NORM INEQUALITIES FOR THE NONCOMMUTATIVE ČEBYŠEV FUNCTIONAL IN BANACH ALGEBRAS

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Abstract. Let $\mathcal{B}$ be a complex Banach algebra. For two continuous functions $x,\ y:[a,b]\to\mathcal{B}$ we define the *noncommutative Čebyšev functional*

$$D(x,y):=(b-a)\int_a^b x(t)\,y(t)\,dt-\int_a^b x(t)\,dt\int_a^b y(t)\,dt.$$

In this paper we show among other that if x, y are strongly differentiable, then

$$\|D(x,y)\|\le\frac{1}{4}(b-a)^2\times\begin{cases}\|x'\|_{[a,b],1}\,\|y'\|_{[a,b],1}\,,\\ \\ \frac{1}{3}(b-a)^2\,\|x'\|_{[a,b],\infty}\,\|y'\|_{[a,b],\infty}\\ \\ \frac{1}{2}(b-a)\,\|x'\|_{[a,b],1}\,\|y'\|_{[a,b],\infty}\,,\end{cases}$$

where

$$\|z'\|_{[a,b],1}:=\int_a^b\|z'(t)\|\,dt\text{ and }\|z'\|_{[a,b],\infty}:=\sup_{t\in(a,b)}\|z'(t)\|$$

for a strongly differentiable function z on (a,b). Some applications for analytic functions of elements in Banach algebras with examples for exponential function are also given.

1. Introduction

For two Lebesgue integrable functions $f,\ g:[a,b]\to\mathbb{C}$, in order to compare the integral mean of the product with the product of the integral means, we consider the *Čebyšev functional* defined by

$$D(f,g):=(b-a)\int_a^b f(t)\,g(t)\,dt-\int_a^b f(t)\,dt\int_a^b g(t)\,dt.$$

In 1934, G. Grüss [24] showed that

$$|D(f,g)|\le\frac{1}{4}(b-a)^2(M-m)(N-n)\,,\tag{1.1}$$

provided m, M, n, N are real numbers with the property that

$$-\infty<m\le f\le M<\infty,\quad -\infty<n\le g\le N<\infty\quad\text{a.e. on}\quad[a,b]\,.\tag{1.2}$$

The constant $\frac{1}{4}$ is best possible in (1.1) in the sense that it cannot be replaced by a smaller one.

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Another lesser known inequality for $D(f,g)$ was derived in 1882 by Čebyšev [7] under the assumption that f', g' exist and are continuous on $[a,b]$, and is given by

$$|D(f,g)| \leq \frac{1}{12} \|f'\|_{\infty} \|g'\|_{\infty} (b-a)^4, \tag{1.3}$$

where $\|f'\|_{\infty} := \sup_{t\in[a,b]} |f'(t)| < \infty$.

The constant $\frac{1}{12}$ cannot be improved in general in (1.3).

Čebyšev's inequality (1.3) also holds if f, $g : [a,b] \to \mathbb{R}$ are assumed to be absolutely continuous and f', $g' \in L_{\infty}[a,b]$.

In 1970, A. M. Ostrowski [28] proved, amongst others, the following result that is in a sense a combination of the Čebyšev and Grüss results:

$$|D(f,g)| \leq \frac{1}{8} (b-a)^3 (M-m) \|g'\|_{\infty}, \tag{1.4}$$

provided f is Lebesgue integrable on $[a,b]$ and satisfying (1.2) while $g : [a,b] \to \mathbb{R}$ is absolutely continuous and $g' \in L_{\infty}[a,b]$. Here the constant $\frac{1}{8}$ is also sharp.

In 1973, A. Lupaş [25] (see also [27, p. 210]) obtained the following result as well:

$$|D(f,g)| \leq \frac{1}{\pi^2} \|f'\|_2 \|g'\|_2 (b-a)^3, \tag{1.5}$$

provided f, g are absolutely continuous and f', $g' \in L_2[a,b]$.

Here the constant $\frac{1}{\pi^2}$ is the best possible as well.

