Generalized Canavati Fractional Ostrowski, Opial and Grüss type inequalities for Banach algebra valued functions

George A. Anastassiou
Department of Mathematical Sciences
University of Memphis
Memphis, TN 38152, U.S.A.
ganastss@memphis.edu

Abstract

Using generalized Canavati fractional left and right vectorial Taylor formulae we establish mixed fractional Ostrowski, Opial and Grüss type inequalities involving several Banach algebra valued functions. The estimates are with respect to all norms $\|\cdot\|_p$, $1 \leq p \leq \infty$. We provide also applications.

2020 Mathematics Subject Classification : 26A33, 26D10, 26D15.
Keywords and Phrases: generalized Canavati fractional derivative, generalized Canavati fractional inequalities, Ostrowski-Opial-Grüss inequalities, Banach algebra.

1 Introduction

The following result motivates our work.

Theorem 1 *(1938, Ostrowski [9]) Let $f : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b) whose derivative $f' : (a,b) \to \mathbb{R}$ is bounded on (a,b), i.e., $\|f'\|_\infty^{\sup} := \sup_{t\in(a,b)} |f'(t)| < +\infty$. Then*

$$\left| \frac{1}{b-a} \int_a^b f(t)\,dt - f(x) \right| \leq \left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right] (b-a) \|f'\|_\infty^{\sup}, \qquad (1)$$

for any $x \in [a,b]$. The constant $\frac{1}{4}$ is the best possible.

RGMIA Res. Rep. Coll. 24 (2021), Art. 137, 21 pp. Received 22/10/21

Ostrowski type inequalities have great applications to integral approximations in Numerical Analysis.

We presented also ([3], Ch. 8,9) mixed fractional Ostrowski and Grüss inequalities for several functions for various norms.

See also the monographs written by the author [1], Chapters 24-26 and [2], Chapters 2-6.

In this article we generalize [3], Ch. 8,9 for several Banach algebra valued functions by employing their Canavati type generalized left and right fractional derivatives and our integrals are of Bochner type [7]. Motivation came also from [6].

We are also inspired by Z. Opial [8], 1960, famous inequality.

Theorem 2 *Let $x(t) \in C^1([0,h])$ be such that $x(0) = x(h) = 0$, and $x(t) > 0$ in $(0,h)$. Then*

$$\int_0^h |x(t)\, x'(t)|\, dt \leq \frac{h}{4} \int_0^h (x'(t))^2\, dt. \tag{2}$$

In (2), the constant $\frac{h}{4}$ is the best possible. Inequality (2) holds as equality for the optimal function

$$x(t) = \begin{cases} ct, & 0 \leq t \leq \frac{h}{2}, \\ c(h-t), & \frac{h}{2} \leq t \leq h, \end{cases} \tag{3}$$

where $c > 0$ is an arbitrary constant.

Opial-type inequalities are used a lot in proving uniqueness of solutions to differential equations and also to give upper bounds to their solutions.

For an extensive study about fractional Opial inequalities see the author's monograph [1].

In this work we also derive Opial type inequalities for Banach algebra valued functions with respect to their Canavati type generalized left and right fractional derivatives.

We include applications for Ostrowski and Opial inequalities. We finish the article with related Grüss type inequalities.

2 Background on Vectorial generalized Canavati fractional calculus

All in this section come from [5], pp. 109-115 and [4].

Let $g : [a,b] \to \mathbb{R}$ be a strictly increasing function. such that $g \in C^1([a,b])$, and $g^{-1} \in C^n([g(a), g(b)])$, $n \in \mathbb{N}$, $(X, \|\cdot\|)$ is a Banach space. Let $f \in C^n([a,b], X)$, and call $l := f \circ g^{-1} : [g(a), g(b)] \to X$. It is clear that $l, l', ..., l^{(n)}$ are continuous functions from $[g(a), g(b)]$ into $f([a,b]) \subseteq X$.

Let $\nu \geq 1$ such that $[\nu] = n$, $n \in \mathbb{N}$ as above, where $[\cdot]$ is the integral part of the number.

Clearly when $0 < \nu < 1$, $[\nu] = 0$.

I) Let $h \in C([g(a), g(b)], X)$, we define the left Riemann-Liouville Bochner fractional integral as

$$\left(J_{\nu}^{z_0} h\right)(z) := \frac{1}{\Gamma(\nu)} \int_{z_0}^{z} (z-t)^{\nu-1} h(t)\, dt, \tag{4}$$

for $g(a) \leq z_0 \leq z \leq g(b)$, where Γ is the gamma function; $\Gamma(\nu) = \int_0^\infty e^{-t} t^{\nu-1} dt$. We set $J_0^{z_0} h = h$.

Let $\alpha := \nu - [\nu]$ $(0 < \alpha < 1)$. We define the subspace $C_{g(x_0)}^{\nu}([g(a), g(b)], X)$ of $C^{[\nu]}([g(a), g(b)], X)$, where $x_0 \in [a, b]$ as:

$$C_{g(x_0)}^{\nu}([g(a), g(b)], X) =$$

$$\left\{ h \in C^{[\nu]}([g(a), g(b)], X) : J_{1-\alpha}^{g(x_0)} h^{([\nu])} \in C^1([g(x_0), g(b)], X) \right\}. \tag{5}$$

So let $h \in C_{g(x_0)}^{\nu}([g(a), g(b)], X)$, we define the left g-generalized X-valued fractional derivative of h of order ν, of Canavati type, over $[g(x_0), g(b)]$ as

$$D_{g(x_0)}^{\nu} h := \left(J_{1-\alpha}^{g(x_0)} h^{([\nu])} \right)'. \tag{6}$$

Clearly, for $h \in C_{g(x_0)}^{\nu}([g(a), g(b)], X)$, there exists

$$\left(D_{g(x_0)}^{\nu} h \right)(z) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dz} \int_{g(x_0)}^{z} (z-t)^{-\alpha} h^{([\nu])}(t)\, dt, \tag{7}$$

for all $g(x_0) \leq z \leq g(b)$.

In particular, when $f \circ g^{-1} \in C_{g(x_0)}^{\nu}([g(a), g(b)], X)$, we have that

$$\left(D_{g(x_0)}^{\nu} \left(f \circ g^{-1} \right) \right)(z) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dz} \int_{g(x_0)}^{z} (z-t)^{-\alpha} \left(f \circ g^{-1} \right)^{([\nu])}(t)\, dt, \tag{8}$$

for all $g(x_0) \leq z \leq g(b)$. We have that $D_{g(x_0)}^{n} \left(f \circ g^{-1} \right) = \left(f \circ g^{-1} \right)^{(n)}$ and $D_{g(x_0)}^{0} \left(f \circ g^{-1} \right) = f \circ g^{-1}$, see [4].

By [4], we have for $f \circ g^{-1} \in C_{g(x_0)}^{\nu}([g(a), g(b)], X)$, where $x_0 \in [a, b]$ the following left generalized g-fractional, of Canavati type,X-valued Taylor's formula:

Theorem 3 *Let $f \circ g^{-1} \in C_{g(x_0)}^{\nu}([g(a), g(b)], X)$, where $x_0 \in [a, b]$ is fixed.*

(i) If $\nu \geq 1$, then

$$f(x) - f(x_0) = \sum_{k=1}^{[\nu]-1} \frac{\left(f \circ g^{-1} \right)^{(k)} (g(x_0))}{k!} (g(x) - g(x_0))^k +$$

$$\frac{1}{\Gamma(\nu)} \int_{g(x_0)}^{g(x)} (g(x) - t)^{\nu-1} \left(D_{g(x_0)}^{\nu} \left(f \circ g^{-1} \right) \right)(t) \, dt, \tag{9}$$

for all $x_0 \leq x \leq b$.

(ii) If $0 < \nu < 1$, *we get*

$$f(x) = \frac{1}{\Gamma(\nu)} \int_{g(x_0)}^{g(x)} (g(x) - t)^{\nu-1} \left(D_{g(x_0)}^{\nu} \left(f \circ g^{-1} \right) \right)(t) \, dt, \tag{10}$$

for all $x_0 \leq x \leq b$.

