# REVERSES OF HERMITE-HADAMARD TYPE INEQUALITIES FOR THE SQUARE NORM IN HILBERT SPACES

SILVESTRU SEVER DRAGOMIR[1,2]

Abstract. Let $(H; \langle \cdot, \cdot \rangle)$ be a complex Hilbert space and $f : [0, \infty) \to \mathbb{R}$ be convex on $[0, \infty)$. If $x, y \in H$ and there exists the constants $0 \le m < M$ such that $m \le \|(1-t)x + ty\|^2 \le M$ for all $t \in [0,1]$, then

$$
\begin{aligned}
&(0 \le) \int_0^1 f\left(\|(1-t)x+ty\|^2\right) dt - f\left(\frac{\|x\|^2 + \operatorname{Re}\langle x, y\rangle + \|y\|^2}{3}\right) \\
&\le \frac{M - \frac{1}{3}\left(\|x\|^2 + \operatorname{Re}\langle x, y\rangle + \|y\|^2\right)}{M-m} f(m) \\
&+ \frac{\frac{1}{3}\left(\|x\|^2 + \operatorname{Re}\langle x, y\rangle + \|y\|^2\right) - m}{M-m} f(M) \\
&- f\left(\frac{\|x\|^2 + \operatorname{Re}\langle x, y\rangle + \|y\|^2}{3}\right) \\
&\le \frac{2}{M-m}\left\{\frac{1}{2}(M-m) + \left|\frac{\|x\|^2 + \operatorname{Re}\langle x, y\rangle + \|y\|^2}{3} - \frac{m+M}{2}\right|\right\} \\
&\times \left[\frac{f(m)+f(M)}{2} - f\left(\frac{m+M}{2}\right)\right] \\
&\le 2\left[\frac{f(m)+f(M)}{2} - f\left(\frac{m+M}{2}\right)\right].
\end{aligned}
$$

Some examples for power functions and exponential are also provided.

## 1. Introduction

Let $\mathbb{R}$ be the set of real numbers and $I \subseteq \mathbb{R}$ be an interval. A function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is said to be *convex* in the classical sense if it satisfies the following inequality

$$f(tx + (1-t)t) \le tf(x) + (1-t)f(y)$$

for all $x, y \in I$ and $t \in [0,1]$.

Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ and $a, b \in I$ with $a < b$. Then the inequality

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)dx \le \frac{f(a)+f(b)}{2} \tag{1.1}$$

holds if $f$ is convex, and is known in the literature as *Hermite-Hadamard inequality*, after the name of C. Hermite and J. Hadamard (see [13]). The inequalities in (1.1) hold in reversed direction if $f$ is a concave function.

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A vast literature related to (1.1) have been produced by a large number of mathematicians [11] since it is considered to be one of the most famous inequality for convex functions due to its usefulness and many applications in various branches of Pure and Applied Mathematics, such as Numerical Analysis [2], Information Theory [1], Operator Theory [6], [7] and others.

Let $X$ be a vector space over the real or complex number field $\mathbb{K}$ and $x,y\in X,\ x\neq y$. Define the segment

$$[x,y]:=\{(1-t)x+ty,\ t\in[0,1]\}.$$

We consider the function $f:[x,y]\to\mathbb{R}$ and the associated function

$$g(x,y):[0,1]\to\mathbb{R},\ g(x,y)(t):=f[(1-t)x+ty],\ t\in[0,1].$$

Note that $f$ is convex on $[x,y]$ if and only if $g(x,y)$ is convex on $[0,1]$.

For any convex function defined on a segment $[x,y]\subset X$, we have the *Hermite-Hadamard integral inequality* (see [3, p. 2], [4, p. 2])

$$f\left(\frac{x+y}{2}\right)\leq\int_0^1 f[(1-t)x+ty]dt\leq\frac{f(x)+f(y)}{2}, \tag{1.2}$$

which can be derived from the classical Hermite-Hadamard inequality (1.1) for the convex function $g(x,y):[0,1]\to\mathbb{R}$.

Since $f(x)=\|x\|^p$ ($x\in X$ and $1\leq p<\infty$) is a convex function, then for any $x,y\in X$ we have the following norm inequality from (1.2) (see [12, p. 106])

$$\left\|\frac{x+y}{2}\right\|^p\leq\int_0^1\|(1-t)x+ty\|^p dt\leq\frac{\|x\|^p+\|y\|^p}{2}. \tag{1.3}$$

Assume that $(H;\langle\cdot,\cdot\rangle)$ is a complex Hilbert space that generates the norm $\|\cdot\|$.

In the recent paper [9] we proved the following Hermite-Hadamard type inequalities:

**Theorem 1.** *Let $f:[0,\infty)\to\mathbb{R}$ be convex (concave) on $[0,\infty)$. If $x,\ y\in H$ with* $\operatorname{Re}\langle x,y\rangle\geq 0,$ *then*

$$\begin{aligned} &f\left(\frac{\|x\|^2+\operatorname{Re}\langle x,y\rangle+\|y\|^2}{3}\right) \\ &\leq(\geq)\int_0^1 f\left(\|(1-t)\,x+ty\|^2\right)dt \\ &\leq(\geq)\frac{1}{3}\left[f\left(\|x\|^2\right)+f\left[\operatorname{Re}\langle x,y\rangle\right]+f\left(\|y\|^2\right)\right]. \end{aligned} \tag{1.4}$$

Motivated by the above results we provide in this paper some upper bounds for the positive difference

$$\int_0^1 f\left(\|(1-t)\,x+ty\|^2\right)dt-f\left(\frac{\|x\|^2+\operatorname{Re}\langle x,y\rangle+\|y\|^2}{3}\right)$$

in the case of convex functions $f:[0,\infty)\to\mathbb{R}$ and some assumptions for the vectors $x,\ y\in H$. Some examples for power function, logarithm and exponential are also provided.