In [3], P. Cerone and S. S. Dragomir proved the following inequalities:

$$|D(f,g)| \leq (b-a) \times \begin{cases} \inf\limits_{\gamma\in\mathbb{R}} \|g-\gamma\|_{\infty} \int_a^b \left| f(t) - \frac{1}{b-a}\int_a^b f(s)\,ds \right| dt, \\ \\ \inf\limits_{\gamma\in\mathbb{R}} \|g-\gamma\|_{q} \left(\int_a^b \left| f(t) - \frac{1}{b-a}\int_a^b f(s)\,ds \right|^p dt \right)^{\frac{1}{p}} \\ \qquad\qquad \text{where } p>1,\ 1/p+1/q=1. \end{cases} \tag{1.6}$$

For $\gamma = 0$, we get from the first inequality in (1.6)

$$|D(f,g)| \leq (b-a) \|g\|_{\infty} \int_a^b \left| f(t) - \frac{1}{b-a}\int_a^b f(s)\,ds \right| dt \tag{1.7}$$

for which the constant 1 cannot be replaced by a smaller constant.

If $m \leq g \leq M$ for a.e. $x \in [a,b]$, then $\left\| g - \frac{m+M}{2} \right\|_{\infty} \leq \frac{1}{2}(M-m)$ and by the first inequality in (1.6) we can deduce the following result obtained by Cheng and Sun [5]

$$|D(f,g)| \leq \frac{1}{2} (b-a)(M-m) \int_a^b \left| f(t) - \frac{1}{b-a}\int_a^b f(s)\,ds \right| dt. \tag{1.8}$$

The constant $\frac{1}{2}$ is best in (1.8) as shown by Cerone and Dragomir in [4].

The following result holds [15].

Theorem 1. *Let $f : [a,b] \to \mathbb{C}$ be of bounded variation on $[a,b]$ and $g : [a,b] \to \mathbb{C}$ a Lebesgue integrable function on $[a,b]$. Then*

$$|D(f,g)| \leq \frac{1}{2} (b-a) \bigvee_a^b (f) \int_a^b \left| g(t) - \frac{1}{b-a}\int_a^b g(s)\,ds \right| dt \tag{1.9}$$

where $\bigvee_a^b (f)$ *denotes the total variation of* f *on the interval* $[a,b]$. *The constant* $\frac{1}{2}$ *is best possible in (1.9).*

For more recent upper bounds related to the Čebyšev functional see [3], [4] and [10]-[15].

In order to obtain similar results for two functions with values in Banach algebras, we need the following preparations.

Let $\mathcal{B}$ be an algebra. An *algebra norm* on $\mathcal{B}$ is a map $\|\cdot\| : \mathcal{B}\to[0,\infty)$ such that $(\mathcal{B}, \|\cdot\|)$ is a normed space, and, further: $\|ab\| \leq \|a\| \|b\|$ for any $a, b \in \mathcal{B}$. The normed algebra $(\mathcal{B}, \|\cdot\|)$ is a *Banach algebra* if $\|\cdot\|$ is a *complete norm.* We assume that the Banach algebra is *unital,* this means that $\mathcal{B}$ has an identity 1 and that $\|1\| = 1$.

Let $\mathcal{B}$ be a unital algebra. An element $a \in \mathcal{B}$ is *invertible* if there exists an element $b \in \mathcal{B}$ with $ab = ba = 1$. The element b is unique; it is called the *inverse* of a and written a^{-1} or $\frac{1}{a}$. The set of invertible elements of $\mathcal{B}$ is denoted by $\text{Inv}(\mathcal{B})$. If $a, b \in \text{Inv}(\mathcal{B})$ then $ab \in \text{Inv}(\mathcal{B})$ and $(ab)^{-1} = b^{-1}a^{-1}$.

For a unital Banach algebra we also have:

(i) If $a \in \mathcal{B}$ and $\lim_{n\to\infty} \|a^n\|^{1/n} < 1$, then $1 - a \in \text{Inv}(\mathcal{B})$;
(ii) $\{b \in \mathcal{B}\colon\ \|1 - b\| < 1\} \subset \text{Inv}(\mathcal{B})$;
(iii) $\text{Inv}(\mathcal{B})$ is an *open subset* of $\mathcal{B}$;
(iv) The map $\text{Inv}(\mathcal{B}) \ni a \longmapsto a^{-1} \in \text{Inv}(\mathcal{B})$ is continuous.