II) Let $h \in C([g(a), g(b)], X)$, we define the right Riemann-Liouville Bochner fractional integral as

$$\left(J_{z_0-}^{\nu} h \right)(z) := \frac{1}{\Gamma(\nu)} \int_{z}^{z_0} (t - z)^{\nu-1} h(t) \, dt, \tag{11}$$

for $g(a) \leq z \leq z_0 \leq g(b)$. We set $J_{z_0-}^{0} h = h$.

Let $\alpha := \nu - [\nu]$ $(0 < \alpha < 1)$. We define the subspace $C_{g(x_0)-}^{\nu}([g(a), g(b)], X)$ of $C^{[\nu]}([g(a), g(b)], X)$, where $x_0 \in [a, b]$ as:

$$C_{g(x_0)-}^{\nu}([g(a), g(b)], X) :=$$

$$\left\{ h \in C^{[\nu]}([g(a), g(b)], X) : J_{g(x_0)-}^{1-\alpha} h^{([\nu])} \in C^1([g(a), g(x_0)], X) \right\}. \tag{12}$$

So let $h \in C_{g(x_0)-}^{\nu}([g(a), g(b)], X)$, we define the right g-generalized X-valued fractional derivative of h of order ν, of Canavati type, over $[g(a), g(x_0)]$ as

$$D_{g(x_0)-}^{\nu} h := (-1)^{n-1} \left(J_{g(x_0)-}^{1-\alpha} h^{([\nu])} \right)'. \tag{13}$$

Clearly, for $h \in C_{g(x_0)-}^{\nu}([g(a), g(b)], X)$, there exists

$$\left(D_{g(x_0)-}^{\nu} h \right)(z) = \frac{(-1)^{n-1}}{\Gamma(1-\alpha)} \frac{d}{dz} \int_{z}^{g(x_0)} (t - z)^{-\alpha} h^{([\nu])}(t) \, dt, \tag{14}$$

for all $g(a) \leq z \leq g(x_0) \leq g(b)$.

In particular, when $f \circ g^{-1} \in C_{g(x_0)-}^{\nu}([g(a), g(b)], X)$, we have that

$$\left(D_{g(x_0)-}^{\nu} \left(f \circ g^{-1} \right) \right)(z) = \frac{(-1)^{n-1}}{\Gamma(1-\alpha)} \frac{d}{dz} \int_{z}^{g(x_0)} (t - z)^{-\alpha} \left(f \circ g^{-1} \right)^{([\nu])}(t) \, dt, \tag{15}$$

for all $g(a) \leq z \leq g(x_0) \leq g(b)$.

We get that

$$\left(D_{g(x_0)-}^{n} \left(f \circ g^{-1} \right) \right)(z) = (-1)^n \left(f \circ g^{-1} \right)^{(n)}(z) \tag{16}$$

and $\left(D^{0}_{g(x_0)-}\left(f\circ g^{-1}\right)\right)(z)=\left(f\circ g^{-1}\right)(z)$, all $z\in[g(a),g(b)]$, see [4].

By [4], we have for $f\circ g^{-1}\in C^{\nu}_{g(x_0)-}([g(a),g(b)],X)$, where $x_0\in[a,b]$ is fixed, the following right generalized g-fractional, of Canavati type, X-valued Taylor's formula:

Theorem 4 *Let $f\circ g^{-1}\in C^{\nu}_{g(x_0)-}([g(a),g(b)],X)$, where $x_0\in[a,b]$ is fixed.*
(i) If $\nu\geq 1$, then

$$f(x)-f(x_0)=\sum_{k=1}^{[\nu]-1}\frac{\left(f\circ g^{-1}\right)^{(k)}(g(x_0))}{k!}(g(x)-g(x_0))^{k}+$$

$$\frac{1}{\Gamma(\nu)}\int_{g(x)}^{g(x_0)}(t-g(x))^{\nu-1}\left(D^{\nu}_{g(x_0)-}\left(f\circ g^{-1}\right)\right)(t)\,dt, \tag{17}$$

for all $a\leq x\leq x_0$,
(ii) If $0<\nu<1$, we get

$$f(x)=\frac{1}{\Gamma(\nu)}\int_{g(x)}^{g(x_0)}(t-g(x))^{\nu-1}\left(D^{\nu}_{g(x_0)-}\left(f\circ g^{-1}\right)\right)(t)\,dt, \tag{18}$$

all $a\leq x\leq x_0$.

III) Denote by

$$D^{m\nu}_{g(x_0)}=D^{\nu}_{g(x_0)}D^{\nu}_{g(x_0)}...D^{\nu}_{g(x_0)}\quad (m\text{-times}),\ m\in\mathbb{N}. \tag{19}$$

We mention the following modified and generalized left X-valued fractional Taylor's formula of Canavati type:

Theorem 5 *Let $f\in C^{1}([a,b],X)$, $g\in C^{1}([a,b])$, strictly increasing: $g^{-1}\in C^{1}([g(a),g(b)])$. Assume that $\left(D^{i\nu}_{g(x_0)}\left(f\circ g^{-1}\right)\right)\in C^{\nu}_{g(x_0)}([g(a),g(b)],X)$, $0<\nu<1$, $x_0\in[a,b]$, for $i=0,1,...,m$. Then*

$$f(x)=\frac{1}{\Gamma((m+1)\nu)}\int_{g(x_0)}^{g(x)}(g(x)-z)^{(m+1)\nu-1}\left(D^{(m+1)\nu}_{g(x_0)}\left(f\circ g^{-1}\right)\right)(z)\,dz, \tag{20}$$

all $x_0\leq x\leq b$.

IV) Denote by

$$D^{m\nu}_{g(x_0)-}=D^{\nu}_{g(x_0)-}D^{\nu}_{g(x_0)-}...D^{\nu}_{g(x_0)-}\quad (m\text{ times}),\ m\in\mathbb{N}. \tag{21}$$

We mention the following modified and generalized right X-valued fractional Taylor's formula of Canavati type:

Theorem 6 *Let* $f \in C^1([a,b], X)$, $g \in C^1([a,b])$, *strictly increasing:* $g^{-1} \in C^1([g(a), g(b)])$. *Assume that* $\left(D^{i\nu}_{g(x_0)-}\left(f \circ g^{-1}\right)\right) \in C^{\nu}_{g(x_0)-}([g(a), g(b)], X)$, $0 < \nu < 1$, $x_0 \in [a,b]$, *for all* $i = 0, 1, ..., m$. *Then*

$$f(x) = \frac{1}{\Gamma((m+1)\nu)} \int_{g(x)}^{g(x_0)} (z - g(x))^{(m+1)\nu - 1} \left(D^{(m+1)\nu}_{g(x_0)-}\left(f \circ g^{-1}\right)\right)(z)\, dz, \tag{22}$$

all $a \leq x \leq x_0 \leq b$.

3 Banach Algebras background

All here come from [10].

We need

Definition 7 *([10], p. 245) A complex algebra is a vector space* A *over the complex filed* $\mathbb{C}$ *in which a multiplication is defined that satisfies*

$$x(yz) = (xy)z, \tag{23}$$

$$(x+y)z = xz + yz, \;\; x(y+z) = xy + xz, \tag{24}$$

and

$$\alpha(xy) = (\alpha x)y = x(\alpha y), \tag{25}$$

for all x, y *and* z *in* A *and for all scalars* α.

Additionally if A *is a Banach space with respect to a norm that satisfies the multiplicative inequality*

$$\|xy\| \leq \|x\| \|y\| \quad (x \in A,\ y \in A) \tag{26}$$

and if A *contains a unit element* e *such that*

$$xe = ex = x \quad (x \in A) \tag{27}$$

and

$$\|e\| = 1, \tag{28}$$

then A *is called a Banach algebra.*

A *is commutative iff* $xy = yx$ *for all* $x, y \in A$.

We make

Remark 8 *Commutativity of* A *will be explicited stated when needed.*

There exists at most one $e \in A$ *that satisfies (27).*

Inequality (26) makes multiplication to be continuous, more precisely left and right continuous, see [10], p. 246.