## 2. Main Results

We have the following reverses of the Hermite-Hadamard type inequalities:

**Theorem 2.** *Let $f:[0,\infty)\to\mathbb{R}$ be convex on $[0,\infty)$. If $x,\ y\in H$ and there exists the constants $0\le m<M$ such that $m\le\|(1-t)x+ty\|^2\le M$ for all $t\in[0,1]$, then*

$$\begin{aligned}
0 &\le \frac{2}{M-m}\left\{\frac{1}{2}(M-m)-\int_0^1\left|\|(1-t)x+ty\|^2-\frac{m+M}{2}\right|dt\right\} \\
&\times\left[\frac{f(m)+f(M)}{2}-f\left(\frac{m+M}{2}\right)\right] \\
&\le \frac{M-\frac{1}{3}\left(\|x\|^2+\operatorname{Re}\langle x,y\rangle+\|y\|^2\right)}{M-m}f(m) \\
&+\frac{\frac{1}{3}\left(\|x\|^2+\operatorname{Re}\langle x,y\rangle+\|y\|^2\right)-m}{M-m}f(M) \\
&-\int_0^1 f\left(\|(1-t)x+ty\|^2\right)dt \\
&\le \frac{2}{M-m}\left\{\frac{1}{2}(M-m)+\int_0^1\left|\|(1-t)x+ty\|^2-\frac{m+M}{2}\right|dt\right\} \\
&\times\left[\frac{f(m)+f(M)}{2}-f\left(\frac{m+M}{2}\right)\right] \\
&\le 2\left[\frac{f(m)+f(M)}{2}-f\left(\frac{m+M}{2}\right)\right].
\end{aligned}\tag{2.1}$$

*Proof.* We use the double inequality, see [5]

$$\begin{aligned}
&2\min\{\alpha,1-\alpha\}\left[\frac{f(m)+f(M)}{2}-f\left(\frac{m+M}{2}\right)\right] \\
&\le(1-\alpha)f(m)+\alpha f(M)-f((1-\alpha)m+\alpha M) \\
&\le 2\max\{\alpha,1-\alpha\}\left[\frac{f(m)+f(M)}{2}-f\left(\frac{m+M}{2}\right)\right],
\end{aligned}\tag{2.2}$$

for $\alpha\in[0,1]$.

Let $s\in[m,M]$ and take $\alpha=\frac{s-m}{M-m}\in[0,1]$ to get

$$\begin{aligned}
&2\min\left\{\frac{s-m}{M-m},\frac{M-s}{M-m}\right\}\left[\frac{f(m)+f(M)}{2}-f\left(\frac{m+M}{2}\right)\right] \\
&\le\frac{M-s}{M-m}f(m)+\frac{s-m}{M-m}f(M)-f\left(\frac{M-s}{M-m}m+\frac{s-m}{M-m}M\right) \\
&\le 2\max\left\{\frac{s-m}{M-m},\frac{M-s}{M-m}\right\}\left[\frac{f(m)+f(M)}{2}-f\left(\frac{m+M}{2}\right)\right],
\end{aligned}$$

namely

$$0 \leq \frac{2}{M-m}\left\{\frac{1}{2}(M-m)-\left|s-\frac{m+M}{2}\right|\right\} \times\left[\frac{f(m)+f(M)}{2}-f\left(\frac{m+M}{2}\right)\right] \leq \frac{M-s}{M-m} f(m)+\frac{s-m}{M-m} f(M)-f(s) \leq \frac{2}{M-m}\left\{\frac{1}{2}(M-m)+\left|s-\frac{m+M}{2}\right|\right\} \times\left[\frac{f(m)+f(M)}{2}-f\left(\frac{m+M}{2}\right)\right] . \tag{2.3}$$

If we take $s=\|(1-t) x+t y\|^{2}, t \in[0,1]$ in (2.3) we get

$$0 \leq \frac{2}{M-m}\left\{\frac{1}{2}(M-m)-\left|\|(1-t) x+t y\|^{2}-\frac{m+M}{2}\right|\right\} \times\left[\frac{f(m)+f(M)}{2}-f\left(\frac{m+M}{2}\right)\right] \leq \frac{M-\|(1-t) x+t y\|^{2}}{M-m} f(m)+\frac{\|(1-t) x+t y\|^{2}-m}{M-m} f(M) -f\left(\|(1-t) x+t y\|^{2}\right) \leq \frac{2}{M-m}\left\{\frac{1}{2}(M-m)+\left|\|(1-t) x+t y\|^{2}-\frac{m+M}{2}\right|\right\} \times\left[\frac{f(m)+f(M)}{2}-f\left(\frac{m+M}{2}\right)\right] \leq 2\left[\frac{f(m)+f(M)}{2}-f\left(\frac{m+M}{2}\right)\right] . \tag{2.4}$$

If we take the integral over $t \in[0,1]$, then we get

$$0 \leq \frac{2}{M-m}\left\{\frac{1}{2}(M-m)-\int_{0}^{1}\left|\|(1-t) x+t y\|^{2}-\frac{m+M}{2}\right| d t\right\} \times\left[\frac{f(m)+f(M)}{2}-f\left(\frac{m+M}{2}\right)\right] \leq \frac{M-\int_{0}^{1}\|(1-t) x+t y\|^{2} d t}{M-m} f(m)+\frac{\int_{0}^{1}\|(1-t) x+t y\|^{2} d t-m}{M-m} f(M) -\int_{0}^{1} f\left(\|(1-t) x+t y\|^{2}\right) d t \leq \frac{2}{M-m}\left\{\frac{1}{2}(M-m)+\int_{0}^{1}\left|\|(1-t) x+t y\|^{2}-\frac{m+M}{2}\right| d t\right\} \times\left[\frac{f(m)+f(M)}{2}-f\left(\frac{m+M}{2}\right)\right] \leq 2\left[\frac{f(m)+f(M)}{2}-f\left(\frac{m+M}{2}\right)\right] . \tag{2.5}$$