For simplicity, we denote $z1$, where $z \in \mathbb{C}$ and 1 is the identity of $\mathcal{B}$, by z. The *resolvent set* of $a \in \mathcal{B}$ is defined by

$$\rho(a) := \{z \in \mathbb{C} :\ z - a \in \text{Inv}(\mathcal{B})\};$$

the *spectrum* of a is $\sigma(a)$, the complement of $\rho(a)$ in $\mathbb{C}$, and the *resolvent function* of a is $R_a : \rho(a) \to \text{Inv}(\mathcal{B})$, $R_a(z) := (z - a)^{-1}$. For each $z, w \in \rho(a)$ we have the identity

$$R_a(w) - R_a(z) = (z - w) R_a(z) R_a(w).$$

We also have that

$$\sigma(a) \subset \{z \in \mathbb{C} :\ |z| \leq \|a\|\}.$$

The *spectral radius* of a is defined as

$$\nu(a) = \sup\{|z| : z \in \sigma(a)\}.$$

Let $\mathcal{B}$ a unital Banach algebra and $a \in \mathcal{B}$. Then

(i) The resolvent set $\rho(a)$ is open in $\mathbb{C}$;
(ii) For any *bounded linear functionals* $\lambda : \mathcal{B} \to\mathbb{C}$, the function $\lambda \circ R_a$ is analytic on $\rho(a)$;
(iii) The spectrum $\sigma(a)$ is compact and nonempty in $\mathbb{C}$;
(iv) For each $n \in \mathbb{N}$ and $r > \nu(a)$, we have $a^n = \frac{1}{2\pi i}\int_{|\xi|=r} \xi^n (\xi - a)^{-1} d\xi$;
(v) We have $\nu(a) = \lim_{n\to\infty} \|a^n\|^{1/n}$.

Let $\mathcal{B}$ be a unital Banach algebra, $a \in \mathcal{B}$ and G be a domain of $\mathbb{C}$ with $\sigma(a) \subset G$. If $f : G \to \mathbb{C}$ is analytic on G, we define an element $f(a)$ in $\mathcal{B}$ by

$$f(a) := \frac{1}{2\pi i}\int_\delta f(\xi)(\xi - a)^{-1} d\xi, \tag{1.10}$$

where $\delta \subset G$ is taken to be close rectifiable curve in G and such that $\sigma(a) \subset \operatorname{ins}(\delta)$, the inside of δ.

It is well known (see for instance [6, pp. 201-204]) that $f(a)$ does not depend on the choice of δ and the *Spectral Mapping Theorem* (SMT)

$$\sigma(f(a)) = f(\sigma(a)) \tag{1.11}$$

holds.

Concerning other basic definitions and facts in the theory of Banach algebras, the reader can consult the classical books [8] and [29].

For some recent norm inequalities for functions on Banach algebras, see [19], [2] and [16]-[23].

2. Main Results

For two continuous functions $x, y : [a,b] \to \mathcal{B}$ we define the *noncommutative Čebyšev functional*

$$D(x,y) := (b-a)\int_a^b x(t)\,y(t)\,dt - \int_a^b x(t)\,dt \int_a^b y(t)\,dt.$$

We have the following result of interest:

Theorem 2. *Let $x, y : [a,b] \to \mathcal{B}$ be a strongly differentiable functions on the interval (a,b). Then*

$$\|D(x,y)\| \le D\left(\int_a^{\cdot} \|x'(u)\|\,du, \int_a^{\cdot} \|y'(u)\|\,du\right) \le \frac{1}{4}(b-a)^2 \|x'\|_{[a,b],1} \|y'\|_{[a,b],1}, \tag{2.1}$$

where $\|z'\|_{[a,b],1} := \int_a^b \|z'(u)\|\,du$.

Proof. Observe that

$$\begin{aligned}
&\int_a^b \int_a^b [x(t) - x(s)]\,[y(t) - y(s)]\,dtds \\
&= \int_a^b \int_a^b (x(t)\,y(t) - x(s)\,y(t) - x(t)\,y(s) + x(s)\,y(s))\,dtds \\
&= (b-a)\int_a^b x(t)\,y(t)\,dt - \int_a^b x(s)\,ds \int_a^b y(t)\,dt \\
&- \int_a^b x(t)\,dt \int_a^b y(s)\,ds + (b-a)\int_a^b x(s)\,y(s)\,ds \\
&= 2(b-a)\int_a^b x(t)\,y(t)\,dt - 2\int_a^b x(t)\,dt \int_a^b y(t)\,dt = 2D(x,y),
\end{aligned}$$

which give the Korkine's noncommutative identity for functions with values in Banach algebras

$$D(x,y) = \frac{1}{2}\int_a^b \int_a^b [x(t) - x(s)]\,[y(t) - y(s)]\,dtds.$$

For Korkine's classical identity for real-valued functions, see [27, p. 242].

