{\infty\}$.

For a μ-measurable function $w:\Omega\to\mathbb{R}$, with $w(x)\ge 0$ for μ-a.e. $x\in\Omega$, consider the Lebesgue space $L_w(\Omega,\mu):=\{h:\Omega\to\mathbb{R},\ h$ is μ-measurable and $\int_\Omega w(x)|h(x)|\,d\mu(x)<\infty\}$. Assume $\int_\Omega w(x)\,d\mu(x)=1$. In order to simplify the notation for the integrals, we do not write the variable, namely, instead of $\int_\Omega w(x)\,d\mu(x)$ we simply write $\int_\Omega wd\mu$.

If $h,\ k:\Omega\to\mathbb{R}$ are μ-measurable functions and $h,\ k,\ hk\in L_w(\Omega,\mu)$, then we may consider the weighted Čebyšev functional in the following form

$$D_w(h,k):=\int_\Omega whkd\mu-\int_\Omega whd\mu\int_\Omega wkd\mu.$$

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The following result is known in the literature as the Grüss inequality, see for instance [7]:

$$|D_w(h,k)| \leq \frac{1}{4}(\Gamma-\gamma)(\Delta-\delta), \tag{1.1}$$

provided

$$-\infty<\gamma\leq h\leq\Gamma<\infty,\quad -\infty<\delta\leq k\leq\Delta<\infty \tag{1.2}$$

μ-a.e. on Ω. The constant $\frac{1}{4}$ is sharp in the sense that it cannot be replaced by a smaller constant.

For classical and more recent upper bounds related to the Čebyšev functional see [3]-[7] and [10]-[18].

Assume that $\mathbf{a}=(a_1,...,a_n)$ and $\mathbf{b}=(b_1,...,b_n)$ are n-tuples of real numbers and $\mathbf{p}=(p_1,...,p_n)$ a density distribution, namely $p_k\geq 0$ with $P_n:=\sum_{k=1}^n p_k=1$. We consider the *weighted Čebyšev's functional*

$$D_{\mathbf{p}}(\mathbf{a},\mathbf{b}) := \sum_{k=1}^n p_k a_k b_k - \sum_{k=1}^n p_k a_k \sum_{k=1}^n p_k b_k.$$

In the case when $\mathbf{p}=\left(\frac{1}{n},...,\frac{1}{n}\right)$ is the *uniform distribution* we also consider the *Čebyšev's functional*

$$D(\mathbf{a},\mathbf{b}) := \frac{1}{n}\sum_{k=1}^n a_k b_k - \frac{1}{n}\sum_{k=1}^n a_k \frac{1}{n}\sum_{k=1}^n b_k.$$

In [7] Cerone and Dragomir proved among others the following *refinement of Grüss' discrete inequality:*

Theorem 1. *Assume that* $\mathbf{a}=(a_1,...,a_n)$ *and* $\mathbf{b}=(b_1,...,b_n)$ *are n-tuples of real numbers and so that* $-\infty<\gamma\leq a_k\leq\Gamma<\infty$, $-\infty<\delta\leq b_k\leq\Delta<\infty$ *for all* $k\in\{1,...,n\}$, *then*

$$\begin{aligned}|D_{\mathbf{p}}(\mathbf{a},\mathbf{b})| &\leq \frac{1}{2}(\Delta-\delta)\sum_{k=1}^n p_k\left|b_k-\sum_{j=1}^n p_j b_j\right| \\ &\leq \frac{1}{2}(\Delta-\delta)\left[\sum_{k=1}^n p_k b_k^2 - \left(\sum_{k=1}^n p_k b_k\right)^2\right]^{\frac{1}{2}} \\ &\leq \frac{1}{4}(\Delta-\delta)(\Gamma-\gamma),\end{aligned} \tag{1.3}$$

The constants $\frac{1}{2}$ and $\frac{1}{4}$ are best possible.

In particular, we have

$$\begin{aligned}|D(\mathbf{a},\mathbf{b})| &\leq \frac{1}{2}(\Delta-\delta)\frac{1}{n}\sum_{k=1}^n\left|b_k-\frac{1}{n}\sum_{j=1}^n b_j\right| \\ &\leq \frac{1}{2}(\Delta-\delta)\left[\frac{1}{n}\sum_{k=1}^n b_k^2 - \left(\frac{1}{n}\sum_{k=1}^n b_k\right)^2\right]^{\frac{1}{2}} \\ &\leq \frac{1}{4}(\Delta-\delta)(\Gamma-\gamma).\end{aligned} \tag{1.4}$$

In order to extend these results for p-Schatten norms we need the following preparations.

Let $(H;\langle .,.\rangle)$ be a complex Hilbert space and $\mathcal{B}(H)$ the Banach algebra of all bounded linear operators on H. If $\{e_i\}_{i\in I}$ an orthonormal basis of H, we say that $A\in\mathcal{B}(H)$ is of *trace class* if

$$\|A\|_1 := \sum_{i\in I}\langle |A| e_i, e_i\rangle < \infty. \tag{1.5}$$

The definition of $\|A\|_1$ does not depend on the choice of the orthonormal basis $\{e_i\}_{i\in I}$. We denote by $\mathcal{B}_1(H)$ the set of trace class operators in $\mathcal{B}(H)$.

We define the *trace* of a trace class operator $A\in\mathcal{B}_1(H)$ to be

$$\operatorname{tr}(A) := \sum_{i\in I}\langle Ae_i, e_i\rangle, \tag{1.6}$$

where $\{e_i\}_{i\in I}$ an orthonormal basis of H. Note that this coincides with the usual definition of the trace if H is finite-dimensional. We observe that the series (1.6) converges absolutely and it is independent from the choice of basis.

The following result collects some properties of the trace:

Theorem 2. *We have:*

(i) If $A\in\mathcal{B}_1(H)$ *then* $A^*\in\mathcal{B}_1(H)$ *and*

$$\operatorname{tr}(A^*) = \overline{\operatorname{tr}(A)}; \tag{1.7}$$

(ii) If $A\in\mathcal{B}_1(H)$ *and* $T\in\mathcal{B}(H)$, *then* AT, $TA\in\mathcal{B}_1(H)$ *and*

$$\operatorname{tr}(AT) = \operatorname{tr}(TA) \text{ and } |\operatorname{tr}(AT)| \le \|A\|_1\|T\|; \tag{1.8}$$

(iii) $\operatorname{tr}(\cdot)$ *is a bounded linear functional on* $\mathcal{B}_1(H)$ *with* $\|\operatorname{tr}\| = 1$;

(iv) If A, $B\in\mathcal{B}_2(H)$ *then* AB, $BA\in\mathcal{B}_1(H)$ *and* $\operatorname{tr}(AB) = \operatorname{tr}(BA)$;

(v) $\mathcal{B}_{fin}(H)$, *the space of operators of finite rank, is a dense subspace of* $\mathcal{B}_1(H)$.