Multiplication in A is not necessarily the numerical multiplication, it is something more general and it is defined abstractly, that is for $x, y \in A$ we have $xy \in A$, e.g. composition or convolution, etc.

For nice examples about Banach algebras see [10], p. 247-248, § 10.3.

We also make

Remark 9 *Next we mention about integration of A-valued functions, see [10], p. 259, § 10.22:*

If A is a Banach algebra and f is a continuous A-valued function on some compact Hausdorff space Q on which a complex Borel measure μ is defined, then $\int f d\mu$ exists and has all the properties that were discussed in Chapter 3 of [10], simply because A is a Banach space. However, an additional property can be added to these, namely: If $x \in A$, then

$$x \int_Q f \ d\mu = \int_Q x f(p) \ d\mu(p) \tag{29}$$

and

$$\left(\int_Q f \ d\mu \right) x = \int_Q f(p)\, x \ d\mu(p). \tag{30}$$

The Bochner integrals we will involve in our article follow (29) and (30). Also, let $f \in C([a,b], X)$, where $[a,b] \subset \mathbb{R}$, $(X, \|\cdot\|)$ is a Banach space. By [5], p. 3, f is Bochner integrable.

4 Main Results

We start with mixed generalized Canavati type fractional Ostrowski type inequalities for several functions over a Banach algebra. A uniform estimate follows.

Theorem 10 *Let $(A, \|\cdot\|)$ be a Banach algebra, $x_0 \in [a,b] \subset \mathbb{R}$, $\nu \geq 1$, $n = [\nu]$, $f_i \in C^n([a,b], A)$, $i = 1, ..., r \in \mathbb{N} - \{1\}$; $g \in C^1([a,b])$ strictly increasing, such that $g^{-1} \in C^n([g(a), g(b)])$, with $\left(f_i \circ g^{-1}\right)^{(k)}(g(x_0)) = 0$, $k = 1, ..., n-1$; $i = 1, ..., r$. Assume futher that $f \circ g^{-1} \in C^{\nu}_{g(x_0)-}([g(a), g(b)], A) \cap C^{\nu}_{g(x_0)}([g(a), g(b)], A)$.*

Denote by

$$K(f_1, ..., f_r)(x_0) :=$$
$$\sum_{i=1}^{r} \left[\int_a^b \left(\prod_{\substack{j=1 \\ j \neq i}}^{r} f_j(x) \right) f_i(x)\, dx - \left(\int_a^b \left(\prod_{\substack{j=1 \\ j \neq i}}^{r} f_j(x) \right) dx \right) f_i(x_0) \right]. \tag{31}$$

Then

$$\|K(f_1,...,f_r)(x_0)\| \leq \frac{1}{\Gamma(\nu+1)} \sum_{i=1}^{r} \left[\left[\left\| \left\| D^{\nu}_{g(x_0)-}\left(f_i \circ g^{-1}\right) \right\| \right\|_{\infty,[g(a),g(x_0)]} \right.\right.$$

$$\left.(g(x_0)-g(a))^{\nu} \left(\int_a^{x_0} \left(\prod_{\substack{j=1 \\ j\neq i}}^{r} \|f_j(x)\| \right) dx \right) \right] + \tag{32}$$

$$\left.\left[\left\| \left\| D^{\nu}_{g(x_0)}\left(f_i \circ g^{-1}\right) \right\| \right\|_{\infty,[g(x_0),g(b)]} (g(b)-g(x_0))^{\nu} \left(\int_{x_0}^{b} \left(\prod_{\substack{j=1 \\ j\neq i}}^{r} \|f_j(x)\| \right) dx \right) \right] \right].$$

Proof. Since $\left(f_i \circ g^{-1}\right)^{(k)}(g(x_0)) = 0$, $k = 1,...,[\nu]-1$; $i = 1,...,r$; we have by Theorem 3 that

$$f_i(x) - f_i(x_0) = \frac{1}{\Gamma(\nu)} \int_{g(x_0)}^{g(x)} (g(x)-t)^{\nu-1} \left(D^{\nu}_{g(x_0)}\left(f_i \circ g^{-1}\right) \right)(t)\,dt, \tag{33}$$

$\forall\ x \in [x_0, b]$,

and by Theorem 4 that

$$f_i(x) - f_i(x_0) = \frac{1}{\Gamma(\nu)} \int_{g(x)}^{g(x_0)} (t-g(x))^{\nu-1} \left(D^{\nu}_{g(x_0)-}\left(f_i \circ g^{-1}\right) \right)(t)\,dt, \tag{34}$$

$\forall\ x \in [a, x_0]$, for all $i = 1,...,r$.

Left multiplying (33) and (34) with $\left(\prod_{\substack{j=1 \\ j\neq i}}^{r} f_j(x)\right)$ we get, respectively,

$$\left(\prod_{\substack{j=1 \\ j\neq i}}^{r} f_j(x) \right) f_i(x) - \left(\prod_{\substack{j=1 \\ j\neq i}}^{r} f_j(x) \right) f_i(x_0) =$$

$$\frac{\left(\prod_{\substack{j=1 \\ j\neq i}}^{r} f_j(x)\right)}{\Gamma(\nu)} \int_{g(x_0)}^{g(x)} (g(x)-t)^{\nu-1} \left(D^{\nu}_{g(x_0)}\left(f_i \circ g^{-1}\right) \right)(t)\,dt, \tag{35}$$

$\forall\ x \in [x_0, b]$,

and

$$\left(\prod_{\substack{j=1 \\ j\neq i}}^{r} f_j(x) \right) f_i(x) - \left(\prod_{\substack{j=1 \\ j\neq i}}^{r} f_j(x) \right) f_i(x_0) = \tag{36}$$

$$\frac{\left(\prod_{\substack{j=1\\j\neq i}}^{r} f_j(x)\right)}{\Gamma(\nu)} \int_{g(x)}^{g(x_0)} (t-g(x))^{\nu-1} \left(D_{g(x_0)-}^{\nu}\left(f_i \circ g^{-1}\right)\right)(t)\,dt,$$

$\forall\ x \in [a, x_0]$, for all $i = 1, ..., r$.

Adding (35) and (36) as separate groups, we obtain

$$\sum_{i=1}^{r}\left(\prod_{\substack{j=1\\j\neq i}}^{r} f_j(x)\right) f_i(x) - \sum_{i=1}^{r}\left(\prod_{\substack{j=1\\j\neq i}}^{r} f_j(x)\right) f_i(x_0) =$$

$$\frac{1}{\Gamma(\nu)} \sum_{i=1}^{r}\left(\prod_{\substack{j=1\\j\neq i}}^{r} f_j(x)\right) \int_{g(x_0)}^{g(x)} (g(x)-t)^{\nu-1} \left(D_{g(x_0)}^{\nu}\left(f_i \circ g^{-1}\right)\right)(t)\,dt, \quad (37)$$

$\forall\ x \in [x_0, b]$,

and

$$\sum_{i=1}^{r}\left(\prod_{\substack{j=1\\j\neq i}}^{r} f_j(x)\right) f_i(x) - \sum_{i=1}^{r}\left(\prod_{\substack{j=1\\j\neq i}}^{r} f_j(x)\right) f_i(x_0) =$$

$$\frac{1}{\Gamma(\nu)} \sum_{i=1}^{r}\left(\prod_{\substack{j=1\\j\neq i}}^{r} f_j(x)\right) \int_{g(x)}^{g(x_0)} (t-g(x))^{\nu-1} \left(D_{g(x_0)-}^{\nu}\left(f_i \circ g^{-1}\right)\right)(t)\,dt, \quad (38)$$

$\forall\ x \in [a, x_0]$.