If we take the norm and use the integral's properties, we get

$$\|D(x,y)\| \le \frac{1}{2}\int_a^b\int_a^b \|[x(t)-x(s)][y(t)-y(s)]\|\,dtds \tag{2.2}$$
$$\le \frac{1}{2}\int_a^b\int_a^b \|x(t)-x(s)\|\,\|y(t)-y(s)\|\,dtds.$$

Observe that for $s,\,t\in[a,b]$

$$x(t)-x(s)=\int_s^t x'(u)\,du,\;\; y(t)-y(s)=\int_s^t y'(u)\,du,$$

which implies that

$$\begin{aligned}\|x(t)-x(s)\|\,\|y(t)-y(s)\| &= \left\|\int_s^t x'(u)\,du\right\|\left\|\int_s^t y'(u)\,du\right\| \\ &\le \left|\int_s^t \|x'(u)\|\,du\right|\left|\int_s^t \|y'(u)\|\,du\right| \\ &= \left|\int_s^t \|x'(u)\|\,du\int_s^t \|y'(u)\|\,du\right| \\ &= \int_s^t \|x'(u)\|\,du\int_s^t \|y'(u)\|\,du,\end{aligned}$$

for all $s,\,t\in[a,b]$.

By (2.2) we get

$$\|D(x,y)\| \le \frac{1}{2}\int_a^b\int_a^b\left(\int_s^t \|x'(u)\|\,du\right)\left(\int_s^t \|y'(u)\|\,du\right)dtds. \tag{2.3}$$

Since

$$\begin{aligned}&\int_s^t \|x'(u)\|\,du\int_s^t \|y'(u)\|\,du \\ &=\left(\int_a^t \|x'(u)\|\,du-\int_a^s \|x'(u)\|\,du\right)\left(\int_a^t \|y'(u)\|\,du-\int_a^s \|y'(u)\|\,du\right),\end{aligned}$$

hence by Korkine's identity for real valued functions $f(t)=\int_a^t \|x'(u)\|\,du$ and $g(t)=\int_a^t \|y'(u)\|\,du$, we have

$$\begin{aligned}&\frac{1}{2}\int_a^b\int_a^b\left(\int_a^t \|x'(u)\|\,du-\int_a^s \|x'(u)\|\,du\right) \\ &\times\left(\int_a^t \|y'(u)\|\,du-\int_a^s \|y'(u)\|\,du\right) \\ &=(b-a)\int_a^b\left(\int_a^t \|x'(u)\|\,du\right)\left(\int_a^t \|y'(u)\|\,du\right)dt \\ &-\int_a^b\left(\int_a^t \|x'(u)\|\,du\right)dt\int_a^b\left(\int_a^t \|y'(u)\|\,du\right)dt \\ &=D\left(\int_a^{\cdot} \|x'(u)\|\,du,\int_a^{\cdot} \|y'(u)\|\,du\right).\end{aligned} \tag{2.4}$$

By utilising (2.3) and (2.4), we deduce the first inequality in (2.1).

Observe that

$$0 \le \int_a^t \|x'(u)\| \, du \le \int_a^b \|x'(u)\| \, du$$

and

$$0 \le \int_a^t \|y'(u)\| \, du \le \int_a^b \|y'(u)\| \, du$$

for all $t \in [a,b]$, then by Grüss's inequality for the functions $f(t) = \int_a^t \|x'(u)\| \, du$ and $g(t) = \int_a^t \|y'(u)\| \, du$, $t \in [a,b]$, we get the last part of (2.1). □

We have the noncommutative Čebyšev's inequality:

Corollary 1. *Let $x, y : [a,b] \to \mathcal{B}$ be strongly differentiable functions on the interval (a,b) with*

$$\|x'\|_{[a,b],\infty} := \sup_{u \in (a,b)} \|x'(u)\|, \quad \|y'\|_{[a,b],\infty} < \infty,$$

then

$$\|D(x,y)\| \le \frac{1}{12}(b-a)^4 \|x'\|_{[a,b],\infty} \|y'\|_{[a,b],\infty}. \tag{2.5}$$

Proof. If we use Čebyšev's inequality (1.3) for $f(t) = \int_a^t \|x'(u)\| \, du$ and $g(t) = \int_a^t \|y'(u)\| \, du$, $t \in [a,b]$, then we get

$$\begin{aligned} 0 &\le D\left(\int_a^{\cdot} \|x'(u)\| \, du, \int_a^{\cdot} \|y'(u)\| \, du\right) \\ &\le \frac{1}{12}(b-a)^4 \|f'\|_\infty \|g'\|_\infty = \frac{1}{12}(b-a)^4 \|x'\|_{[a,b],\infty} \|y'\|_{[a,b],\infty}, \end{aligned}$$

which by the first inequality in (2.1) gives the desired result (2.5). □

By the use of Ostrowski's inequality (1.4) we derive:

Corollary 2. *Let $x, y : [a,b] \to \mathcal{B}$ be strongly differentiable functions on the interval (a,b) with $\|y'\|_{[a,b],\infty} < \infty$, then*

$$\|D(x,y)\| \le \frac{1}{8}(b-a)^3 \|x'\|_{[a,b],1} \|y'\|_{[a,b],\infty}. \tag{2.6}$$

For a strongly differentiable function z on (a,b), we define

$$\|z'\|_{[a,b],2} := \left(\int_a^b \|z'(u)\|^2 \, du\right)^{1/2}.$$

By the use of Lupaş inequality for $f(t) = \int_a^t \|x'(u)\| \, du$ and $g(t) = \int_a^t \|y'(u)\| \, du$, $t \in [a,b]$, we get:

Corollary 3. *Let $x, y : [a,b] \to \mathcal{B}$ be strongly differentiable functions on the interval (a,b) with $\|x'\|_{[a,b],2}, \|y'\|_{[a,b],2} < \infty$, then*

$$\|D(x,y)\| \le \frac{1}{\pi^2}(b-a)^3 \|x'\|_{[a,b],2} \|y'\|_{[a,b],2}. \tag{2.7}$$

Observe that for $f(t)=\int_a^t \|x'(u)\|\,du$, we get integrating by parts that

$$\begin{aligned}
&\int_a^b \left| f(t) - \frac{1}{b-a}\int_a^b f(s)\,ds \right| dt \\
&= \int_a^b \left| \int_a^t \|x'(u)\|\,du - \frac{1}{b-a}\int_a^b \left(\int_a^s \|x'(u)\|\,du\right) ds \right| dt \\
&= \int_a^b \left| \int_a^t \|x'(u)\|\,du - \frac{1}{b-a}\left(\left(\int_a^b \|x'(u)\|\,du\right) b - \int_a^b \|x'(s)\|\,s ds\right) \right| dt
\end{aligned}$$

$$\begin{aligned}
&= \int_a^b \left| \int_a^t \|x'(u)\|\,du - \frac{1}{b-a}\left(\int_a^b (b-u)\|x'(u)\|\,du\right) \right| dt \\
&= \frac{1}{b-a}\int_a^b \left| (b-a)\int_a^t \|x'(u)\|\,du - \int_a^b (b-u)\|x'(u)\|\,du \right| dt \\
&= \frac{1}{b-a}\int_a^b \left| \int_a^t (u-a)\|x'(u)\|\,du - \int_t^b (b-u)\|x'(u)\|\,du \right| dt.
\end{aligned}$$

By utilising (1.8) for $f(t)=\int_a^t \|x'(u)\|\,du$ and $g(t)=\int_a^t \|y'(u)\|\,du$, $t\in[a,b]$, we get:

Corollary 4. *Let $x, y : [a,b] \to \mathcal{B}$ be strongly differentiable functions on the interval (a,b), then*

$$\begin{aligned}
\|D(x,y)\| \le{}& \frac{1}{2}\|y'\|_{[a,b],1} \\
&\times \int_a^b \left| \int_a^t (u-a)\|x'(u)\|\,du - \int_t^b (b-u)\|x'(u)\|\,du \right| dt.
\end{aligned} \tag{2.8}$$

Remark 1. *We observe that*

$$\begin{aligned}
&\int_a^b \left| \int_a^t (u-a)\|x'(u)\|\,du - \int_t^b (b-u)\|x'(u)\|\,du \right| dt \\
&\le \int_a^b \left[\left| \int_a^t (u-a)\|x'(u)\|\,du \right| + \left| \int_t^b (b-u)\|x'(u)\|\,du \right| \right] dt \\
&\le \int_a^b \left[\int_a^t (u-a)\|x'(u)\|\,du + \int_t^b (b-u)\|x'(u)\|\,du \right] dt
\end{aligned}$$

$$\begin{aligned}
&= \left[\int_a^t (u-a)\|x'(u)\|\,du + \int_t^b (b-u)\|x'(u)\|\,du \right] t \Bigg|_a^b \\
&- \int_a^b t\left((t-a)\|x'(t)\| - (b-t)\|x'(t)\|\right) dt
\end{aligned}$$

$$= b\int_a^b (u-a)\left\|x'(u)\right\| du - a\int_a^b (b-u)\left\|x'(u)\right\| du$$
$$-\int_a^b t\left((t-a)\left\|x'(t)\right\| - (b-t)\left\|x'(t)\right\|\right) dt$$
$$= 2\int_a^b (b-t)(t-a)\left\|x'(t)\right\| dt$$

and by (2.8) we get

$$\left\|D(x,y)\right\| \leq \left\|y'\right\|_{[a,b],1} \int_a^b (b-t)(t-a)\left\|x'(t)\right\| dt. \tag{2.9}$$