An operator $A\in\mathcal{B}(H)$ is said to belong to the *von Neumann-Schatten class* $\mathcal{B}_p(H)$, $1\le p<\infty$ if the *p-Schatten norm* is finite [21, p. 60-64]

$$\|A\|_p := \left[\operatorname{tr}(|A|^p)\right]^{1/p} < \infty.$$

For $1<p<q<\infty$ we have that

$$\mathcal{B}_1(H)\subset\mathcal{B}_p(H)\subset\mathcal{B}_q(H)\subset\mathcal{B}(H) \tag{1.9}$$

and

$$\|A\|_1 \ge \|A\|_p \ge \|A\|_q \ge \|A\|. \tag{1.10}$$

For $p\ge 1$ the functional $\|\cdot\|_p$ is a *norm* on the $*$-ideal $\mathcal{B}_p(H)$ and $\left(\mathcal{B}_p(H), \|\cdot\|_p\right)$ is a Banach space.

Also, see for instance [21, p. 60-64],

$$\|A\|_p = \|A^*\|_p,\ A\in\mathcal{B}_p(H) \tag{1.11}$$

$$\|AB\|_p \le \|A\|_p\|B\|_p,\ A,B\in\mathcal{B}_p(H) \tag{1.12}$$

and

$$\|AB\|_p \le \|A\|_p\|B\|,\ \|BA\|_p \le \|B\|\|A\|_p,\ A\in\mathcal{B}_p(H),\ B\in\mathcal{B}(H). \tag{1.13}$$

This implies that

$$\|CAB\|_p \le \|C\| \|A\|_p \|B\|, \ A \in \mathcal{B}_p(H), \ B, \ C \in \mathcal{B}(H). \tag{1.14}$$

In terms of *p-Schatten norm* we have the *Hölder inequality* for $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$

$$(|\mathrm{tr}(AB)| \le) \|AB\|_1 \le \|A\|_p \|B\|_q, \ A \in \mathcal{B}_p(H), \ B \in \mathcal{B}_q(H). \tag{1.15}$$

For the theory of trace functionals and their applications the reader is referred to [20] and [21].

For some classical trace inequalities see [8], [9], and [19], which are continuations of the work of Bellman [1].

2. 1-Schatten Norm Inequalities

Let $\mathbf{p} = (p_1, ..., p_n)$ be a probability density distribution. For $\mathbf{A} = (A_1, ..., A_n)$, $\mathbf{B} = (B_1, ..., B_n) \in B(H)$ we define the functional

$$D_{\mathbf{p}}(\mathbf{A}, \mathbf{B}) := \sum_{k=1}^{n} p_k A_k B_k - \sum_{k=1}^{n} p_k A_k \sum_{k=1}^{n} p_k B_k. \tag{2.1}$$

The following inequality of Grüss type holds.

Theorem 3. *For $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, assume that $\{A_k\}_{k=0}^{n} \subset B_p(H)$, $\{B_k\}_{k=0}^{n} \subset B_q(H)$ are sequences of operators and let $\mathbf{p} = (p_1, ..., p_n)$ be a probability density distribution, then we have the inequality:*

$$\begin{aligned} &\|D_{\mathbf{p}}(\mathbf{A}, \mathbf{B})\|_1 \\ &\le D_{\mathbf{p}}(\mathbf{V}_{\mathbf{A},p}, \mathbf{V}_{\mathbf{B},q}) \\ &\le \frac{1}{2} \sum_{k=1}^{n-1} \|\Delta A_k\|_p \sum_{k=1}^{n} p_k \left| \sum_{l=1}^{k-1} \|\Delta B_l\|_q - \sum_{j=1}^{n} p_j \sum_{l=1}^{j-1} \|\Delta B_l\|_q \right| \\ &\le \frac{1}{2} \sum_{k=1}^{n-1} \|\Delta A_k\|_p \left[\sum_{k=1}^{n} p_k \left(\sum_{l=1}^{k-1} \|\Delta B_l\|_q \right)^2 - \left(\sum_{k=1}^{n} p_k \sum_{l=1}^{k-1} \|\Delta B_l\|_q \right)^2 \right]^{\frac{1}{2}} \\ &\le \frac{1}{4} \sum_{k=1}^{n-1} \|\Delta A_k\|_p \sum_{k=1}^{n-1} \|\Delta B_k\|_q, \end{aligned} \tag{2.2}$$

where

$$\mathbf{V}_{\mathbf{A},p} := \left(0, \|\Delta A_1\|_p, ..., \sum_{k=1}^{n-1} \|\Delta A_k\|_p \right), \ \mathbf{V}_{\mathbf{B},q} := \left(0, \|\Delta B_1\|_q, ..., \sum_{k=1}^{n-1} \|\Delta B_k\|_q \right)$$

and $\Delta A_k := A_{k+1} - A_k$ $(k = 1, ..., n-1)$ are the usual forward differences.

In particular, if $\{A_k\}_{k=0}^{n}, \{B_k\}_{k=0}^{n} \subset B_2(H)$, *then for* $p=q=2$ *we have*

$$\begin{aligned}
&\|D_{\mathbf{p}}(\mathbf{A},\mathbf{B})\|_1 \\
&\le D_{\mathbf{p}}(\mathbf{V}_{\mathbf{A},2},\mathbf{V}_{\mathbf{B},2}) \\
&\le \frac{1}{2}\sum_{k=1}^{n-1}\|\Delta A_k\|_2 \sum_{k=1}^{n} p_k \left|\sum_{l=1}^{k-1}\|\Delta B_l\|_2 - \sum_{j=1}^{n} p_j \sum_{l=1}^{j-1}\|\Delta B_l\|_2\right| \\
&\le \frac{1}{2}\sum_{k=1}^{n-1}\|\Delta A_k\|_2 \left[\sum_{k=1}^{n} p_k \left(\sum_{l=1}^{k-1}\|\Delta B_l\|_2\right)^2 - \left(\sum_{k=1}^{n} p_k \sum_{l=1}^{k-1}\|\Delta B_l\|_2\right)^2\right]^{\frac{1}{2}} \\
&\le \frac{1}{4}\sum_{k=1}^{n-1}\|\Delta A_k\|_2 \sum_{k=1}^{n-1}\|\Delta B_k\|_2 .
\end{aligned} \tag{2.3}$$

Proof. Let us start with the following identity in Banach algebra $B(H)$ which can be proved by direct computation

$$\begin{aligned}
\sum_{k=1}^{n} p_k A_k B_k - \sum_{k=1}^{n} p_k A_k \sum_{k=1}^{n} p_k B_k &= \frac{1}{2}\sum_{i,j=1}^{n} p_i p_j (A_j - A_i)(B_j - B_i) \\
&= \sum_{1\le i<j\le n} p_i p_j (A_j - A_i)(B_j - B_i).
\end{aligned}$$

As $i<j$, we can write $A_j - A_i = \sum_{k=i}^{j-1}\Delta A_k$ and $B_j - B_i = \sum_{l=i}^{j-1}\Delta B_l$. Using the generalized triangle inequality and the property (1.15), we have successively:

$$\begin{aligned}
\|D_{\mathbf{p}}(\mathbf{A},\mathbf{B})\|_1 &= \left\|\sum_{1\le i<j\le n} p_i p_j \sum_{k=i}^{j-1}\Delta A_k \sum_{l=k}^{j-1}\Delta B_l\right\|_1 \\
&\le \sum_{1\le i<j\le n} p_i p_j \left\|\sum_{k=i}^{j-1}\Delta A_k \sum_{l=i}^{j-1}\Delta B_l\right\|_1 \\
&\le \sum_{1\le i<j\le n} p_i p_j \left\|\sum_{k=i}^{j-1}\Delta A_k\right\|_p \left\|\sum_{l=i}^{j-1}\Delta B_l\right\|_q \\
&\le \sum_{1\le i<j\le n} p_i p_j \sum_{k=i}^{j-1}\|\Delta A_k\|_p \sum_{l=i}^{j-1}\|\Delta B_l\|_q .
\end{aligned} \tag{2.4}$$