Since

$$\begin{aligned}
&\int_0^1 \|(1-t)x+ty\|^2 dt \\
&= \int_0^1 \left[(1-t)^2 \|x\|^2 + 2t(1-t)\operatorname{Re}\langle x,y\rangle + t^2\|y\|^2\right] dt \\
&= \frac{1}{3}\left(\|x\|^2 + \operatorname{Re}\langle x,y\rangle + \|y\|^2\right),
\end{aligned}$$

hence by (2.5) we get (2.1). □

**Corollary 1.** *Let $f:[0,\infty)\to\mathbb{R}$ be convex on $[0,\infty)$ with $f(0)=0$. If $x,\ y\in H$ and there exists the constant $0<M$ such that $\|x\|^2,\ \|y\|^2\leq M$, then*

$$\begin{aligned}
(2.6)\qquad 0 &\leq \frac{2}{M}\left\{\frac{1}{2}M - \int_0^1 \left|\|(1-t)x+ty\|^2 - \frac{M}{2}\right| dt\right\}\left[\frac{f(M)}{2} - f\left(\frac{M}{2}\right)\right] \\
&\leq \frac{\frac{1}{3}\left(\|x\|^2 + \operatorname{Re}\langle x,y\rangle + \|y\|^2\right)}{M} f(M) \\
&- \int_0^1 f\left(\|(1-t)x+ty\|^2\right) dt \\
&\leq \frac{2}{M}\left\{\frac{1}{2}M + \int_0^1 \left|\|(1-t)x+ty\|^2 - \frac{M}{2}\right| dt\right\}\left[\frac{f(M)}{2} - f\left(\frac{M}{2}\right)\right] \\
&\leq f(M) - 2f\left(\frac{M}{2}\right).
\end{aligned}$$

*Proof.* It follows by Theorem 2 for $m=0$ and observing that

$$\|(1-t)x+ty\|^2 \leq (1-t)\|x\|^2 + t\|y\|^2 \leq (1-t)M + tM = M.$$

□

**Theorem 3.** *Let $f:[0,\infty)\to\mathbb{R}$ be convex on $[0,\infty)$. If $x,\ y\in H$ and there exists the constants $0\leq m<M$ such that $m\leq\|(1-t)x+ty\|^2\leq M$ for all $t\in[0,1]$, then*

$$\begin{aligned}
(2.7)\qquad (0\leq) &\int_0^1 f\left(\|(1-t)x+ty\|^2\right) dt - f\left(\frac{\|x\|^2 + \operatorname{Re}\langle x,y\rangle + \|y\|^2}{3}\right) \\
&\leq \frac{M - \frac{1}{3}\left(\|x\|^2 + \operatorname{Re}\langle x,y\rangle + \|y\|^2\right)}{M-m} f(m) \\
&+ \frac{\frac{1}{3}\left(\|x\|^2 + \operatorname{Re}\langle x,y\rangle + \|y\|^2\right) - m}{M-m} f(M) \\
&- f\left(\frac{\|x\|^2 + \operatorname{Re}\langle x,y\rangle + \|y\|^2}{3}\right)
\end{aligned}$$

$$\leq \frac{2}{M-m}\left\{\frac{1}{2}(M-m)+\left|\frac{\|x\|^2+\operatorname{Re}\langle x,y\rangle+\|y\|^2}{3}-\frac{m+M}{2}\right|\right\}$$
$$\times\left[\frac{f(m)+f(M)}{2}-f\left(\frac{m+M}{2}\right)\right]$$
$$\leq 2\left[\frac{f(m)+f(M)}{2}-f\left(\frac{m+M}{2}\right)\right].$$

*Proof.* From (2.1) we have

$$\int_0^1 f\left(\|(1-t)x+ty\|^2\right)dt \leq \frac{M-\frac{1}{3}\left(\|x\|^2+\operatorname{Re}\langle x,y\rangle+\|y\|^2\right)}{M-m}f(m)$$
$$+\frac{\frac{1}{3}\left(\|x\|^2+\operatorname{Re}\langle x,y\rangle+\|y\|^2\right)-m}{M-m}f(M).$$

By subtracting $f\left(\frac{1}{3}\left(\|x\|^2+\operatorname{Re}\langle x,y\rangle+\|y\|^2\right)\right)$ in both sides, we get

$$(0\leq)\int_0^1 f\left(\|(1-t)x+ty\|^2\right)dt-f\left(\frac{1}{3}\left(\|x\|^2+\operatorname{Re}\langle x,y\rangle+\|y\|^2\right)\right) \tag{2.8}$$
$$\leq \frac{M-\frac{1}{3}\left(\|x\|^2+\operatorname{Re}\langle x,y\rangle+\|y\|^2\right)}{M-m}f(m)$$
$$+\frac{\frac{1}{3}\left(\|x\|^2+\operatorname{Re}\langle x,y\rangle+\|y\|^2\right)-m}{M-m}f(M)$$
$$-f\left(\frac{1}{3}\left(\|x\|^2+\operatorname{Re}\langle x,y\rangle+\|y\|^2\right)\right),$$

which proves the first part of (2.8).