Theorem 3. *Let $x,\ y:[a,b]\to\mathcal{B}$ be continuous functions on the interval (a,b), then*

$$\left\|D(x,y)\right\| \leq \begin{cases} \inf_{v\in\mathcal{B}}\left\|x-v\right\|_{[a,b],\infty}\int_a^b \left\|(b-a)\,y(t)-\int_a^b y(s)\,ds\right\| dt, \\[2ex] \inf_{v\in\mathcal{B}}\left\|x-v\right\|_{[a,b],q}\left(\int_a^b \left\|(b-a)\,y(t)-\int_a^b y(s)\,ds\right\|^p dt\right)^{1/p}, \\[2ex] \inf_{v\in\mathcal{B}}\left\|x-v\right\|_{[a,b],1}\sup_{t\in[a,b]}\left\|(b-a)\,y(t)-\int_a^b y(s)\,ds\right\| \end{cases} \tag{2.10}$$

for all $p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1.$

Proof. For all $v\in\mathcal{B}$ we have

$$\int_a^b [x(t)-v]\left[(b-a)\,y(t)-\int_a^b y(s)\,ds\right] dt$$
$$= \int_a^b x(t)\left[(b-a)\,y(t)-\int_a^b y(s)\,ds\right] dt$$
$$- v\int_a^b \left[(b-a)\,y(t)-\int_a^b y(s)\,ds\right] dt$$
$$= (b-a)\int_a^b x(t)\,y(t)\,dt - \int_a^b x(t)\,dt\int_a^b y(s)\,ds$$
$$- v\left[(b-a)\int_a^b y(t)\,dt-(b-a)\int_a^b y(s)\,ds\right] = D(x,y).$$

Taking the norm in this equality, we get by Hölder's inequality that

$$\left\|D(x,y)\right\| \leq \int_a^b \left\|[x(t)-v]\left[(b-a)\,y(t)-\int_a^b y(s)\,ds\right]\right\| dt$$

$$
\begin{aligned}
&\le \int_a^b \|x(t)-v\| \left\|(b-a)\,y(t)-\int_a^b y(s)\,ds\right\| dt \\
&\le \begin{cases}
\sup_{t\in[a,b]}\|x(t)-v\| \int_a^b \left\|(b-a)\,y(t)-\int_a^b y(s)\,ds\right\| dt, \\[2ex]
\left(\int_a^b \|x(t)-v\|^q\,dt\right)^{1/q} \left(\int_a^b \left\|(b-a)\,y(t)-\int_a^b y(s)\,ds\right\|^p dt\right)^{1/p}, \\[2ex]
\int_a^b \|x(t)-v\| \sup_{t\in[a,b]} \left\|(b-a)\,y(t)-\int_a^b y(s)\,ds\right\|
\end{cases}
\end{aligned}
$$

for all $v \in \mathcal{B}$.

By taking the infimum over $v \in \mathcal{B}$, we obtain the desired result (2.10). □

Corollary 5. *With the assumptions of Theorem 3 and if there exists $v \in \mathcal{B}$ and $M > 0$ such that*

$$\|x(t)-v\| \le M \text{ for all } t \in [a,b],$$

then

$$\|D(x,y)\| \le M \int_a^b \left\|(b-a)\,y(t)-\int_a^b y(s)\,ds\right\| dt. \tag{2.11}$$

The proof is obvious from the first branch of (2.10).