Now, observe that

$$\begin{aligned}
&\sum_{1\le i<j\le n} p_i p_j \sum_{k=i}^{j-1}\|\Delta A_k\|_p \sum_{l=i}^{j-1}\|\Delta B_l\|_q \\
&= \sum_{1\le i<j\le n} p_i p_j \left(\sum_{k=1}^{j-1}\|\Delta A_k\|_p - \sum_{k=1}^{i-1}\|\Delta A_k\|_p\right)\left(\sum_{l=1}^{j-1}\|\Delta B_l\|_q - \sum_{l=1}^{i-1}\|\Delta B_l\|_q\right) \\
&= \frac{1}{2}\sum_{i,j=1}^{n} p_i p_j \left(\sum_{k=1}^{j-1}\|\Delta A_k\|_p - \sum_{k=1}^{i-1}\|\Delta A_k\|_p\right)\left(\sum_{l=1}^{j-1}\|\Delta B_l\|_q - \sum_{l=1}^{i-1}\|\Delta B_l\|_q\right) \\
&= D_{\mathbf{p}}(\mathbf{V}_{\mathbf{A},p},\mathbf{V}_{\mathbf{B},q}),
\end{aligned}$$

which prove the first inequality in (2.2).

Now, if we use the Cerone-Dragomir inequality (1.3) for $(\mathbf{V}_{\mathbf{A},p}, \mathbf{V}_{\mathbf{B},q})$ we have

$$\begin{aligned}
&\left|D_{\mathbf{p}}\left(\mathbf{V}_{\mathbf{A},p}, \mathbf{V}_{\mathbf{B},q}\right)\right| \\
&\leq \frac{1}{2}\sum_{k=1}^{n-1}\left\|\Delta A_k\right\|_p \sum_{k=1}^{n} p_k \left|\sum_{l=1}^{k-1}\left\|\Delta B_l\right\|_q - \sum_{j=1}^{n} p_j \sum_{l=1}^{j-1}\left\|\Delta B_l\right\|_q\right| \\
&\leq \frac{1}{2}\sum_{k=1}^{n-1}\left\|\Delta A_k\right\|_p \left[\sum_{k=1}^{n} p_k \left(\sum_{l=1}^{k-1}\left\|\Delta B_l\right\|_q\right)^2 - \left(\sum_{k=1}^{n} p_k \sum_{l=1}^{k-1}\left\|\Delta B_l\right\|_q\right)^2\right]^{\frac{1}{2}} \\
&\leq \frac{1}{4}\sum_{k=1}^{n-1}\left\|\Delta A_k\right\|_p \sum_{k=1}^{n-1}\left\|\Delta B_k\right\|_q,
\end{aligned}$$

which proves the second part of (2.2). □

Remark 1. *In the case of uniformly distributed probabilities, we get*

$$\begin{aligned}
&\left\|D_n\left(\mathbf{A}, \mathbf{B}\right)\right\|_1 \\
&\leq D_n\left(\mathbf{V}_{\mathbf{A},p}, \mathbf{V}_{\mathbf{B},q}\right) \\
&\leq \frac{1}{2}\sum_{k=1}^{n-1}\left\|\Delta A_k\right\|_p \frac{1}{n}\sum_{k=1}^{n}\left|\sum_{l=1}^{k-1}\left\|\Delta B_l\right\|_q - \frac{1}{n}\sum_{l=1}^{j-1}\left\|\Delta B_l\right\|_q\right| \\
&\leq \frac{1}{2}\sum_{k=1}^{n-1}\left\|\Delta A_k\right\|_p \left[\frac{1}{n}\sum_{k=1}^{n}\left(\sum_{l=1}^{k-1}\left\|\Delta B_l\right\|_q\right)^2 - \left(\frac{1}{n}\sum_{k=1}^{n}\sum_{l=1}^{k-1}\left\|\Delta B_l\right\|_q\right)^2\right]^{\frac{1}{2}} \\
&\leq \frac{1}{4}\sum_{k=1}^{n-1}\left\|\Delta A_k\right\|_p \sum_{k=1}^{n-1}\left\|\Delta B_k\right\|_q .
\end{aligned} \tag{2.5}$$

Further, we have:

Theorem 4. *For $p,\ q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, assume that $\{A_k\}_{k=0}^{n} \subset B_p(H)$, $\{B_k\}_{k=0}^{n} \subset B_q(H)$ are sequences of operators and let $\mathbf{p} = (p_1, ..., p_n)$ be a probability density distribution, then we have the inequality:*

$$\begin{aligned}
\left\|D_{\mathbf{p}}\left(\mathbf{A}, \mathbf{B}\right)\right\|_1 &\leq \max_{k\in\{1,\dots,n-1\}}\left\|\Delta A_k\right\|_p D_{\mathbf{p}}\left(\mathbf{e}, \mathbf{V}_{\mathbf{B},q}\right) \\
&\leq \frac{1}{2}\sum_{k=1}^{n} p_k\left|k - \sum_{j=1}^{n} j p_j\right| \max_{k\in\{1,\dots,n-1\}}\left\|\Delta A_k\right\|_p \sum_{k=1}^{n-1}\left\|\Delta B_k\right\|_q \\
&\leq \frac{1}{2}\left[\sum_{k=1}^{n} k^2 p_k - \left(\sum_{k=1}^{n} k p_k\right)^2\right]^{\frac{1}{2}} \\
&\times \max_{k\in\{1,\dots,n-1\}}\left\|\Delta A_k\right\|_p \sum_{k=1}^{n-1}\left\|\Delta B_k\right\|_q \\
&\leq \frac{1}{4}(n-1)\max_{k\in\{1,\dots,n-1\}}\left\|\Delta A_k\right\|_p \sum_{k=1}^{n-1}\left\|\Delta B_k\right\|_q,
\end{aligned} \tag{2.6}$$

where $\mathbf{e} := (1, 2, ..., n)$.