From the second part of (2.3) for $s=\frac{1}{3}\left(\|x\|^2+\operatorname{Re}\langle x,y\rangle+\|y\|^2\right)$ we get

$$\frac{M-\frac{1}{3}\left(\|x\|^2+\operatorname{Re}\langle x,y\rangle+\|y\|^2\right)}{M-m}f(m) \tag{2.9}$$
$$+\frac{\frac{1}{3}\left(\|x\|^2+\operatorname{Re}\langle x,y\rangle+\|y\|^2\right)-m}{M-m}f(M)$$
$$-f\left(\frac{1}{3}\left(\|x\|^2+\operatorname{Re}\langle x,y\rangle+\|y\|^2\right)\right)$$
$$\leq \frac{2}{M-m}\left\{\frac{1}{2}(M-m)+\left|\frac{\|x\|^2+\operatorname{Re}\langle x,y\rangle+\|y\|^2}{3}-\frac{m+M}{2}\right|\right\}$$
$$\times\left[\frac{f(m)+f(M)}{2}-f\left(\frac{m+M}{2}\right)\right]$$
$$\leq 2\left[\frac{f(m)+f(M)}{2}-f\left(\frac{m+M}{2}\right)\right].$$

By making use of (2.8) and (2.9) we derive the desired result (2.7). □

**Corollary 2.** *Let $f:[0,\infty)\to\mathbb{R}$ be convex on $[0,\infty)$ with $f(0)=0$. If $x,\ y\in H$ and there exists the constant $0<M$ such that $\|x\|^2,\ \|y\|^2\le M$, then*

$$
\begin{aligned}
(2.10)\qquad & (0\le)\int_0^1 f\left(\|(1-t)x+ty\|^2\right)dt-f\left(\frac{\|x\|^2+\operatorname{Re}\langle x,y\rangle+\|y\|^2}{3}\right)\\
&\le\frac{\frac{1}{3}\left(\|x\|^2+\operatorname{Re}\langle x,y\rangle+\|y\|^2\right)}{M}f(M)\\
&-f\left(\frac{\|x\|^2+\operatorname{Re}\langle x,y\rangle+\|y\|^2}{3}\right)\\
&\le\frac{2}{M}\left\{\frac{1}{2}M+\left|\frac{\|x\|^2+\operatorname{Re}\langle x,y\rangle+\|y\|^2}{3}-\frac{M}{2}\right|\right\}\left[\frac{1}{2}f(M)-f\left(\frac{M}{2}\right)\right]\\
&\le f(M)-2f\left(\frac{M}{2}\right).
\end{aligned}
$$

From a different perspective we also have:

**Theorem 4.** *Let $f:[0,\infty)\to\mathbb{R}$ be convex on $[0,\infty)$. If $x,\ y\in H$ and there exists the constants $0\le m<M$ such that $m\le\|(1-t)x+ty\|^2\le M$ for all $t\in[0,1]$, then*

$$
\begin{aligned}
(2.11)\qquad & (0\le)\int_0^1 f\left(\|(1-t)x+ty\|^2\right)dt-f\left(\frac{\|x\|^2+\operatorname{Re}\langle x,y\rangle+\|y\|^2}{3}\right)\\
&\le\frac{M-\frac{1}{3}\left(\|x\|^2+\operatorname{Re}\langle x,y\rangle+\|y\|^2\right)}{M-m}f(m)\\
&+\frac{\frac{1}{3}\left(\|x\|^2+\operatorname{Re}\langle x,y\rangle+\|y\|^2\right)-m}{M-m}f(M)\\
&-f\left(\frac{\|x\|^2+\operatorname{Re}\langle x,y\rangle+\|y\|^2}{3}\right)\\
&\le\frac{f'_-(M)-f'_+(m)}{M-m}\left[\frac{1}{3}\left(\|x\|^2+\operatorname{Re}\langle x,y\rangle+\|y\|^2\right)-m\right]\\
&\times\left[M-\frac{1}{3}\left(\|x\|^2+\operatorname{Re}\langle x,y\rangle+\|y\|^2\right)\right]\\
&\le\frac{1}{4}\left[f'_-(M)-f'_+(m)\right](M-m).
\end{aligned}
$$

*Proof.* We use the inequality, see [10]

$$
\begin{aligned}
&(1-\alpha)f(m)+\alpha f(M)-f((1-\alpha)m+\alpha M)\\
&\le\alpha(1-\alpha)(M-m)\left[f'_-(M)-f'_+(m)\right]
\end{aligned}
$$

for $\alpha\in[0,1]$.

Let $s\in[m,M]$ and take $\alpha=\frac{s-m}{M-m}\in[0,1]$ to get

$$
(2.12)\quad \frac{M-s}{M-m}f(m)+\frac{s-m}{M-m}f(M)-f(s)\le\frac{f'_-(M)-f'_+(m)}{M-m}(s-m)(M-s).
$$

Now, if we take $s=\frac{1}{3}\left(\|x\|^2+\operatorname{Re}\langle x,y\rangle+\|y\|^2\right)$ in (2.12) we get

$$\begin{aligned}
&\frac{M-\frac{1}{3}\left(\|x\|^2+\operatorname{Re}\langle x,y\rangle+\|y\|^2\right)}{M-m}f(m)\\
&+\frac{\frac{1}{3}\left(\|x\|^2+\operatorname{Re}\langle x,y\rangle+\|y\|^2\right)-m}{M-m}f(M)\\
&-f\left(\frac{1}{3}\left(\|x\|^2+\operatorname{Re}\langle x,y\rangle+\|y\|^2\right)\right)\\
&\le\frac{f'_-(M)-f'_+(m)}{M-m}\left(\frac{1}{3}\left(\|x\|^2+\operatorname{Re}\langle x,y\rangle+\|y\|^2\right)-m\right)\\
&\le\left(M-\frac{1}{3}\left(\|x\|^2+\operatorname{Re}\langle x,y\rangle+\|y\|^2\right)\right)\\
&\le\frac{1}{4}\left[f'_-(M)-f'_+(m)\right](M-m)
\end{aligned}$$