Next, we integrate (37) and (38) with respect to $x \in [a, b]$. We have

$$\sum_{i=1}^{r}\int_{x_0}^{b}\left(\prod_{\substack{j=1\\j\neq i}}^{r} f_j(x)\right) f_i(x)\,dx - \sum_{i=1}^{r}\left(\int_{x_0}^{b}\left(\prod_{\substack{j=1\\j\neq i}}^{r} f_j(x)\right) dx\right) f_i(x_0) =$$

$$\frac{1}{\Gamma(\nu)} \sum_{i=1}^{r}\left[\int_{x_0}^{b}\left(\prod_{\substack{j=1\\j\neq i}}^{r} f_j(x)\right)\left(\int_{g(x_0)}^{g(x)} (g(x)-t)^{\nu-1} \left(D_{g(x_0)}^{\nu}\left(f_i \circ g^{-1}\right)\right)(t)\,dt\right) dx\right], \quad (39)$$

and

$$\sum_{i=1}^{r}\int_{a}^{x_0}\left(\prod_{\substack{j=1\\j\neq i}}^{r} f_j(x)\right) f_i(x)\,dx - \sum_{i=1}^{r}\left(\int_{a}^{x_0}\left(\prod_{\substack{j=1\\j\neq i}}^{r} f_j(x)\right) dx\right) f_i(x_0) =$$

$$\frac{1}{\Gamma(\nu)}\sum_{i=1}^{r}\left[\int_a^{x_0}\left(\prod_{\substack{j=1\\ j\neq i}}^{r} f_j(x)\right)\left(\int_{g(x)}^{g(x_0)}(t-g(x))^{\nu-1}\left(D_{g(x_0)-}^{\nu}\left(f_i\circ g^{-1}\right)\right)(t)\,dt\right)dx\right], \tag{40}$$

Finally, adding (39) and (40) we obtain the useful identity

$$K(f_1,...,f_r)(x_0):=$$

$$\sum_{i=1}^{r}\left[\int_a^b\left(\prod_{\substack{j=1\\ j\neq i}}^{r} f_j(x)\right)f_i(x)\,dx-\left(\int_a^b\left(\prod_{\substack{j=1\\ j\neq i}}^{r} f_j(x)\right)dx\right)f_i(x_0)\right]=$$

$$\frac{1}{\Gamma(\nu)}\sum_{i=1}^{r}\left[\left[\int_a^{x_0}\left(\prod_{\substack{j=1\\ j\neq i}}^{r} f_j(x)\right)\left(\int_{g(x)}^{g(x_0)}(t-g(x))^{\nu-1}\left(D_{g(x_0)-}^{\nu}\left(f_i\circ g^{-1}\right)\right)(t)\,dt\right)dx\right]\right.$$

$$\left.+\left[\int_{x_0}^b\left(\prod_{\substack{j=1\\ j\neq i}}^{r} f_j(x)\right)\left(\int_{g(x_0)}^{g(x)}(g(x)-t)^{\nu-1}\left(D_{g(x_0)}^{\nu}\left(f_i\circ g^{-1}\right)\right)(t)\,dt\right)dx\right]\right]. \tag{41}$$

Therefore, we get that

$$\|K(f_1,...,f_r)(x_0)\|=$$

$$\left\|\sum_{i=1}^{r}\left[\int_a^b\left(\prod_{\substack{j=1\\ j\neq i}}^{r} f_j(x)\right)f_i(x)\,dx-\left(\int_a^b\left(\prod_{\substack{j=1\\ j\neq i}}^{r} f_j(x)\right)dx\right)f_i(x_0)\right]\right\|\leq\frac{1}{\Gamma(\nu)}$$

$$\sum_{i=1}^{r}\left[\left\|\left[\int_a^{x_0}\left(\prod_{\substack{j=1\\ j\neq i}}^{r} f_j(x)\right)\left(\int_{g(x)}^{g(x_0)}(t-g(x))^{\nu-1}\left(D_{g(x_0)-}^{\nu}\left(f_i\circ g^{-1}\right)\right)(t)\,dt\right)dx\right]\right\|\right. \tag{42}$$

$$\left.+\left\|\left[\int_{x_0}^b\left(\prod_{\substack{j=1\\ j\neq i}}^{r} f_j(x)\right)\left(\int_{g(x_0)}^{g(x)}(g(x)-t)^{\nu-1}\left(D_{g(x_0)}^{\nu}\left(f_i\circ g^{-1}\right)\right)(t)\,dt\right)dx\right]\right\|\right]\leq$$

$$\frac{1}{\Gamma(\nu)}\sum_{i=1}^{r}\left[\left[\int_a^{x_0}\left\|\left(\prod_{\substack{j=1\\ j\neq i}}^{r} f_j(x)\right)\left(\int_{g(x)}^{g(x_0)}(t-g(x))^{\nu-1}\left(D_{g(x_0)-}^{\nu}\left(f_i\circ g^{-1}\right)\right)(t)\,dt\right)\right\|dx\right]\right. \tag{43}$$

$$+\left[\int_{x_0}^{b}\left\|\left(\prod_{\substack{j=1\\ j\neq i}}^{r} f_j(x)\right)\left(\int_{g(x_0)}^{g(x)}(g(x)-t)^{\nu-1}\left(D_{g(x_0)}^{\nu}\left(f_i\circ g^{-1}\right)\right)(t)\,dt\right)\right\|dx\right]\right]\leq$$

$$\frac{1}{\Gamma(\nu)}\sum_{i=1}^{r}\left[\left[\int_{a}^{x_0}\left(\prod_{\substack{j=1\\ j\neq i}}^{r}\|f_j(x)\|\right)\left(\int_{g(x)}^{g(x_0)}(t-g(x))^{\nu-1}\left\|\left(D_{g(x_0)-}^{\nu}\left(f_i\circ g^{-1}\right)\right)(t)\right\|dt\right)dx\right]\right.$$

$$\left.+\left[\int_{x_0}^{b}\left(\prod_{\substack{j=1\\ j\neq i}}^{r}\|f_j(x)\|\right)\left(\int_{g(x_0)}^{g(x)}(g(x)-t)^{\nu-1}\left\|\left(D_{g(x_0)}^{\nu}\left(f_i\circ g^{-1}\right)\right)(t)\right\|dt\right)dx\right]\right]=:(*)\,. \tag{44}$$

Hence it holds

$$\|K(f_1,...,f_r)(x_0)\|\leq(*)\,. \tag{45}$$

We have that

$$(*)\leq\frac{1}{\Gamma(\nu+1)}$$

$$\sum_{i=1}^{r}\left[\left[\left\|\left\|\left(D_{g(x_0)-}^{\nu}\left(f_i\circ g^{-1}\right)\right)\right\|\right\|_{\infty,[g(a),g(x_0)]}\int_{a}^{x_0}\left(\prod_{\substack{j=1\\ j\neq i}}^{r}\|f_j(x)\|\right)(g(x_0)-g(x))^{\nu}\,dx\right]\right.$$

$$\left.+\left[\left\|\left\|\left(D_{g(x_0)}^{\nu}\left(f_i\circ g^{-1}\right)\right)\right\|\right\|_{\infty,[g(x_0),g(b)]}\int_{x_0}^{b}\left(\prod_{\substack{j=1\\ j\neq i}}^{r}\|f_j(x)\|\right)(g(x)-g(x_0))^{\nu}\,dx\right]\right]\leq \tag{46}$$

$$\frac{1}{\Gamma(\nu+1)}\sum_{i=1}^{r}\left[\left[\left\|\left\|\left(D_{g(x_0)-}^{\nu}\left(f_i\circ g^{-1}\right)\right)\right\|\right\|_{\infty,[g(a),g(x_0)]}\right.\right.$$

$$\left.(g(x_0)-g(a))^{\nu}\left(\int_{a}^{x_0}\left(\prod_{\substack{j=1\\ j\neq i}}^{r}\|f_j(x)\|\right)dx\right)\right]+ \tag{47}$$

$$\left.\left[\left\|\left\|\left(D_{g(x_0)}^{\nu}\left(f_i\circ g^{-1}\right)\right)\right\|\right\|_{\infty,[g(x_0),g(b)]}(g(b)-g(x_0))^{\nu}\left(\int_{x_0}^{b}\left(\prod_{\substack{j=1\\ j\neq i}}^{r}\|f_j(x)\|\right)dx\right)\right]\right],$$

proving (32). ■

Next comes an L_1 estimate.