In particular, if $\{A_k\}_{k=0}^n, \{B_k\}_{k=0}^n \subset B_2(H)$, *then for* $p = q = 2$ *we have*

$$\begin{aligned}
(2.7) \qquad \|D_{\mathbf{p}}(\mathbf{A}, \mathbf{B})\|_1 &\le \max_{k\in\{1,...,n-1\}} \|\Delta A_k\|_2 \, D_{\mathbf{p}}(\mathbf{e}, \mathbf{V}_{\mathbf{B},2}) \\
&\le \frac{1}{2}\sum_{k=1}^n p_k \left| k - \sum_{j=1}^n j p_j \right| \max_{k\in\{1,...,n-1\}} \|\Delta A_k\|_2 \sum_{k=1}^{n-1} \|\Delta B_k\|_2 \\
&\le \frac{1}{2}\left[\sum_{k=1}^n k^2 p_k - \left(\sum_{k=1}^n k p_k\right)^2\right]^{\frac{1}{2}} \\
&\times \max_{k\in\{1,...,n-1\}} \|\Delta A_k\|_2 \sum_{k=1}^{n-1} \|\Delta B_k\|_2 \\
&\le \frac{1}{4}(n-1) \max_{k\in\{1,...,n-1\}} \|\Delta A_k\|_2 \sum_{k=1}^{n-1} \|\Delta B_k\|_2 .
\end{aligned}$$

Proof. From (2.4) we also derive

$$\begin{aligned}
(2.8) \qquad & \|D_{\mathbf{p}}(\mathbf{A}, \mathbf{B})\|_1 \\
&\le \sum_{1\le i<j\le n} p_i p_j \sum_{k=i}^{j-1} \|\Delta A_k\|_p \sum_{l=i}^{j-1} \|\Delta B_l\|_q \\
&\le \sum_{1\le i<j\le n} p_i p_j (j-i) \max_{k\in\{i,...,j-1\}} \|\Delta A_k\|_p \sum_{l=i}^{j-1} \|\Delta B_l\|_q \\
&\le \max_{k\in\{1,...,n-1\}} \|\Delta A_k\|_p \sum_{1\le i<j\le n} p_i p_j (j-i) \sum_{l=i}^{j-1} \|\Delta B_l\|_q \\
&= \max_{k\in\{1,...,n-1\}} \|\Delta A_k\|_p \sum_{1\le i<j\le n} p_i p_j (j-i) \left(\sum_{l=1}^{j-1} \|\Delta B_l\|_q - \sum_{l=1}^{i-1} \|\Delta B_l\|_q\right) \\
&= \max_{k\in\{1,...,n-1\}} \|\Delta A_k\|_p \frac{1}{2} \sum_{i,j=1}^{n} p_i p_j (j-i) \left(\sum_{l=1}^{j-1} \|\Delta B_l\|_q - \sum_{l=1}^{i-1} \|\Delta B_l\|_q\right) \\
&= \max_{k\in\{1,...,n-1\}} \|\Delta A_k\|_p \, D_{\mathbf{p}}(\mathbf{e}, \mathbf{V}_{\mathbf{B},q}),
\end{aligned}$$

where $\mathbf{e} = (1, 2, ..., n)$. This proves the first inequality in (2.6).

Now, by the use of (1.3) we get for $\mathbf{b}=\mathbf{e}$ and $\mathbf{a}=\mathbf{V}_{\mathbf{B},q}$ that

$$|D_{\mathbf{p}}(\mathbf{e},\mathbf{V}_{\mathbf{B},q})| \leq \frac{1}{2}\sum_{k=1}^{n-1}\|\Delta B_k\|_q \sum_{k=1}^{n} p_k \left| k-\sum_{j=1}^{n} jp_j \right| \tag{2.9}$$
$$\leq \frac{1}{2}\sum_{k=1}^{n-1}\|\Delta B_k\|_q \left[\sum_{k=1}^{n} k^2 p_k - \left(\sum_{k=1}^{n} kp_k\right)^2\right]^{\frac{1}{2}}$$
$$\leq \frac{1}{4}(n-1)\sum_{k=1}^{n-1}\|\Delta B_k\|_q,$$

which proves the last part of (2.6). □

Corollary 1. *If* $\mathbf{A}=(A_1,...,A_n)\in(B_p(H))^n$ *and* $\mathbf{B}=(B_1,...,B_n)\in(B_q(H))^n$ *for* $p,\ q>1$ *with* $\frac{1}{p}+\frac{1}{q}=1$, *then*

$$\|D_n(\mathbf{A},\mathbf{B})\|_1 \tag{2.10}$$
$$\leq \max_{k\in\{1,...,n-1\}}\|\Delta A_k\|_p D_n(\mathbf{e},\mathbf{V}_{\mathbf{B},q})$$
$$\leq \frac{1}{2}\left\lfloor\frac{n+1}{2}\right\rfloor\left(1-\frac{1}{n}\left\lfloor\frac{n+1}{2}\right\rfloor\right)\max_{k\in\{1,...,n-1\}}\|\Delta A_k\|_p\sum_{k=1}^{n-1}\|\Delta B_k\|_q$$
$$\leq \frac{1}{8}n\max_{k\in\{1,...,n-1\}}\|\Delta A_k\|_p\sum_{k=1}^{n-1}\|\Delta B_k\|_q,$$

where $\mathbf{V}_{\mathbf{B},q}:=\left(0,\|\Delta B_1\|_q,...,\sum_{k=1}^{n-1}\|\Delta B_k\|_q\right)$.

In particular, if $\mathbf{A},\ \mathbf{B}\in(B_2(H))^n$, *then*

$$\|D_n(\mathbf{A},\mathbf{B})\|_1 \tag{2.11}$$
$$\leq \max_{k\in\{1,...,n-1\}}\|\Delta A_k\|_p D_n(\mathbf{e},\mathbf{V}_{\mathbf{B},2})$$
$$\leq \frac{1}{2}\left\lfloor\frac{n+1}{2}\right\rfloor\left(1-\frac{1}{n}\left\lfloor\frac{n+1}{2}\right\rfloor\right)\max_{k\in\{1,...,n-1\}}\|\Delta A_k\|_2\sum_{k=1}^{n-1}\|\Delta B_k\|_2$$
$$\leq \frac{1}{8}n\max_{k\in\{1,...,n-1\}}\|\Delta A_k\|_2\sum_{k=1}^{n-1}\|\Delta B_k\|_2,$$

where $\mathbf{V}_{\mathbf{B},2}:=\left(0,\|\Delta B_1\|_2,...,\sum_{k=1}^{n-1}\|\Delta B_k\|_2\right)$.

Proof. From (2.6) we get for the uniform distribution that

$$\|D_n(\mathbf{A},\mathbf{B})\|_1 \leq \max_{k\in\{1,...,n-1\}}\|\Delta A_k\|_p D_n(\mathbf{e},\mathbf{V}_{\mathbf{B},q}) \tag{2.12}$$
$$\leq \frac{1}{2n}\sum_{k=1}^{n}\left|k-\frac{1}{n}\sum_{j=1}^{n} j\right|\max_{k\in\{1,...,n-1\}}\|\Delta A_k\|_p\sum_{k=1}^{n-1}\|\Delta B_k\|_q.$$