and by (2.8) we get (2.11). □

**Corollary 3.** *Let $f:[0,\infty)\to\mathbb{R}$ be convex on $[0,\infty)$ with $f(0)=0$. If $x,\ y\in H$ and there exists the constant $0<M$ such that $\|x\|^2,\ \|y\|^2\le M$, then*

$$\begin{aligned}
(2.13)\qquad &(0\le)\int_0^1 f\left(\|(1-t)x+ty\|^2\right)dt-f\left(\frac{\|x\|^2+\operatorname{Re}\langle x,y\rangle+\|y\|^2}{3}\right)\\
&\le\frac{\frac{1}{3}\left(\|x\|^2+\operatorname{Re}\langle x,y\rangle+\|y\|^2\right)}{M}f(M)\\
&-f\left(\frac{\|x\|^2+\operatorname{Re}\langle x,y\rangle+\|y\|^2}{3}\right)\\
&\le\frac{f'_-(M)-f'_+(0)}{3M}\left(\|x\|^2+\operatorname{Re}\langle x,y\rangle+\|y\|^2\right)\\
&\times\left[M-\frac{1}{3}\left(\|x\|^2+\operatorname{Re}\langle x,y\rangle+\|y\|^2\right)\right]\\
&\le\frac{1}{4}\left[f'_-(M)-f'_+(0)\right]M.
\end{aligned}$$

Finally, we also have:

**Theorem 5.** *Let $f:[0,\infty)\to\mathbb{R}$ be twice differentiable and so that there exists the constants $0\le D$ such that $0\le f''(t)\le D$ for any $t\in(m,M)\subset[0,\infty)$. If $x,\ y\in H$ such that $m\le\|(1-t)x+ty\|^2\le M$ for all $t\in[0,1]$, then*

$$(2.14)\qquad (0\le)\int_0^1 f\left(\|(1-t)x+ty\|^2\right)dt-f\left(\frac{\|x\|^2+\operatorname{Re}\langle x,y\rangle+\|y\|^2}{3}\right)$$

$$\le \frac{M-\frac{1}{3}\left(\|x\|^2+\operatorname{Re}\langle x,y\rangle+\|y\|^2\right)}{M-m}f(m)$$
$$+\frac{\frac{1}{3}\left(\|x\|^2+\operatorname{Re}\langle x,y\rangle+\|y\|^2\right)-m}{M-m}f(M)$$
$$-f\left(\frac{\|x\|^2+\operatorname{Re}\langle x,y\rangle+\|y\|^2}{3}\right)$$

$$\le \frac{1}{2}D\left[\frac{1}{3}\left(\|x\|^2+\operatorname{Re}\langle x,y\rangle+\|y\|^2\right)-m\right]$$
$$\times\left[M-\frac{1}{3}\left(\|x\|^2+\operatorname{Re}\langle x,y\rangle+\|y\|^2\right)\right]$$
$$\le \frac{1}{8}D(M-m)^2.$$

*Proof.* If there exists the constants $0\le d$, $D$ such that

$$d\le f''(t)\le D \text{ for any } t\in(m,M),$$

then, see for instance [8],

$$\frac{1}{2}\nu(1-\nu)d(M-m)^2\le(1-\nu)f(m)+\nu f(M)-f((1-\nu)m+\nu M)\le\frac{1}{2}\nu(1-\nu)D(M-m)^2 \tag{2.15}$$

for any $\nu\in[0,1]$.

Let $s\in[m,M]$ and take $\alpha=\frac{s-m}{M-m}\in[0,1]$ in (2.15) to get

$$\frac{1}{2}d(s-m)(M-s)\le\frac{M-s}{M-m}f(m)+\frac{s-m}{M-m}f(M)-f(s)\le\frac{1}{2}D(s-m)(M-s). \tag{2.16}$$

Now, if we take $s=\frac{1}{3}\left(\|x\|^2+\operatorname{Re}\langle x,y\rangle+\|y\|^2\right)$ in (2.16), then we get the inequality of interest

$$\frac{1}{2}d\left(\frac{1}{3}\left(\|x\|^2+\operatorname{Re}\langle x,y\rangle+\|y\|^2\right)-m\right) \tag{2.17}$$
$$\times\left(M-\frac{1}{3}\left(\|x\|^2+\operatorname{Re}\langle x,y\rangle+\|y\|^2\right)\right)$$
$$\le \frac{M-\frac{1}{3}\left(\|x\|^2+\operatorname{Re}\langle x,y\rangle+\|y\|^2\right)}{M-m}f(m)$$
$$+\frac{\frac{1}{3}\left(\|x\|^2+\operatorname{Re}\langle x,y\rangle+\|y\|^2\right)-m}{M-m}f(M)$$
$$-f\left(\frac{1}{3}\left(\|x\|^2+\operatorname{Re}\langle x,y\rangle+\|y\|^2\right)\right)$$

$$\leq \frac{1}{2}D\left(\frac{1}{3}\left(\|x\|^2+\operatorname{Re}\langle x,y\rangle+\|y\|^2\right)-m\right)$$
$$\times\left(M-\frac{1}{3}\left(\|x\|^2+\operatorname{Re}\langle x,y\rangle+\|y\|^2\right)\right)$$
$$\leq \frac{1}{8}D\left(M-m\right)^2.$$