Corollary 6. *Let x, $y : [a,b] \to \mathcal{B}$ be continuous functions on the interval $[a,b]$ and x strongly differentiable on (a,b), then for p, $q > 1$ with $\frac{1}{p}+\frac{1}{q}=1$,*

$$
\begin{aligned}
\|D(x,y)\| &\le \sup_{t\in[a,b]} \left\|(b-a)\,y(t)-\int_a^b y(s)\,ds\right\| \\
&\quad\times \left[\int_a^{\frac{a+b}{2}} (t-a)\|x'(t)\|\,dt + \int_{\frac{a+b}{2}}^b (b-t)\|x'(t)\|\,dt\right] \\
&\le \sup_{t\in[a,b]} \left\|(b-a)\,y(t)-\int_a^b y(s)\,ds\right\| \\
&\quad\times \begin{cases}
\frac{1}{8}(b-a)^2 \left[\|x'\|_{[a,\frac{a+b}{2}],\infty} + \|x'\|_{[\frac{a+b}{2},b],\infty}\right], \\[2ex]
\frac{1}{(q+1)^{1/q}2^{1+1/q}} \left[\|x'\|_{[a,\frac{a+b}{2}],p} + \|x'\|_{[\frac{a+b}{2},b],p}\right], \\[2ex]
\frac{1}{2}(b-a)\|x'\|_{[a,b],1}.
\end{cases}
\end{aligned}
\tag{2.12}
$$

Proof. We have

$$
\begin{aligned}
&\int_a^b \left\|x(t)-x\left(\frac{a+b}{2}\right)\right\| dt \\
&= \int_a^b \left\|\int_{\frac{a+b}{2}}^t x'(s)\,ds\right\| dt \le \int_a^b \left|\int_{\frac{a+b}{2}}^t \|x'(s)\|\,ds\right| dt \\
&= \int_a^{\frac{a+b}{2}} \left(\int_t^{\frac{a+b}{2}} \|x'(s)\|\,ds\right) dt + \int_{\frac{a+b}{2}}^b \left(\int_{\frac{a+b}{2}}^t \|x'(s)\|\,ds\right) dt
\end{aligned}
$$

$$= \left(\int_t^{\frac{a+b}{2}} \|x'(s)\| \, ds \right) t \Big|_a^{\frac{a+b}{2}} + \int_a^{\frac{a+b}{2}} t \|x'(t)\| \, dt$$
$$+ \left(\int_{\frac{a+b}{2}}^{t} \|x'(s)\| \, ds \right) t \Big|_{\frac{a+b}{2}}^{b} - \int_{\frac{a+b}{2}}^{b} t \|x'(t)\| \, dt$$
$$= \int_a^{\frac{a+b}{2}} t \|x'(t)\| \, dt - a \int_a^{\frac{a+b}{2}} \|x'(s)\| \, ds$$
$$+ b \int_{\frac{a+b}{2}}^{b} \|x'(s)\| \, ds - \int_{\frac{a+b}{2}}^{b} t \|x'(t)\| \, dt$$
$$= \int_a^{\frac{a+b}{2}} (t-a) \|x'(t)\| \, dt + \int_{\frac{a+b}{2}}^{b} (b-t) \|x'(t)\| \, dt,$$

which, by the third branch of (2.10), gives the first part of (2.12).

The last part follows by Hölder's inequality. □

3. Applications for Analytic Functions

Assume that $f : G \to \mathbb{C}$ is analytic on the convex domain G and $x, y \in \mathcal{B}$, $x \neq y$ with $\sigma(x), \sigma(y) \subset G$. By the convexity of G we have that $\sigma((1-t)x + ty) \subset G$ for all $t \in [0,1]$ and we can define the auxiliary function $f_{x,y} : [0,1] \to \mathcal{B}$ by

$$f_{x,y}(t) := f((1-t)x + ty), \; t \in [0,1]. \tag{3.1}$$

Lemma 1. *Assume that $f : G \to \mathbb{C}$ is analytic on the convex domain G and x, $y \in \mathcal{B}$, $x \neq y$ with $\sigma(x), \sigma(y) \subset G$. The function $f_{x,y} : [0,1] \to \mathcal{B}$ is differentiable on $(0,1)$ as a function of t and we have*

$$f_{x,y}^{'}(t) = D(f)((1-t)x + ty)(y-x) \tag{3.2}$$

for all $t \in (0,1)$, where $D(f)(\cdot)(\cdot)$ is the Fréchet derivative of function f as a function defined on the Banach algebra $\mathcal{B}$ by equation (1.10).