Theorem 11 *All as in Theorem 10. Then*

$$\left\| K\left(f_1,...,f_r\right)(x_0)\right\| \leq \frac{1}{\Gamma(\nu)}$$

$$\sum_{i=1}^{r}\left[\left[\left\|\left\|\left(D^{\nu}_{g(x_0)-}\left(f_i\circ g^{-1}\right)\right)\right\|\right\|_{L_1([g(a),g(x_0)])}\int_a^{x_0}\left(\prod_{\substack{j=1\\ j\neq i}}^{r}\left\|f_j(x)\right\|\right)\left(g(x_0)-g(x)\right)^{\nu-1}dx\right]\right. \tag{48}$$

$$\left.+\left[\left\|\left\|\left(D^{\nu}_{g(x_0)}\left(f_i\circ g^{-1}\right)\right)\right\|\right\|_{L_1([g(x_0),g(b)])}\int_{x_0}^{b}\left(\prod_{\substack{j=1\\ j\neq i}}^{r}\left\|f_j(x)\right\|\right)\left(g(x)-g(x_0)\right)^{\nu-1}dx\right]\right].$$

Proof. We observe that (by (45)

$$(*) \leq \frac{1}{\Gamma(\nu)}$$

$$\sum_{i=1}^{r}\left[\left[\left\|\left\|\left(D^{\nu}_{g(x_0)-}\left(f_i\circ g^{-1}\right)\right)\right\|\right\|_{L_1([g(a),g(x_0)])}\int_a^{x_0}\left(\prod_{\substack{j=1\\ j\neq i}}^{r}\left\|f_j(x)\right\|\right)\left(g(x_0)-g(x)\right)^{\nu-1}dx\right]\right. \tag{49}$$

$$\left.+\left[\left\|\left\|\left(D^{\nu}_{g(x_0)}\left(f_i\circ g^{-1}\right)\right)\right\|\right\|_{L_1([g(x_0),g(b)])}\int_{x_0}^{b}\left(\prod_{\substack{j=1\\ j\neq i}}^{r}\left\|f_j(x)\right\|\right)\left(g(x)-g(x_0)\right)^{\nu-1}dx\right]\right],$$

proving (48). ■

An L_p estimate follows.

Theorem 12 *All as in Theorem 10. Let now $p,q>1:\frac{1}{p}+\frac{1}{q}=1$. Then*

$$\left\|K\left(f_1,...,f_r\right)(x_0)\right\| \leq \frac{1}{\left(p(\nu-1)+1\right)^{\frac{1}{p}}\Gamma(\nu)}$$

$$\sum_{i=1}^{r}\left[\left[\left\|\left\|\left(D^{\nu}_{g(x_0)-}\left(f_i\circ g^{-1}\right)\right)\right\|\right\|_{q,[g(a),g(x_0)]}\left(\int_a^{x_0}\left(g(x_0)-g(x)\right)^{\nu-\frac{1}{q}}\left(\prod_{\substack{j=1\\ j\neq i}}^{r}\left\|f_j(x)\right\|\right)dx\right)\right]\right. \tag{50}$$

$$\left.+\left[\left\|\left\|\left(D^{\nu}_{g(x_0)}\left(f_i\circ g^{-1}\right)\right)\right\|\right\|_{q,[g(x_0),g(b)]}\left(\int_{x_0}^{b}\left(g(x)-g(x_0)\right)^{\nu-\frac{1}{q}}\left(\prod_{\substack{j=1\\ j\neq i}}^{r}\left\|f_j(x)\right\|\right)dx\right)\right]\right].$$

Proof. We have that

$$(*) \overset{(45)}{\leq} \frac{1}{\Gamma(\nu)} \sum_{i=1}^{r} \left[\left[\int_a^{x_0} \left(\prod_{\substack{j=1 \\ j \neq i}}^{r} \|f_j(x)\| \right) \left(\int_{g(x)}^{g(x_0)} (t - g(x))^{p(\nu-1)} dt \right)^{\frac{1}{p}} \right.\right.$$

$$\left. \left(\int_{g(x)}^{g(x_0)} \left\| \left(D_{g(x_0)-}^{\nu} \left(f_i \circ g^{-1} \right) \right)(t) \right\|^q dt \right)^{\frac{1}{q}} dx \right] +$$

$$\left[\int_{x_0}^{b} \left(\prod_{\substack{j=1 \\ j \neq i}}^{r} \|f_j(x)\| \right) \left(\int_{g(x_0)}^{g(x)} (g(x) - t)^{p(\nu-1)} dt \right)^{\frac{1}{p}} \right.$$

$$\left.\left.\left(\int_{g(x)}^{g(x_0)} \left\| \left(D_{g(x_0)}^{\nu} \left(f_i \circ g^{-1} \right) \right)(t) \right\|^q dt \right)^{\frac{1}{q}} dx \right] \right] \leq \tag{51}$$

$$\frac{1}{\Gamma(\nu)} \sum_{i=1}^{r} \left[\left[\int_a^{x_0} \left(\prod_{\substack{j=1 \\ j \neq i}}^{r} \|f_j(x)\| \right) \frac{(g(x_0) - g(x))^{\nu - 1 + \frac{1}{p}}}{(p(\nu-1)+1)^{\frac{1}{p}}} \left\| \left\| \left(D_{g(x_0)-}^{\nu} \left(f_i \circ g^{-1} \right) \right) \right\| \right\|_{q,[g(a),g(x_0)]} dx \right] \right.$$

$$\left. + \left[\int_{x_0}^{b} \left(\prod_{\substack{j=1 \\ j \neq i}}^{r} \|f_j(x)\| \right) \frac{(g(x) - g(x_0))^{\nu - 1 + \frac{1}{p}}}{(p(\nu-1)+1)^{\frac{1}{p}}} \left\| \left\| \left(D_{g(x_0)}^{\nu} \left(f_i \circ g^{-1} \right) \right)(z) \right\| \right\|_{q,[g(x_0),g(b)]} dx \right] \right]$$

$$= \frac{1}{(p(\nu-1)+1)^{\frac{1}{p}} \Gamma(\nu)}$$

$$\sum_{i=1}^{r} \left[\left\| \left\| \left(D_{g(x_0)-}^{\nu} \left(f_i \circ g^{-1} \right) \right) \right\| \right\|_{q,[g(a),g(x_0)]} \left(\int_a^{x_0} (g(x_0) - g(x))^{\nu - \frac{1}{q}} \left(\prod_{\substack{j=1 \\ j \neq i}}^{r} \|f_j(x)\| \right) dx \right) \right. \tag{52}$$

$$\left. + \left\| \left\| \left(D_{g(x_0)}^{\nu} \left(f_i \circ g^{-1} \right) \right) \right\| \right\|_{q,[g(x_0),g(b)]} \left(\int_{x_0}^{b} (g(x) - g(x_0))^{\nu - \frac{1}{q}} \left(\prod_{\substack{j=1 \\ j \neq i}}^{r} \|f_j(x)\| \right) dx \right) \right],$$

proving (50). ■

Next we present a left generalized Canavati Opial type inequality:

Theorem 13 *Let $(A, \|\cdot\|)$ be a Banach algebra, $x_0 \in [a,b] \subset \mathbb{R}$, $\nu \geq 1$, $n = [\nu]$, $f \in C^n([a,b], A)$; $g \in C^1([a,b])$, strictly increasing, such that $g^{-1} \in C^n([g(a), g(b)])$, with $\left(f \circ g^{-1}\right)^{(k)}(g(x_0)) = 0$, $k = 0, 1, ..., n-1$. Assume* *further that* $f \circ g^{-1} \in C^{\nu}_{g(x_0)}([g(a), g(b)], A)$*. Let also* $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$*. Then*

$$\int_{g(x_0)}^{z} \left\| \left(\left(f \circ g^{-1} \right)(w) \right) \left(\left(D^{\nu}_{g(x_0)} \left(f \circ g^{-1} \right) \right)(w) \right) \right\| dw \leq \tag{53}$$

$$\frac{2^{-\frac{1}{q}} \left(z - g(x_0) \right)^{\nu + \frac{1}{p} - \frac{1}{q}}}{\Gamma(\nu) \left[\left(p(\nu - 1) + 1 \right) \left(p(\nu - 1) + 2 \right) \right]^{\frac{1}{p}}} \left(\int_{g(x_0)}^{z} \left\| \left(D^{\nu}_{g(x_0)} \left(f \circ g^{-1} \right) \right)(t) \right\|^{q} dt \right)^{\frac{2}{q}},$$

for all $g(x_0) \leq z \leq g(b)$.