By utilizing (2.8) and (2.17) we derive (2.14). □

**Corollary 4.** *Let $f:[0,\infty)\to\mathbb{R}$ be twice differentiable with $f(0)=0$. If $x,\, y\in H$ and there exists the constant $0<M$ such that $\|x\|^2,\ \|y\|^2\leq M$ and so that there exists the constant $0\leq D$ such that $0\leq f''(t)\leq D$ for any $t\in(0,M)$ then*

$$(0\leq)\int_0^1 f\left(\|(1-t)\,x+ty\|^2\right)dt-f\left(\frac{\|x\|^2+\operatorname{Re}\langle x,y\rangle+\|y\|^2}{3}\right) \tag{2.18}$$
$$\leq\frac{\frac{1}{3}\left(\|x\|^2+\operatorname{Re}\langle x,y\rangle+\|y\|^2\right)}{M}f(M)$$
$$-f\left(\frac{\|x\|^2+\operatorname{Re}\langle x,y\rangle+\|y\|^2}{3}\right)$$
$$\leq\frac{1}{6}D\left(\|x\|^2+\operatorname{Re}\langle x,y\rangle+\|y\|^2\right)\left[M-\frac{1}{3}\left(\|x\|^2+\operatorname{Re}\langle x,y\rangle+\|y\|^2\right)\right]$$
$$\leq\frac{1}{8}DM^2.$$

## 3. Some Examples

We consider the convex function $f:[0,\infty)\to[0,\infty)$, $f(s)=s^p$, $p\geq 1$ and assume that $x,\, y\in H$ such that $\|x\|^2,\ \|y\|^2\leq M$ with $0<M$, then by (2.10) we get

$$(0\leq)\int_0^1\|(1-t)\,x+ty\|^{2p}\,dt-\left(\frac{\|x\|^2+\operatorname{Re}\langle x,y\rangle+\|y\|^2}{3}\right)^p \tag{3.1}$$
$$\leq\frac{1}{3}\left(\|x\|^2+\operatorname{Re}\langle x,y\rangle+\|y\|^2\right)M^{p-1}-\left(\frac{\|x\|^2+\operatorname{Re}\langle x,y\rangle+\|y\|^2}{3}\right)^p$$
$$\leq\left(\frac{2^{p-1}-1}{2^{p-1}}\right)M^{p-1}\left\{\frac{1}{2}M+\left|\frac{\|x\|^2+\operatorname{Re}\langle x,y\rangle+\|y\|^2}{3}-\frac{M}{2}\right|\right\}$$
$$\leq\left(\frac{2^{p-1}-1}{2^{p-1}}\right)M^p.$$

From (2.13) we get

$$(0\leq)\int_0^1\|(1-t)\,x+ty\|^{2p}\,dt-\left(\frac{\|x\|^2+\operatorname{Re}\langle x,y\rangle+\|y\|^2}{3}\right)^p \tag{3.2}$$
$$\leq\frac{\|x\|^2+\operatorname{Re}\langle x,y\rangle+\|y\|^2}{3}M^{p-1}-\left(\frac{\|x\|^2+\operatorname{Re}\langle x,y\rangle+\|y\|^2}{3}\right)^p$$

$$\leq \frac{1}{3}pM^{p-2}\left(\|x\|^2+\operatorname{Re}\langle x,y\rangle+\|y\|^2\right)$$
$$\times\left[M-\frac{1}{3}\left(\|x\|^2+\operatorname{Re}\langle x,y\rangle+\|y\|^2\right)\right]$$
$$\leq \frac{1}{4}pM^{p},$$

while from (2.18) we derive for $p\geq 2$ that

$$(0\leq)\int_0^1\|(1-t)\,x+ty\|^{2p}\,dt-\left(\frac{\|x\|^2+\operatorname{Re}\langle x,y\rangle+\|y\|^2}{3}\right)^p \tag{3.3}$$
$$\leq \frac{\|x\|^2+\operatorname{Re}\langle x,y\rangle+\|y\|^2}{3}M^{p-1}-\left(\frac{\|x\|^2+\operatorname{Re}\langle x,y\rangle+\|y\|^2}{3}\right)^p$$
$$\leq \frac{1}{6}p\,(p-1)\,M^{p-2}\left(\|x\|^2+\operatorname{Re}\langle x,y\rangle+\|y\|^2\right)$$
$$\times\left[M-\frac{1}{3}\left(\|x\|^2+\operatorname{Re}\langle x,y\rangle+\|y\|^2\right)\right]$$
$$\leq \frac{1}{8}M^{p}.$$

Consider the convex function $f(s)=\exp(\alpha s)-1,\ s,\ \alpha\in\mathbb{R}$ where $\alpha\neq 0$ and assume that $x,\ y\in H$ such that $\|x\|^2,\ \|y\|^2\leq M$ with $0<M$, then by (2.10) we get

$$(0\leq)\int_0^1\exp\left(\alpha\|(1-t)\,x+ty\|^2\right)dt-\exp\left(\alpha\frac{\|x\|^2+\operatorname{Re}\langle x,y\rangle+\|y\|^2}{3}\right) \tag{3.4}$$
$$\leq \frac{1}{3}\left(\|x\|^2+\operatorname{Re}\langle x,y\rangle+\|y\|^2\right)\left[\frac{\exp(\alpha M)-1}{M}\right]$$
$$-\exp\left[\left(\alpha\frac{\|x\|^2+\operatorname{Re}\langle x,y\rangle+\|y\|^2}{3}\right)\right]+1$$
$$\leq \frac{2}{M}\left\{\frac{1}{2}M+\left|\frac{\|x\|^2+\operatorname{Re}\langle x,y\rangle+\|y\|^2}{3}-\frac{M}{2}\right|\right\}$$
$$\times\left[\frac{1}{2}\exp(\alpha M)-\exp\left(\alpha\frac{M}{2}\right)-\frac{1}{2}\right]$$
$$\leq\left[\exp\left(\alpha\frac{M}{2}\right)-1\right]^2.$$