Now, observe that, if $\lfloor \cdot \rfloor$ is the integer part function, then

$$
\begin{aligned}
&\sum_{k=1}^{n}\left|k-\frac{n+1}{2}\right| \\
&=\sum_{k=1}^{\lfloor \frac{n+1}{2}\rfloor}\left|k-\frac{n+1}{2}\right|+\sum_{k=\lfloor \frac{n+1}{2}\rfloor+1}^{n}\left|k-\frac{n+1}{2}\right| \\
&=\sum_{k=1}^{\lfloor \frac{n+1}{2}\rfloor}\left(\frac{n+1}{2}-k\right)+\sum_{k=\lfloor \frac{n+1}{2}\rfloor+1}^{n}\left(k-\frac{n+1}{2}\right) \\
&=\frac{n+1}{2}\left\lfloor \frac{n+1}{2}\right\rfloor-\sum_{k=1}^{\lfloor \frac{n+1}{2}\rfloor}k+\sum_{k=\lfloor \frac{n+1}{2}\rfloor+1}^{n}k-\frac{n+1}{2}\left(n-\left\lfloor \frac{n+1}{2}\right\rfloor\right)
\end{aligned}
$$

$$
\begin{aligned}
&=2\frac{n+1}{2}\left\lfloor \frac{n+1}{2}\right\rfloor-\frac{n+1}{2}n-\sum_{k=1}^{\lfloor \frac{n+1}{2}\rfloor}k+\sum_{k=1}^{n}k-\sum_{k=1}^{\lfloor \frac{n+1}{2}\rfloor}k \\
&=2\frac{n+1}{2}\left\lfloor \frac{n+1}{2}\right\rfloor-\frac{n+1}{2}n+\frac{n+1}{2}n-2\sum_{k=1}^{\lfloor \frac{n+1}{2}\rfloor}k \\
&=2\frac{n+1}{2}\left\lfloor \frac{n+1}{2}\right\rfloor-2\frac{\lfloor \frac{n+1}{2}\rfloor\left(\lfloor \frac{n+1}{2}\rfloor+1\right)}{2}= \\
&=(n+1)\left\lfloor \frac{n+1}{2}\right\rfloor-\left\lfloor \frac{n+1}{2}\right\rfloor^{2}-\left\lfloor \frac{n+1}{2}\right\rfloor \\
&=\left\lfloor \frac{n+1}{2}\right\rfloor\left[(n+1)-\left\lfloor \frac{n+1}{2}\right\rfloor-1\right]=\left\lfloor \frac{n+1}{2}\right\rfloor\left(n-\left\lfloor \frac{n+1}{2}\right\rfloor\right).
\end{aligned}
$$

By using (2.12), we derive (2.10). □

Theorem 5. *For $p,\ q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, assume that $\{A_k\}_{k=0}^{n}\subset B_p(H)$, $\{B_k\}_{k=0}^{n}\subset B_q(H)$ are sequences of operators and let $\mathbf{p}=(p_1,...,p_n)$ be a probability density distribution. If $\alpha,\ \beta>1$ with $\frac{1}{\alpha}+\frac{1}{\beta}=1$, then we have the inequality:*

$$
\begin{aligned}
\|D_{\mathbf{p}}(\mathbf{A},\mathbf{B})\|_1 &\le \left[D_{\mathbf{p}}(\mathbf{e},\mathbf{V}_{\mathbf{A},p,\alpha})\right]^{1/\alpha}\left[D_{\mathbf{p}}(\mathbf{e},\mathbf{V}_{\mathbf{B},q,\beta})\right]^{1/\beta} \\
&\le \frac{1}{2}\sum_{k=1}^{n}p_k\left|k-\sum_{j=1}^{n}jp_j\right| \\
&\quad\times\left(\sum_{k=1}^{n-1}\|\Delta A_k\|_p^{\alpha}\right)^{1/\alpha}\left(\sum_{k=1}^{n-1}\|\Delta B_k\|_q^{\beta}\right)^{1/\beta}
\end{aligned}
\tag{2.13}
$$

$$\leq \frac{1}{2}\left[\sum_{k=1}^{n}k^{2}p_{k}-\left(\sum_{k=1}^{n}kp_{k}\right)^{2}\right]^{\frac{1}{2}}$$
$$\times\left(\sum_{k=1}^{n-1}\left\|\Delta A_{k}\right\|_{p}^{\alpha}\right)^{1/\alpha}\left(\sum_{k=1}^{n-1}\left\|\Delta B_{k}\right\|_{q}^{\beta}\right)^{1/\beta}$$
$$\leq \frac{1}{4}\left(n-1\right)\left(\sum_{k=1}^{n-1}\left\|\Delta A_{k}\right\|_{p}^{\alpha}\right)^{1/\alpha}\left(\sum_{k=1}^{n-1}\left\|\Delta B_{k}\right\|_{q}^{\beta}\right)^{1/\beta},$$

where

$$\mathbf{V}_{\mathbf{A},p,\alpha} := \left(0,\left\|\Delta A_{1}\right\|_{p}^{\alpha},...,\sum_{k=1}^{n-1}\left\|\Delta A_{k}\right\|_{p}^{\alpha}\right),$$
$$\mathbf{V}_{\mathbf{B},q,\beta} := \left(0,\left\|\Delta B_{1}\right\|_{q}^{\beta},...,\sum_{k=1}^{n-1}\left\|\Delta B_{k}\right\|_{q}^{\beta}\right).$$

In particular, if $\{A_k\}_{k=0}^{n}, \{B_k\}_{k=0}^{n} \subset B_2(H)$, *then*

$$\left\|D_{\mathbf{p}}\left(\mathbf{A},\mathbf{B}\right)\right\|_{1} \leq \left[D_{\mathbf{p}}\left(\mathbf{e},\mathbf{V}_{\mathbf{A},2,\alpha}\right)\right]^{1/\alpha}\left[D_{\mathbf{p}}\left(\mathbf{e},\mathbf{V}_{\mathbf{B},2,\beta}\right)\right]^{1/\beta} \tag{2.14}$$
$$\leq \frac{1}{2}\sum_{k=1}^{n}p_{k}\left|k-\sum_{j=1}^{n}jp_{j}\right|$$
$$\times\left(\sum_{k=1}^{n-1}\left\|\Delta A_{k}\right\|_{2}^{\alpha}\right)^{1/\alpha}\left(\sum_{k=1}^{n-1}\left\|\Delta B_{k}\right\|_{2}^{\beta}\right)^{1/\beta}$$
$$\leq \frac{1}{2}\left[\sum_{k=1}^{n}k^{2}p_{k}-\left(\sum_{k=1}^{n}kp_{k}\right)^{2}\right]^{\frac{1}{2}}$$
$$\times\left(\sum_{k=1}^{n-1}\left\|\Delta A_{k}\right\|_{2}^{\alpha}\right)^{1/\alpha}\left(\sum_{k=1}^{n-1}\left\|\Delta B_{k}\right\|_{2}^{\beta}\right)^{1/\beta}$$
$$\leq \frac{1}{4}\left(n-1\right)\left(\sum_{k=1}^{n-1}\left\|\Delta A_{k}\right\|_{2}^{\alpha}\right)^{1/\alpha}\left(\sum_{k=1}^{n-1}\left\|\Delta B_{k}\right\|_{2}^{\beta}\right)^{1/\beta}.$$

Proof. As above we have

$$\left\|D_{\mathbf{p}}\left(\mathbf{A},\mathbf{B}\right)\right\|_{1} \leq \sum_{1\leq i<j\leq n}p_{i}p_{j}\sum_{k=i}^{j-1}\left\|\Delta A_{k}\right\|_{p}\sum_{l=i}^{j-1}\left\|\Delta B_{l}\right\|_{q}. \tag{2.15}$$