Proof. By (9) and asumptions we get that

$$\left(f \circ g^{-1} \right)(z) = \frac{1}{\Gamma(\nu)} \int_{g(x_0)}^{z} (z - t)^{\nu - 1} \left(D^{\nu}_{g(x_0)} \left(f \circ g^{-1} \right) \right)(t)\, dt, \tag{54}$$

for all $g(x_0) \leq z \leq g(b)$.

By Hölder's inequality we obtain

$$\left\| \left(f \circ g^{-1} \right)(z) \right\| \leq \frac{1}{\Gamma(\nu)} \int_{g(x_0)}^{z} (z - t)^{\nu - 1} \left\| \left(D^{\nu}_{g(x_0)} \left(f \circ g^{-1} \right) \right)(t) \right\| dt \leq$$

$$\frac{1}{\Gamma(\nu)} \left(\int_{g(x_0)}^{z} (z - t)^{p(\nu - 1)} dt \right)^{\frac{1}{p}} \left(\int_{g(x_0)}^{z} \left\| \left(D^{\nu}_{g(x_0)} \left(f \circ g^{-1} \right) \right)(t) \right\|^{q} dt \right)^{\frac{1}{q}} =$$

$$\frac{1}{\Gamma(\nu)} \frac{\left(z - g(x_0) \right)^{\frac{p(\nu - 1) + 1}{p}}}{\left(p(\nu - 1) + 1 \right)^{\frac{1}{p}}} \left(\int_{g(x_0)}^{z} \left\| \left(D^{\nu}_{g(x_0)} \left(f \circ g^{-1} \right) \right)(t) \right\|^{q} dt \right)^{\frac{1}{q}}. \tag{55}$$

Call

$$\varphi(z) := \int_{g(x_0)}^{z} \left\| \left(D^{\nu}_{g(x_0)} \left(f \circ g^{-1} \right) \right)(t) \right\|^{q} dt, \tag{56}$$

$\varphi(g(x_0)) = 0$.

Thus

$$\varphi'(z) = \left\| \left(D^{\nu}_{g(x_0)} \left(f \circ g^{-1} \right) \right)(z) \right\|^{q} \geq 0, \tag{57}$$

and

$$\left(\varphi'(z) \right)^{\frac{1}{q}} = \left\| \left(D^{\nu}_{g(x_0)} \left(f \circ g^{-1} \right) \right)(z) \right\| \geq 0, \tag{58}$$

$\forall\, z \in [g(x_0), g(b)]$.

Consequently, we get

$$\left\| \left(\left(f \circ g^{-1} \right)(w) \right) |||| \left(\left(D^{\nu}_{g(x_0)} \left(f \circ g^{-1} \right) \right)(w) \right) \right\| \leq$$

$$\frac{1}{\Gamma(\nu)} \frac{\left(w - g(x_0) \right)^{\frac{p(\nu - 1) + 1}{p}}}{\left(p(\nu - 1) + 1 \right)^{\frac{1}{p}}} \left(\varphi(w) \varphi'(w) \right)^{\frac{1}{q}}, \tag{59}$$

$\forall\, w \in [g(x_0), g(b)]$.

Then

$$\int_{g(x_0)}^{z} \left\| \left(\left(f \circ g^{-1} \right)(w) \right) \left(\left(D_{g(x_0)}^{\nu} \left(f \circ g^{-1} \right) \right)(w) \right) \right\| dw \overset{(26)}{\leq}$$

$$\int_{g(x_0)}^{z} \left\| \left(\left(f \circ g^{-1} \right)(w) \right) \right\| \left\| \left(\left(D_{g(x_0)}^{\nu} \left(f \circ g^{-1} \right) \right)(w) \right) \right\| dw \leq$$

$$\frac{1}{\Gamma(\nu)\left(p(\nu-1)+1\right)^{\frac{1}{p}}} \int_{g(x_0)}^{z} \left(w - g(x_0) \right)^{\frac{p(\nu-1)+1}{p}} \left(\varphi(w)\varphi'(w) \right)^{\frac{1}{q}} dw \leq \quad (60)$$

$$\frac{1}{\Gamma(\nu)\left(p(\nu-1)+1\right)^{\frac{1}{p}}} \left(\int_{g(x_0)}^{z} \left(w - g(x_0) \right)^{p(\nu-1)+1} dw \right)^{\frac{1}{p}} \left(\int_{g(x_0)}^{z} \varphi(w)\, d\varphi(w) \right)^{\frac{1}{q}} =$$

$$\frac{1}{\Gamma(\nu)\left(p(\nu-1)+1\right)^{\frac{1}{p}} \left(p(\nu-1)+2\right)^{\frac{1}{p}}} \left(z - g(x_0) \right)^{\frac{p(\nu-1)+2}{p}} \left(\frac{\varphi^2(z)}{2} \right)^{\frac{1}{q}} =$$

$$\frac{2^{-\frac{1}{q}} \left(z - g(x_0) \right)^{\nu+\frac{1}{p}-\frac{1}{q}}}{\Gamma(\nu)\left[\left(p(\nu-1)+1\right)\left(p(\nu-1)+2\right) \right]^{\frac{1}{p}}} \left(\int_{g(x_0)}^{z} \left\| \left(D_{g(x_0)}^{\nu} \left(f \circ g^{-1} \right) \right)(t) \right\|^{q} dt \right)^{\frac{2}{q}}, \quad (61)$$

for all $g(x_0) \leq z \leq g(b)$, proving (53). ■

It follows the corresponding right side fractional Opial type inequality:

Theorem 14 *All as in Theorem 13, however now it is $f \circ g^{-1} \in C_{g(x_0)-}^{\nu}\left([g(a), g(b)], A \right)$. Then*

$$\int_{z}^{g(x_0)} \left\| \left(\left(f \circ g^{-1} \right)(w) \right) \left(\left(D_{g(x_0)-}^{\nu} \left(f \circ g^{-1} \right) \right)(w) \right) \right\| dw \leq$$

$$\frac{2^{-\frac{1}{q}} \left(g(x_0) - z \right)^{\nu+\frac{1}{p}-\frac{1}{q}}}{\Gamma(\nu)\left[\left(p(\nu-1)+1\right)\left(p(\nu-1)+2\right) \right]^{\frac{1}{p}}} \left(\int_{z}^{g(x_0)} \left\| \left(D_{g(x_0)-}^{\nu} \left(f \circ g^{-1} \right) \right)(t) \right\|^{q} dt \right)^{\frac{2}{q}}, \quad (62)$$

for all $g(a) \leq z \leq g(x_0)$.

Proof. Based on (17), and as similar to the proof of Theorem 13 is omitted. ■

It follows the modified generalized left A-valued fractional Opial inequality:

Theorem 15 *All as in Theorem 5 and let $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$. Here we assume that $\frac{1}{(m+1)q} < \nu < 1$. Then*

$$\int_{g(x_0)}^{z} \left\| \left(\left(f \circ g^{-1} \right)(w) \right) \left(\left(D_{g(x_0)}^{(m+1)\nu} \left(f \circ g^{-1} \right) \right)(w) \right) \right\| dw \leq \quad (63)$$

$$\frac{2^{-\frac{1}{q}} \left(z - g(x_0) \right)^{(m+1)\nu+\frac{1}{p}-\frac{1}{q}}}{\Gamma((m+1)\nu)\left[\left(p((m+1)\nu-1)+1\right)\left(p((m+1)\nu-1)+2\right) \right]^{\frac{1}{p}}}$$

$$\left(\int_{g(x_0)}^{z}\left\|\left(D_{g(x_0)}^{(m+1)\nu}\left(f\circ g^{-1}\right)\right)(t)\right\|^q dt\right)^{\frac{2}{q}},$$

for all $g(x_0)\leq z\leq g(b)$*.*

Proof. As in Theorem 13. ■

The corresponding modified generalized right A-valued fractional Opial inequality comes next:

Theorem 16 *All as in Theorem 6 and let* $p,q>1:\frac{1}{p}+\frac{1}{q}=1$*. Here we assume that* $\frac{1}{(m+1)q}<\nu<1$*. Then*

$$\int_{z}^{g(x_0)}\left\|\left(\left(f\circ g^{-1}\right)(w)\right)\left(\left(D_{g(x_0)-}^{(m+1)\nu}\left(f\circ g^{-1}\right)\right)(w)\right)\right\|dw\leq \tag{64}$$

$$\frac{2^{-\frac{1}{q}}\left(g(x_0)-z\right)^{(m+1)\nu+\frac{1}{p}-\frac{1}{q}}}{\Gamma\left((m+1)\nu\right)\left[\left(p\left((m+1)\nu-1\right)+1\right)\left(p\left((m+1)\nu-1\right)+2\right)\right]^{\frac{1}{p}}}$$

$$\left(\int_{z}^{g(x_0)}\left\|\left(D_{g(x_0)-}^{(m+1)\nu}\left(f\circ g^{-1}\right)\right)(t)\right\|^q dt\right)^{\frac{2}{q}},$$

for all $g(a)\leq z\leq g(x_0)$*.*

Proof. As in Theorem 13. ■

5 Applications

We make

Remark 17 *Assume in this section that* $(A,\|\cdot\|)$ *is a commutative Banach algebra. Then, we get that*

$$K(f_1,...,f_r)(x_0)\overset{(31)}{=}r\int_a^b\left(\prod_{j=1}^{r}f_j(x)\right)dx-\sum_{i=1}^{r}\left(\int_a^b\left(\prod_{\substack{j=1\\j\neq i}}^{r}f_j(x)\right)dx\right)f_i(x_0), \tag{65}$$

$x_0\in[a,b]$.

When $r=2$*, we find*

$$K(f_1,f_2)(x_0)=2\int_a^b f_1(x)f_2(x)dx-f_1(x_0)\int_a^b f_2(x)dx-f_2(x_0)\int_a^b f_1(x)dx, \tag{66}$$

$x_0\in[a,b]$.

We give

Corollary 18 *All as in Theorem 10, A is a commutative Banach algebra, $r = 2$. Then*

$$\|K(f_1, f_2)(x_0)\| \leq \frac{1}{\Gamma(\nu+1)} \sum_{i=1}^{2} \left[\left[\left\| \left\| \left(D^{\nu}_{g(x_0)-} \left(f_i \circ g^{-1} \right) \right) \right\| \right\|_{\infty,[g(a),g(x_0)]} \right.\right.$$

$$\left. (g(x_0) - g(a))^{\nu} \left(\int_a^{x_0} \left(\prod_{\substack{j=1 \\ j \neq i}}^{2} \|f_j(x)\| \right) dx \right) \right] +$$

$$\left.\left[\left\| \left\| \left(D^{\nu}_{g(x_0)} \left(f_i \circ g^{-1} \right) \right) \right\| \right\|_{\infty,[g(x_0),g(b)]} (g(b) - g(x_0))^{\nu} \left(\int_{x_0}^{b} \left(\prod_{\substack{j=1 \\ j \neq i}}^{2} \|f_j(x)\| \right) dx \right) \right] \right]. \tag{67}$$

Proof. By Theorem 10. ■

We also give

Corollary 19 *All as in Corollary 18, for $g(t) = e^t$. Then*

$$\|K(f_1, f_2)(x_0)\| \leq \frac{1}{\Gamma(\nu+1)} \sum_{i=1}^{2} \left[\left[\left\| \left\| D^{\nu}_{e^{x_0}-} (f_i \circ \log) \right\| \right\|_{\infty,[e^a,e^{x_0}]} \right.\right.$$

$$\left. (e^{x_0} - e^a)^{\nu} \left(\int_a^{x_0} \left(\prod_{\substack{j=1 \\ j \neq i}}^{2} \|f_j(x)\| \right) dx \right) \right] + \tag{68}$$

$$\left.\left[\left\| \left\| D^{\nu}_{e^{x_0}} (f_i \circ \log) \right\| \right\|_{\infty,[e^{x_0},e^b]} \left(e^b - e^{x_0}\right)^{\nu} \left(\int_{x_0}^{b} \left(\prod_{\substack{j=1 \\ j \neq i}}^{2} \|f_j(x)\| \right) dx \right) \right] \right].$$

Proof. By Corollary 18. ■

We add also the following:

Corollary 20 *(to Theorem 13) All as in Theorem 13 for $g(t) = e^t$. Then*

$$\int_{e^{x_0}}^{z} \|((f \circ \log)(w)) ((D^{\nu}_{e^{x_0}} (f \circ \log))(w))\| \, dw \leq \tag{69}$$

$$\frac{2^{-\frac{1}{q}} (z - e^{x_0})^{\nu + \frac{1}{p} - \frac{1}{q}}}{\Gamma(\nu) \left[(p(\nu-1)+1)(p(\nu-1)+2) \right]^{\frac{1}{p}}} \left(\int_{e^{x_0}}^{z} \|(D^{\nu}_{e^{x_0}} (f \circ \log))(t)\|^q \, dt \right)^{\frac{2}{q}},$$

for all $e^{x_0} \leq z \leq e^b$.

Proof. By Theorem 13. ■

6 Addendum

We give the following generalized Canavati type fractional Grüss type inequalities for several functions over a Banach algebra.

We start with a uniform estimate.

Theorem 21 *Let* $(A, \|\cdot\|)$ *be a Banach algebra,* $x_0 \in [a,b] \subset \mathbb{R}$, $1 \leq \nu < 2$, $f_i \in C^1([a,b], A)$, $i = 1, ..., r \in \mathbb{N} - \{1\}$; $g \in C^1([a,b])$ *strictly increasing, such that* $g^{-1} \in C^1([g(a), g(b)])$. *Assume futher that* $f \circ g^{-1} \in C^{\nu}_{g(x_0)-}([g(a), g(b)], A) \cap C^{\nu}_{g(x_0)}([g(a), g(b)], A)$.

Assume that

$$M_1(f_1, ..., f_r)(x_0) := \max_{i=1,...,r} \left\{ \sup_{x_0 \in [a,b]} \left\| \left\| D^{\nu}_{g(x_0)-}\left(f_i \circ g^{-1}\right) \right\| \right\|_{\infty,[g(a),g(x_0)]}, \sup_{x_0 \in [a,b]} \left\| \left\| D^{\nu}_{g(x_0)}\left(f_i \circ g^{-1}\right) \right\| \right\|_{\infty,[g(x_0),g(b)]} \right\} < \infty. \tag{70}$$

Denote by

$$\Delta(f_1, ..., f_r) := \int_a^b K(f_1, ..., f_r)(x)\, dx =$$

$$\sum_{i=1}^r \left[(b-a) \left(\int_a^b \left(\prod_{\substack{j=1 \\ j \neq i}}^r f_j(x) \right) f_i(x)\, dx \right) - \left(\int_a^b \left(\prod_{\substack{j=1 \\ j \neq i}}^r f_j(x) \right) dx \right) \left(\int_a^b f_i(x)\, dx \right) \right]. \tag{71}$$

Then

$$\|\Delta(f_1, ..., f_r)\| =$$

$$\left\| \sum_{i=1}^r \left[(b-a) \left(\int_a^b \left(\prod_{\substack{j=1 \\ j \neq i}}^r f_j(x) \right) f_i(x)\, dx \right) - \left(\int_a^b \left(\prod_{\substack{j=1 \\ j \neq i}}^r f_j(x) \right) dx \right) \left(\int_a^b f_i(x)\, dx \right) \right] \right\|$$

$$\leq \frac{M_1(f_1, ..., f_r)(b-a)^2 (g(b) - g(a))^{\nu}}{\Gamma(\nu+1)} \sum_{i=1}^r \left(\left\| \prod_{\substack{j=1 \\ j \neq i}}^r \|f_j(x)\| \right\|_{\infty,[a,b]} \right). \tag{72}$$