From (2.13) we get

$$(0\leq)\int_0^1\exp\left(\alpha\|(1-t)\,x+ty\|^2\right)dt-\exp\left(\alpha\frac{\|x\|^2+\operatorname{Re}\langle x,y\rangle+\|y\|^2}{3}\right) \tag{3.5}$$
$$\leq \frac{1}{3}\left(\|x\|^2+\operatorname{Re}\langle x,y\rangle+\|y\|^2\right)\left[\frac{\exp(\alpha M)-1}{M}\right]$$
$$-\exp\left[\left(\alpha\frac{\|x\|^2+\operatorname{Re}\langle x,y\rangle+\|y\|^2}{3}\right)\right]+1,$$

$$\leq \frac{\alpha\left[\exp\left(\alpha M\right)-1\right]}{3M}\left(\left\Vert x\right\Vert^2+\operatorname{Re}\left\langle x,y\right\rangle+\left\Vert y\right\Vert^2\right)$$
$$\times\left[M-\frac{1}{3}\left(\left\Vert x\right\Vert^2+\operatorname{Re}\left\langle x,y\right\rangle+\left\Vert y\right\Vert^2\right)\right]$$
$$\leq \frac{1}{4}\alpha M\left[\exp\left(\alpha M\right)-1\right],$$

while from (2.18) we derive for $\alpha > 0$ that

$$(0\leq)\int_0^1 \exp\left(\alpha\left\Vert (1-t)x+ty\right\Vert^2\right)dt-\exp\left(\alpha\frac{\left\Vert x\right\Vert^2+\operatorname{Re}\left\langle x,y\right\rangle+\left\Vert y\right\Vert^2}{3}\right) \tag{3.6}$$
$$\leq \frac{1}{3}\left(\left\Vert x\right\Vert^2+\operatorname{Re}\left\langle x,y\right\rangle+\left\Vert y\right\Vert^2\right)\left[\frac{\exp\left(\alpha M\right)-1}{M}\right]$$
$$-\exp\left[\left(\alpha\frac{\left\Vert x\right\Vert^2+\operatorname{Re}\left\langle x,y\right\rangle+\left\Vert y\right\Vert^2}{3}\right)\right]+1$$
$$\leq \frac{1}{6}\alpha^2\exp\left(\alpha M\right)\left(\left\Vert x\right\Vert^2+\operatorname{Re}\left\langle x,y\right\rangle+\left\Vert y\right\Vert^2\right)$$
$$\times\left[M-\frac{1}{3}\left(\left\Vert x\right\Vert^2+\operatorname{Re}\left\langle x,y\right\rangle+\left\Vert y\right\Vert^2\right)\right]$$
$$\leq \frac{1}{8}\alpha^2M^2\exp\left(\alpha M\right).$$

## 4. The Case of Real Numbers

Assume that $f:[0,\infty)\to\mathbb{R}$ is convex and $0\leq a<b$. We observe that $a\leq(1-t)a+tb\leq b$ for all $t\in[0,1]$. By utilizing (2.7) for $x=a$, $y=b$, $m=a^2$, $M=b^2$ and modulus as a norm, we get

$$(0\leq)\frac{1}{b-a}\int_a^b f\left(\left((1-t)a+tb\right)^2\right)dt-f\left(\frac{a^2+ab+b^2}{3}\right) \tag{4.1}$$
$$\leq \frac{b^2-\frac{1}{3}\left(a^2+ab+b^2\right)}{b^2-a^2}f\left(a^2\right)+\frac{\frac{1}{3}\left(a^2+ab+b^2\right)-a^2}{b^2-a^2}f\left(b^2\right)$$
$$-f\left(\frac{a^2+ab+b^2}{3}\right)$$
$$\leq \frac{2}{b^2-a^2}\left\{\frac{1}{2}\left(b^2-a^2\right)+\left|\frac{a^2+ab+b^2}{3}-\frac{a^2+b^2}{2}\right|\right\}$$
$$\times\left[\frac{f\left(a^2\right)+f\left(b^2\right)}{2}-f\left(\frac{a^2+b^2}{2}\right)\right]$$
$$\leq 2\left[\frac{f\left(a^2\right)+f\left(b^2\right)}{2}-f\left(\frac{a^2+b^2}{2}\right)\right].$$

Using the change of variable $(1-t)a+tb=s$, $t\in[0,1]$ we derive

$$\int_0^1 f\left(\left((1-t)a+tb\right)^2\right)dt=\frac{1}{b-a}\int_a^b f\left(s^2\right)ds.$$

Also

$$\frac{b^2-\frac{1}{3}\left(a^2+ab+b^2\right)}{b^2-a^2}=\frac{a+2b}{3\left(a+b\right)},$$
$$\frac{\frac{1}{3}\left(a^2+ab+b^2\right)-a^2}{b^2-a^2}=\frac{2a+b}{3\left(a+b\right)}$$

and

$$\begin{aligned}&\frac{2}{b^2-a^2}\left\{\frac{1}{2}\left(b^2-a^2\right)+\left|\frac{a^2+ab+b^2}{3}-\frac{a^2+b^2}{2}\right|\right\}\\&=1+\frac{2}{b^2-a^2}\frac{1}{6}\left(b-a\right)^2=1+\frac{1}{3}\frac{b-a}{b+a}=\frac{3b+3a+b-a}{3\left(a+b\right)}\\&=\frac{2\left(a+2b\right)}{3\left(a+b\right)}.\end{aligned}$$