By Hölder's inequality for α, $\beta > 1$ with $\frac{1}{\alpha}+\frac{1}{\beta}=1$ we have for $j>i$ that

$$\sum_{k=i}^{j-1}\left\|\Delta A_{k}\right\|_{p} \leq (j-i)^{1/\beta}\left(\sum_{k=i}^{j-1}\left\|\Delta A_{k}\right\|_{p}^{\alpha}\right)^{1/\alpha}$$

and

$$\sum_{k=i}^{j-1}\left\|\Delta B_k\right\|_q \leq (j-i)^{1/\alpha}\left(\sum_{k=i}^{j-1}\left\|\Delta B_k\right\|_q^{\beta}\right)^{1/\beta},$$

which gives, by multiplication, that

$$\sum_{k=i}^{j-1}\left\|\Delta A_k\right\|_p \sum_{k=i}^{j-1}\left\|\Delta B_k\right\|_q \leq (j-i)\left(\sum_{k=i}^{j-1}\left\|\Delta A_k\right\|_p^{\alpha}\right)^{1/\alpha}\left(\sum_{k=i}^{j-1}\left\|\Delta B_k\right\|_q^{\beta}\right)^{1/\beta}.$$

Therefore, by weighted Hölder's inequality we can state

$$\begin{aligned}
(2.16)\qquad & \sum_{1\leq i<j\leq n} p_i p_j \sum_{k=i}^{j-1}\left\|\Delta A_k\right\|_p \sum_{l=i}^{j-1}\left\|\Delta B_l\right\|_q \\
&\leq \sum_{1\leq i<j\leq n} p_i p_j (j-i)\left(\sum_{k=i}^{j-1}\left\|\Delta A_k\right\|_p^{\alpha}\right)^{1/\alpha}\left(\sum_{k=i}^{j-1}\left\|\Delta B_k\right\|_q^{\beta}\right)^{1/\beta} \\
&\leq \left[\sum_{1\leq i<j\leq n} p_i p_j (j-i)\left[\left(\sum_{k=i}^{j-1}\left\|\Delta A_k\right\|_p^{\alpha}\right)^{1/\alpha}\right]^{\alpha}\right]^{1/\alpha} \\
&\times \left[\sum_{1\leq i<j\leq n} p_i p_j (j-i)\left[\left(\sum_{k=i}^{j-1}\left\|\Delta B_k\right\|_q^{\beta}\right)^{1/\beta}\right]^{\beta}\right]^{1/\beta} \\
&= \left[\sum_{1\leq i<j\leq n} p_i p_j (j-i)\sum_{k=i}^{j-1}\left\|\Delta A_k\right\|_p^{\alpha}\right]^{1/\alpha} \\
&\times \left[\sum_{1\leq i<j\leq n} p_i p_j (j-i)\sum_{k=i}^{j-1}\left\|\Delta B_k\right\|_q^{\beta}\right]^{1/\beta}.
\end{aligned}$$

Now, observe that

$$\begin{aligned}
& \sum_{1\leq i<j\leq n} p_i p_j (j-i)\sum_{k=i}^{j-1}\left\|\Delta A_k\right\|_p^{\alpha} \\
&= \frac{1}{2}\sum_{i,j=} p_i p_j (j-i)\sum_{k=i}^{j-1}\left\|\Delta A_k\right\|_p^{\alpha} \\
&= \frac{1}{2}\sum_{i,j=} p_i p_j (j-i)\left(\sum_{k=1}^{j-1}\left\|\Delta A_k\right\|_p^{\alpha} - \sum_{k=1}^{i-1}\left\|\Delta A_k\right\|_p^{\alpha}\right) \\
&= D_{\mathbf{p}}\left(\mathbf{e}, \mathbf{V}_{\mathbf{A},p,\alpha}\right)
\end{aligned}$$

and, similarly,

$$\sum_{1\leq i<j\leq n} p_i p_j (j-i)\sum_{k=i}^{j-1}\left\|\Delta B_k\right\|_q^{\beta} = D_{\mathbf{p}}\left(\mathbf{e}, \mathbf{V}_{\mathbf{B},q,\beta}\right).$$

Therefore, by (2.16)

$$\sum_{1\leq i<j\leq n} p_i p_j \sum_{k=i}^{j-1} \|\Delta A_k\|_p \sum_{l=i}^{j-1} \|\Delta B_l\|_q \leq [D_{\mathbf{p}}(\mathbf{e}, \mathbf{V}_{\mathbf{A},p,\alpha})]^{1/\alpha} [D_{\mathbf{p}}(\mathbf{e}, \mathbf{V}_{\mathbf{B},q,\beta})]^{1/\beta}. \tag{2.17}$$

Utilising (2.15) and (2.17) we derive the first inequality in (2.13).

By (1.3) we have

$$\begin{aligned} |D_{\mathbf{p}}(\mathbf{e}, \mathbf{V}_{\mathbf{A},p,\alpha})| &\leq \frac{1}{2} \sum_{k=1}^{n-1} \|\Delta A_k\|_p^\alpha \sum_{k=1}^{n} p_k \left| k - \sum_{j=1}^{n} j p_j \right| \\ &\leq \frac{1}{2} \sum_{k=1}^{n-1} \|\Delta A_k\|_p^\alpha \left[\sum_{k=1}^{n} k^2 p_k - \left(\sum_{k=1}^{n} k p_k \right)^2 \right]^{\frac{1}{2}} \\ &\leq \frac{1}{4}(n-1) \sum_{k=1}^{n-1} \|\Delta A_k\|_p^\alpha, \end{aligned} \tag{2.18}$$

and

$$\begin{aligned} |D_{\mathbf{p}}(\mathbf{e}, \mathbf{V}_{\mathbf{B},q,\beta})| &\leq \frac{1}{2} \sum_{k=1}^{n-1} \|\Delta B_k\|_q^\beta \sum_{k=1}^{n} p_k \left| k - \sum_{j=1}^{n} j p_j \right| \\ &\leq \frac{1}{2} \sum_{k=1}^{n-1} \|\Delta B_k\|_q^\beta \left[\sum_{k=1}^{n} k^2 p_k - \left(\sum_{k=1}^{n} k p_k \right)^2 \right]^{\frac{1}{2}} \\ &\leq \frac{1}{4}(n-1) \sum_{k=1}^{n-1} \|\Delta B_k\|_q^\beta. \end{aligned} \tag{2.19}$$