Proof. From (32) we get

$$R.H.S.(32) \leq \frac{M_1(f_1, ..., f_r)(g(b) - g(a))^{\nu}}{\Gamma(\nu+1)} \left(\sum_{i=1}^r \int_a^b \left(\prod_{\substack{j=1 \\ j \neq i}}^r \|f_j(x)\| \right) dx \right) \leq$$

$$\frac{M_1(f_1,...,f_r)(g(b)-g(a))^{\nu}(b-a)}{\Gamma(\nu+1)}\left(\sum_{i=1}^{r}\left\|\prod_{\substack{j=1\\ j\neq i}}^{r}\|f_j\|\right\|_{\infty,[a,b]}\right)=:\lambda_1. \quad (73)$$

We have that

$$\|\Delta(f_1,...,f_r)\|\leq\int_a^b\|K(f_1,...,f_r)(x_0)\|dx_0\overset{((32),(73))}{\leq}\int_a^b\lambda_1 dx=$$

$$\frac{M_1(f_1,...,f_r)(g(b)-g(a))^{\nu}(b-a)^2}{\Gamma(\nu+1)}\left(\sum_{i=1}^{r}\left(\left\|\prod_{\substack{j=1\\ j\neq i}}^{r}\|f_j(x)\|\right\|_{\infty,[a,b]}\right)\right), \quad (74)$$

proving (72). ■

Next comes an L_1-estimate.

Theorem 22 *All as in Theorem 21, however now we assume that*

$$M_2(f_1,...,f_r)(x_0):=\max_{i=1,...,r}\left\{\sup_{x_0\in[a,b]}\left\|\left\|D^{\nu}_{g(x_0)-}\left(f_i\circ g^{-1}\right)\right\|\right\|_{L_1([g(a),g(x_0)])},\right.$$

$$\left.\sup_{x_0\in[a,b]}\left\|\left\|D^{\nu}_{g(x_0)}\left(f_i\circ g^{-1}\right)\right\|\right\|_{L_1([g(x_0),g(b)])}\right\}<\infty. \quad (74)$$

Then

$$\|\Delta(f_1,...,f_r)\|\leq$$

$$\leq\frac{M_2(f_1,...,f_r)(b-a)^2(g(b)-g(a))^{\nu-1}}{\Gamma(\nu)}\sum_{i=1}^{r}\left(\left\|\prod_{\substack{j=1\\ j\neq i}}^{r}\|f_j(x)\|\right\|_{\infty,[a,b]}\right). \quad (75)$$

Proof. By (48) we get

$$R.H.S.(48)\leq\frac{M_2(f_1,...,f_r)(g(b)-g(a))^{\nu-1}}{\Gamma(\nu)}\left(\sum_{i=1}^{r}\int_a^b\left(\prod_{\substack{j=1\\ j\neq i}}^{r}\|f_j(x)\|\right)dx\right)\leq$$

$$\frac{M_2(f_1,...,f_r)(g(b)-g(a))^{\nu-1}(b-a)}{\Gamma(\nu)}\left(\sum_{i=1}^{r}\left\|\prod_{\substack{j=1\\ j\neq i}}^{r}\|f_j\|\right\|_{\infty,[a,b]}\right)=:\lambda_2. \quad (76)$$

We have that

$$\|\Delta(f_1,...,f_r)\|\leq\int_a^b\|K(f_1,...,f_r)(x_0)\|dx_0\overset{((48),(76))}{\leq}\int_a^b\lambda_2 dx=$$

$$\frac{M_2(f_1,...,f_r)(g(b)-g(a))^{\nu-1}(b-a)^2}{\Gamma(\nu)}\left(\sum_{i=1}^{r}\left(\left\|\prod_{\substack{j=1\\ j\neq i}}^{r}\|f_j(x)\|\right\|_{\infty,[a,b]}\right)\right), \quad (77)$$

proving (75). ■

An L_p-estimate follows:

Theorem 23 *All as in Theorem 21, however here we assume that*

$$M_3(f_1,...,f_r)(x_0) := \max_{i=1,...,r}\left\{\sup_{x_0\in[a,b]}\left\|\left\|D^{\nu}_{g(x_0)-}\left(f_i\circ g^{-1}\right)\right\|\right\|_{L_q([g(a),g(x_0)])},\right.$$

$$\left.\sup_{x_0\in[a,b]}\left\|\left\|D^{\nu}_{g(x_0)}\left(f_i\circ g^{-1}\right)\right\|\right\|_{L_q([g(x_0),g(b)])}\right\}<\infty. \quad (78)$$

Above it is $p,q>1:\frac{1}{p}+\frac{1}{q}=1$.

Then

$$\|\Delta(f_1,...,f_r)\|\leq$$

$$\leq\frac{M_3(f_1,...,f_r)(b-a)^2(g(b)-g(a))^{\nu-\frac{1}{q}}}{(p(\nu-1)+1)^{\frac{1}{p}}\Gamma(\nu)}\sum_{i=1}^{r}\left(\left\|\prod_{\substack{j=1\\ j\neq i}}^{r}\|f_j(x)\|\right\|_{\infty,[a,b]}\right). \quad (79)$$

Proof. By (50) we get

$$R.H.S.(50)\leq\frac{M_3(f_1,...,f_r)(g(b)-g(a))^{\nu-\frac{1}{q}}}{(p(\nu-1)+1)^{\frac{1}{p}}\Gamma(\nu)}\left(\sum_{i=1}^{r}\int_a^b\left(\prod_{\substack{j=1\\ j\neq i}}^{r}\|f_j(x)\|\right)dx\right)\leq$$

$$\frac{M_3(f_1,...,f_r)(g(b)-g(a))^{\nu-\frac{1}{q}}(b-a)}{(p(\nu-1)+1)^{\frac{1}{p}}\Gamma(\nu)}\left(\sum_{i=1}^{r}\left\|\prod_{\substack{j=1\\ j\neq i}}^{r}\|f_j\|\right\|_{\infty,[a,b]}\right)=:\lambda_3. \quad (80)$$

Hence it holds

$$\|\Delta(f_1,...,f_r)\|\leq\int_a^b\|K(f_1,...,f_r)(x_0)\|\,dx_0\overset{((50),(80))}{\leq}\int_a^b\lambda_3dx=$$

$$\frac{M_3(f_1,...,f_r)(g(b)-g(a))^{\nu-\frac{1}{q}}(b-a)^2}{(p(\nu-1)+1)^{\frac{1}{p}}\Gamma(\nu)}\left(\sum_{i=1}^{r}\left(\left\|\prod_{\substack{j=1\\ j\neq i}}^{r}\|f_j(x)\|\right\|_{\infty,[a,b]}\right)\right), \quad (81)$$

proving (79). ■

References

[1] G.A. Anastassiou, *Fractional Differentiation Inequalities*, Research Monograph, Springer, New York, 2009.

[2] G.A. Anastassiou, *Advances on Fractional Inequalities*, Research Monograph, Springer, New York, 2011.

[3] G.A. Anastassiou, *Intelligent Comparisons: Analytic Inequalities*, Springer, Heidelberg, New York, 2016.

[4] G.A. Anastassiou, *Strong mixed and generalized fractional calculus for Banach space valued functions*, Mat. Vesnik, 69(3) (2017), 176-191.

[5] G.A. Anastassiou, *Intelligent Computations: Abstract Fractional Calculus, Inequalities, Approximations*, Springer, Heidelberg, New York, 2018.

[6] S.S. Dragomir, *Noncommutative Ostrowski type inequalities for functions in Banach algebras*, RGMIA Res. Rep. Coll. 24 (2021), Art. 10, 24 pp.

[7] J. Mikusinski, *The Bochner integral*, Academic Press, New York, 1978.

[8] Z. Opial, *Sur une inegalite*, Ann. Polon. Math. 8(1960), 29-32.

[9] A. Ostrowski, *Über die Absolutabweichung einer differentiabaren Funcktion von ihrem Integralmittelwert*, Comment. Math. Helv., 10 (1938), 226-227.

[10] W. Rudin, *Functional Analysis*, Second Edition, McGraw-Hill, Inc., New York, 1991.