So we get from (4.1) that

$$\begin{aligned}(4.2)\qquad&(0\leq)\,\frac{1}{b-a}\int_a^b f\left(s^2\right)ds-f\left(\frac{a^2+ab+b^2}{3}\right)\\&\leq\frac{a+2b}{3\left(a+b\right)}f\left(a^2\right)+\frac{2a+b}{3\left(a+b\right)}f\left(b^2\right)-f\left(\frac{a^2+ab+b^2}{3}\right)\\&\leq\frac{2\left(a+2b\right)}{3\left(a+b\right)}\left[\frac{f\left(a^2\right)+f\left(b^2\right)}{2}-f\left(\frac{a^2+b^2}{2}\right)\right]\\&\leq2\left[\frac{f\left(a^2\right)+f\left(b^2\right)}{2}-f\left(\frac{a^2+b^2}{2}\right)\right].\end{aligned}$$

From (2.11) we get

$$\begin{aligned}(4.3)\qquad&(0\leq)\,\frac{1}{b-a}\int_a^b f\left(s^2\right)ds-f\left(\frac{a^2+ab+b^2}{3}\right)\\&\leq\frac{a+2b}{3\left(a+b\right)}f\left(a^2\right)+\frac{2a+b}{3\left(a+b\right)}f\left(b^2\right)-f\left(\frac{a^2+ab+b^2}{3}\right)\\&\leq\frac{\left(2a+b\right)\left(a+2b\right)\left(b-a\right)}{9\left(a+b\right)}\left[f'_-\left(b^2\right)-f'_+\left(a^2\right)\right]\\&\leq\frac{1}{4}\left[f'_-\left(b^2\right)-f'_+\left(a^2\right)\right]\left(b^2-a^2\right).\end{aligned}$$

Let $f:[0,\infty)\to\mathbb{R}$ be twice differentiable and so that there exists the constants $0\leq K$ such that $0\leq f''(t)\leq K$ for any $t\in\left(a^2,b^2\right)\subset[0,\infty)$. From (2.14) we get

$$\begin{aligned}(4.4)\qquad&(0\leq)\,\frac{1}{b-a}\int_0^1 f\left(s^2\right)ds-f\left(\frac{a^2+ab+b^2}{3}\right)\\&\leq\frac{a+2b}{3\left(a+b\right)}f\left(a^2\right)+\frac{2a+b}{3\left(a+b\right)}f\left(b^2\right)-f\left(\frac{a^2+ab+b^2}{3}\right)\\&\leq\frac{1}{18}K\left(a+2b\right)\left(b-a\right)^2\left(2a+b\right)\leq\frac{1}{8}K\left(b^2-a^2\right)^2.\end{aligned}$$

Let $p \geq 1$, then $f(x) = x^p$ is convex on $[0,\infty)$ and for $a, b \in [0,\infty)$ with $a < b$ we get from (4.2) that

$$\begin{aligned}(0 \leq)\ & \frac{b^{2p+1} - a^{2p+1}}{(2p+1)(b-a)} - \left(\frac{a^2+ab+b^2}{3}\right)^p \\ &\leq \frac{a+2b}{3(a+b)}a^{2p} + \frac{2a+b}{3(a+b)}b^{2p} - \left(\frac{a^2+ab+b^2}{3}\right)^p \\ &\leq \frac{2(a+2b)}{3(a+b)}\left[\frac{a^{2p}+b^{2p}}{2} - \left(\frac{a^2+b^2}{2}\right)^p\right] \\ &\leq 2\left[\frac{a^{2p}+b^{2p}}{2} - \left(\frac{a^2+b^2}{2}\right)^p\right].\end{aligned} \tag{4.5}$$

From (4.3) we get

$$\begin{aligned}(0 \leq)\ & \frac{b^{2p+1} - a^{2p+1}}{(2p+1)(b-a)} - \left(\frac{a^2+ab+b^2}{3}\right)^p \\ &\leq \frac{a+2b}{3(a+b)}a^{2p} + \frac{2a+b}{3(a+b)}b^{2p} - \left(\frac{a^2+ab+b^2}{3}\right)^p \\ &\leq \frac{p(2a+b)(a+2b)(b-a)}{9(a+b)}\left(b^{2(p-1)} - a^{2(p-1)}\right) \\ &\leq \frac{1}{4}p\left(b^{2(p-1)} - a^{2(p-1)}\right)\left(b^2 - a^2\right),\end{aligned} \tag{4.6}$$

while from (4.4) we get

$$\begin{aligned}(0 \leq)\ & \frac{b^{2p+1} - a^{2p+1}}{(2p+1)(b-a)} - \left(\frac{a^2+ab+b^2}{3}\right)^p \\ &\leq \frac{a+2b}{3(a+b)}a^{2p} + \frac{2a+b}{3(a+b)}b^{2p} - \left(\frac{a^2+ab+b^2}{3}\right)^p \\ &\leq \frac{1}{18}p(p-1)(a+2b)(b-a)^2(2a+b) \\ &\times \begin{cases} a^{2(p-2)}, & p \in [1,2] \\ b^{2(p-2)}, & p \in (2,\infty) \end{cases} \\ &\leq \frac{1}{8}p(p-1)\left(b^2-a^2\right)^2 \times \begin{cases} a^{2(p-2)}, & p \in [1,2] \\ b^{2(p-2)}, & p \in (2,\infty) \end{cases} .\end{aligned} \tag{4.7}$$

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[1]Mathematics, College of Engineering & Science, Victoria University, PO Box 14428, Melbourne City, MC 8001, Australia.

*E-mail address*: sever.dragomir@vu.edu.au

*URL*: http://rgmia.org/dragomir

[2]DST-NRF Centre of Excellence in the Mathematical, and Statistical Sciences, School of Computer Science, & Applied Mathematics, University of the Witwatersrand,, Private Bag 3, Johannesburg 2050, South Africa