Therefore by (2.18) and (2.19) we get

$$\begin{aligned} &[D_{\mathbf{p}}(\mathbf{e}, \mathbf{V}_{\mathbf{A},p,\alpha})]^{1/\alpha} [D_{\mathbf{p}}(\mathbf{e}, \mathbf{V}_{\mathbf{B},q,\beta})]^{1/\beta} \\ &\leq \left[\frac{1}{2} \sum_{k=1}^{n-1} \|\Delta A_k\|_p^\alpha \sum_{k=1}^{n} p_k \left| k - \sum_{j=1}^{n} j p_j \right| \right]^{1/\alpha} \\ &\times \left[\frac{1}{2} \sum_{k=1}^{n-1} \|\Delta B_k\|_q^\beta \sum_{k=1}^{n} p_k \left| k - \sum_{j=1}^{n} j p_j \right| \right]^{1/\beta} \end{aligned}$$

$$\leq \left[\frac{1}{2}\sum_{k=1}^{n-1}\|\Delta A_k\|_p^{\alpha}\left[\sum_{k=1}^{n}k^2p_k-\left(\sum_{k=1}^{n}kp_k\right)^2\right]^{\frac{1}{2}}\right]^{1/\alpha}$$
$$\times\left[\frac{1}{2}\sum_{k=1}^{n-1}\|\Delta B_k\|_q^{\beta}\left[\sum_{k=1}^{n}k^2p_k-\left(\sum_{k=1}^{n}kp_k\right)^2\right]^{1/2}\right]^{1/\beta}$$
$$\leq\left[\frac{1}{4}(n-1)\sum_{k=1}^{n-1}\|\Delta A_k\|_p^{\alpha}\right]^{1/\alpha}\left[\frac{1}{4}(n-1)\sum_{k=1}^{n-1}\|\Delta B_k\|_q^{\beta}\right]^{1/\beta},$$

which proves the second part of (2.13). □

Corollary 2. *If* $\mathbf{A}=(A_1,...,A_n)\in(B_p(H))^n$ *and* $\mathbf{B}=(B_1,...,B_n)\in(B_q(H))^n$ *for* p, $q>1$ *with* $\frac{1}{p}+\frac{1}{q}=1$, *then*

$$\|D_{\mathbf{p}}(\mathbf{A},\mathbf{B})\|_1\leq[D_{\mathbf{p}}(\mathbf{e},\mathbf{V}_{\mathbf{A},p,\alpha})]^{1/\alpha}[D_{\mathbf{p}}(\mathbf{e},\mathbf{V}_{\mathbf{B},q,\beta})]^{1/\beta}$$
$$\leq\frac{1}{2}\left\lfloor\frac{n+1}{2}\right\rfloor\left(1-\frac{1}{n}\left\lfloor\frac{n+1}{2}\right\rfloor\right)$$
$$\times\left(\sum_{k=1}^{n-1}\|\Delta A_k\|_p^{\alpha}\right)^{1/\alpha}\left(\sum_{k=1}^{n-1}\|\Delta B_k\|_q^{\beta}\right)^{1/\beta},\tag{2.20}$$

where α, $\beta>1$ *with* $\frac{1}{\alpha}+\frac{1}{\beta}=1$.

In particular, if $\{A_k\}_{k=0}^n$, $\{B_k\}_{k=0}^n\subset B_2(H)$, *then*

$$\|D_{\mathbf{p}}(\mathbf{A},\mathbf{B})\|_1\leq[D_{\mathbf{p}}(\mathbf{e},\mathbf{V}_{\mathbf{A},2,\alpha})]^{1/\alpha}[D_{\mathbf{p}}(\mathbf{e},\mathbf{V}_{\mathbf{B},2,\beta})]^{1/\beta}$$
$$\leq\frac{1}{2}\left\lfloor\frac{n+1}{2}\right\rfloor\left(1-\frac{1}{n}\left\lfloor\frac{n+1}{2}\right\rfloor\right)$$
$$\times\left(\sum_{k=1}^{n-1}\|\Delta A_k\|_2^{\alpha}\right)^{1/\alpha}\left(\sum_{k=1}^{n-1}\|\Delta B_k\|_2^{\beta}\right)^{1/\beta}.\tag{2.21}$$

Remark 2. *If we take in (2.13)* $\alpha=p$ *and* $\beta=q$, *then we get*

$$\|D_{\mathbf{p}}(\mathbf{A},\mathbf{B})\|_1\leq[D_{\mathbf{p}}(\mathbf{e},\mathbf{V}_{\mathbf{A},p,p})]^{1/p}[D_{\mathbf{p}}(\mathbf{e},\mathbf{V}_{\mathbf{B},q,q})]^{1/q}$$
$$\leq\frac{1}{2}\sum_{k=1}^{n}p_k\left|k-\sum_{j=1}^{n}jp_j\right|$$
$$\times\left(\sum_{k=1}^{n-1}\|\Delta A_k\|_p^{p}\right)^{1/p}\left(\sum_{k=1}^{n-1}\|\Delta B_k\|_q^{q}\right)^{1/q}\tag{2.22}$$

$$\leq \frac{1}{2}\left[\sum_{k=1}^{n} k^2 p_k - \left(\sum_{k=1}^{n} k p_k\right)^2\right]^{\frac{1}{2}}$$
$$\times \left(\sum_{k=1}^{n-1} \|\Delta A_k\|_p^p\right)^{1/p} \left(\sum_{k=1}^{n-1} \|\Delta B_k\|_q^q\right)^{1/q}$$
$$\leq \frac{1}{4}(n-1) \left(\sum_{k=1}^{n-1} \|\Delta A_k\|_p^p\right)^{1/p} \left(\sum_{k=1}^{n-1} \|\Delta B_k\|_q^q\right)^{1/q}.$$

If we use the properties of trace, we derive from (2.22) that

$$\left|\operatorname{tr}\left(\sum_{k=1}^{n} p_k A_k B_k - \sum_{k=1}^{n} p_k A_k \sum_{k=1}^{n} p_k B_k\right)\right| \leq \frac{1}{2}\sum_{k=1}^{n} p_k \left|k - \sum_{j=1}^{n} j p_j\right| \times \left[\operatorname{tr}\left(\sum_{k=1}^{n-1} |\Delta A_k|^p\right)\right]^{1/p} \left[\operatorname{tr}\left(\sum_{k=1}^{n-1} |\Delta B_k|^q\right)\right]^{1/q} \tag{2.23}$$

provided that $\mathbf{A} = (A_1, ..., A_n) \in (B_p(H))^n$ *and* $\mathbf{B} = (B_1, ..., B_n) \in (B_q(H))^n$ *for* $p,\ q > 1$ *with* $\frac{1}{p} + \frac{1}{q} = 1$.

In particular, if $\mathbf{A} = (A_1, ..., A_n)$, $\mathbf{B} = (B_1, ..., B_n) \in (B_2(H))^n$, *then*

$$\left|\operatorname{tr}\left(\sum_{k=1}^{n} p_k A_k B_k - \sum_{k=1}^{n} p_k A_k \sum_{k=1}^{n} p_k B_k\right)\right|^2 \leq \frac{1}{4}\left(\sum_{k=1}^{n} p_k \left|k - \sum_{j=1}^{n} j p_j\right|\right)^2 \operatorname{tr}\left(\sum_{k=1}^{n-1} |\Delta A_k|^2\right) \operatorname{tr}\left(\sum_{k=1}^{n-1} |\Delta B_k|^2\right). \tag{2.24}$$

3. p-Schatten Norms Inequalities

We have the following results in terms of p-Schatten norms:

Theorem 6. *For* $p \geq 1$, *assume that* $\{A_k\}_{k=0}^n$, $\{B_k\}_{k=0}^n \subset B_p(H)$ *are sequences of operators and let* $\mathbf{p} = (p_1, ..., p_n)$ *be a probability density distribution, then we* *have the inequality:*

$$\begin{aligned}
\|D_{\mathbf{p}}(\mathbf{A},\mathbf{B})\|_p &\le \max_{k\in\{1,\dots,n-1\}} \|\Delta A_k\|_p D_{\mathbf{p}}(\mathbf{e},\mathbf{V}_{\mathbf{B},p}) \\
&\le \frac{1}{2}\sum_{k=1}^{n} p_k \left| k-\sum_{j=1}^{n} jp_j\right| \max_{k\in\{1,\dots,n-1\}} \|\Delta A_k\|_p \sum_{k=1}^{n-1}\|\Delta B_k\|_p \\
&\le \frac{1}{2}\left[\sum_{k=1}^{n} k^2 p_k - \left(\sum_{k=1}^{n} kp_k\right)^2\right]^{\frac{1}{2}} \\
&\quad\times \max_{k\in\{1,\dots,n-1\}} \|\Delta A_k\|_p \sum_{k=1}^{n-1}\|\Delta B_k\|_p \\
&\le \frac{1}{4}(n-1) \max_{k\in\{1,\dots,n-1\}} \|\Delta A_k\|_p \sum_{k=1}^{n-1}\|\Delta B_k\|_p .
\end{aligned} \tag{3.1}$$

Proof. Using the generalized triangle inequality and the property (1.12), we have successively:

$$\begin{aligned}
\|D_{\mathbf{p}}(\mathbf{A},\mathbf{B})\|_p &= \left\| \sum_{1\le i<j\le n} p_i p_j \sum_{k=i}^{j-1}\Delta A_k \sum_{l=k}^{j-1}\Delta B_l \right\|_p \\
&\le \sum_{1\le i<j\le n} p_i p_j \left\| \sum_{k=i}^{j-1}\Delta A_k \sum_{l=i}^{j-1}\Delta B_l \right\|_p \\
&\le \sum_{1\le i<j\le n} p_i p_j \left\| \sum_{k=i}^{j-1}\Delta A_k \right\|_p \left\| \sum_{l=i}^{j-1}\Delta B_l \right\|_p \\
&\le \sum_{1\le i<j\le n} p_i p_j \sum_{k=i}^{j-1}\|\Delta A_k\|_p \sum_{l=i}^{j-1}\|\Delta B_l\|_p .
\end{aligned}$$

By utilising a similar argument to the one from the proof of Theorem 4 we derive the desired result (3.1). □

Corollary 3. *For $p\ge 1$, assume that $\{A_k\}_{k=0}^{n}, \{B_k\}_{k=0}^{n} \subset B_p(H)$ are sequences of operators, then we have the inequality:*

$$\begin{aligned}
\|D_n(\mathbf{A},\mathbf{B})\|_p &\le \max_{k\in\{1,\dots,n-1\}} \|\Delta A_k\|_p D_n(\mathbf{e},\mathbf{V}_{\mathbf{B},p}) \\
&\le \frac{1}{2}\left\lfloor \frac{n+1}{2}\right\rfloor \left(1-\frac{1}{n}\left\lfloor \frac{n+1}{2}\right\rfloor\right) \\
&\quad\times \max_{k\in\{1,\dots,n-1\}} \|\Delta A_k\|_p \sum_{k=1}^{n-1}\|\Delta B_k\|_p .
\end{aligned} \tag{3.2}$$

Finally, we also have:

Theorem 7. *For $p\ge 1$, assume that $\{A_k\}_{k=0}^{n}, \{B_k\}_{k=0}^{n} \subset B_p(H)$ are sequences of operators and let $\mathbf{p}=(p_1,\dots,p_n)$ be a probability density distribution. If $\alpha, \beta > 1$*

with $\frac{1}{\alpha}+\frac{1}{\beta}=1$, *then we have the inequality:*

$$\begin{aligned}(3.3)\quad & \left\|D_{\mathbf{p}}\left(\mathbf{A},\mathbf{B}\right)\right\|_{p} \\ & \leq\left[D_{\mathbf{p}}\left(\mathbf{e},\mathbf{V}_{\mathbf{A},p,\alpha}\right)\right]^{1/\alpha}\left[D_{\mathbf{p}}\left(\mathbf{e},\mathbf{V}_{\mathbf{B},q,\beta}\right)\right]^{1/\beta} \\ & \leq\frac{1}{2}\sum_{k=1}^{n}p_{k}\left|k-\sum_{j=1}^{n}jp_{j}\right|\left(\sum_{k=1}^{n-1}\left\|\Delta A_{k}\right\|_{p}^{\alpha}\right)^{1/\alpha}\left(\sum_{k=1}^{n-1}\left\|\Delta B_{k}\right\|_{p}^{\beta}\right)^{1/\beta} \\ & \leq\frac{1}{2}\left[\sum_{k=1}^{n}k^{2}p_{k}-\left(\sum_{k=1}^{n}kp_{k}\right)^{2}\right]^{\frac{1}{2}}\left(\sum_{k=1}^{n-1}\left\|\Delta A_{k}\right\|_{p}^{\alpha}\right)^{1/\alpha}\left(\sum_{k=1}^{n-1}\left\|\Delta B_{k}\right\|_{p}^{\beta}\right)^{1/\beta} \\ & \leq\frac{1}{4}\left(n-1\right)\left(\sum_{k=1}^{n-1}\left\|\Delta A_{k}\right\|_{p}^{\alpha}\right)^{1/\alpha}\left(\sum_{k=1}^{n-1}\left\|\Delta B_{k}\right\|_{p}^{\beta}\right)^{1/\beta}.\end{aligned}$$

In particular,

$$\begin{aligned}(3.4)\quad \left\|D_{\mathbf{p}}\left(\mathbf{A},\mathbf{B}\right)\right\|_{p} & \leq\left[D_{\mathbf{p}}\left(\mathbf{e},\mathbf{V}_{\mathbf{A},p,2}\right)\right]^{1/2}\left[D_{\mathbf{p}}\left(\mathbf{e},\mathbf{V}_{\mathbf{B},q,2}\right)\right]^{1/2} \\ & \leq\frac{1}{2}\sum_{k=1}^{n}p_{k}\left|k-\sum_{j=1}^{n}jp_{j}\right|\left(\sum_{k=1}^{n-1}\left\|\Delta A_{k}\right\|_{p}^{2}\right)^{1/2}\left(\sum_{k=1}^{n-1}\left\|\Delta B_{k}\right\|_{p}^{2}\right)^{1/2} \\ & \leq\frac{1}{2}\left[\sum_{k=1}^{n}k^{2}p_{k}-\left(\sum_{k=1}^{n}kp_{k}\right)^{2}\right]^{\frac{1}{2}}\left(\sum_{k=1}^{n-1}\left\|\Delta A_{k}\right\|_{p}^{2}\right)^{1/2}\left(\sum_{k=1}^{n-1}\left\|\Delta B_{k}\right\|_{p}^{2}\right)^{1/2} \\ & \leq\frac{1}{4}\left(n-1\right)\left(\sum_{k=1}^{n-1}\left\|\Delta A_{k}\right\|_{p}^{2}\right)^{1/2}\left(\sum_{k=1}^{n-1}\left\|\Delta B_{k}\right\|_{p}^{2}\right)^{1/2}.\end{aligned}$$

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