We also have the lateral derivatives

$$f_{x,y}^{'}(0+) = D(f)(x)(y-x) \text{ and } f_{x,y}^{'}(1-) = D(f)(y)(y-x). \tag{3.3}$$

Proof. We prove only for the interior points $t \in (0,1)$. Let h be in a small interval around 0 such that $t + h \in (0,1)$. Then for $h \neq 0$,

$$\frac{f_{x,y}(t+h) - f(t)}{h}$$
$$= \frac{f((1-(t+h))x + (t+h)y) - f((1-t)x + ty)}{h}$$
$$= \frac{f((1-t)x + ty + h(y-x)) - f((1-t)x + ty)}{h}$$

and by taking the limit over $h \to 0$, we get

$$f_{x,y}^{'}(t) = \frac{df_{x,y}(t)}{dt} = \lim_{h \to 0} \frac{f_{x,y}(t+h) - f(t)}{h}$$
$$= \lim_{h \to 0} \left[\frac{f((1-t)x + ty + h(y-x)) - f((1-t)x + ty)}{h} \right]$$
$$= D(f)((1-t)x + ty)(y-x),$$

which proves (3.2).

The proof is similar for the lateral derivatives. □

Lemma 2. *Assume that $f : G \to \mathbb{C}$ is analytic on the domain G and $x \in \mathcal{B}$, with $\sigma(x) \subset G$, then for $v \in \mathcal{B}$ we have*

$$D(f)(x)(v) = \frac{1}{2\pi i}\int_\gamma f(\xi)(\xi - x)^{-1} v(\xi - x)^{-1} d\xi, \tag{3.4}$$

where $\gamma \subset G$ is taken to be close rectifiable curve in G and such that $\sigma(x) \subset \operatorname{ins}(\gamma)$, the inside of γ.

Proof. Let $v \in \mathcal{B}$. Then there exists a small interval around 0 such that for h in this interval $\sigma(x + hv) \subset \operatorname{ins}(\delta) \subset G$. Then

$$\begin{aligned}
& f(x + hv) - f(x) \\
&= \frac{1}{2\pi i}\left(\int_\gamma f(\xi)(\xi - x - hv)^{-1} d\xi - \int_\gamma f(\xi)(\xi - x)^{-1} d\xi\right) \\
&= \frac{1}{2\pi i}\int_\gamma f(\xi)\left[(\xi - x - hv)^{-1} - (\xi - x)^{-1}\right] d\xi \\
&= \frac{1}{2\pi i} h \int_\gamma f(\xi)\left[(\xi - x - hv)^{-1} v(\xi - x)^{-1}\right] d\xi,
\end{aligned}$$

which gives for $h \neq 0$ that

$$\frac{f(x + hv) - f(x)}{h} = \frac{1}{2\pi i}\int_\gamma f(\xi)(\xi - x - hv)^{-1} v(\xi - x)^{-1} d\xi.$$

By taking the limit over $h \to 0$ and using the properties of the integral, we get (3.4). □

Lemma 3. *Assume that $f : G \to \mathbb{C}$ is analytic on the convex domain G and x, $y \in \mathcal{B}$, $x \neq y$ with $\sigma(x), \sigma(y) \subset G$ while $\gamma \subset G$ is taken to be close rectifiable curve in G and such that $\sigma(x), \sigma(y) \subset \operatorname{ins}(\gamma)$. Then*

$$f'_{x,y}(t) = \frac{1}{2\pi i}\int_\gamma f(\xi)(\xi - (1-t)x - ty)^{-1}(y - x)(\xi - (1-t)x - ty)^{-1} d\xi \tag{3.5}$$

for all $t \in (0,1)$.

We also have the lateral derivatives

$$f'_{x,y}(0+) = \frac{1}{2\pi i}\int_\gamma f(\xi)(\xi - x)^{-1}(y - x)(\xi - x)^{-1} d\xi, \tag{3.6}$$

and

$$f'_{x,y}(1-) = \frac{1}{2\pi i}\int_\gamma f(\xi)(\xi - y)^{-1}(y - x)(\xi - y)^{-1} d\xi. \tag{3.7}$$

The proof is obvious by Lemmas 1 and 2.

Lemma 4. *With the assumptions of Lemma 3 we have the bounds*

$$
\begin{aligned}
(3.8)\qquad & \left\| f_{x,y}^{\prime}(t) \right\| \\
& \le \frac{1}{2\pi} \|y-x\| \int_\gamma |f(\xi)| \left\| (\xi-(1-t)x-ty)^{-1} \right\|^2 |d\xi| \\
& \le \frac{1}{2\pi} \|y-x\| \int_\gamma |f(\xi)| (|\xi| - \|(1-t)x+ty\|)^{-2} |d\xi| \\
& \le \frac{1}{2\pi} \|y-x\| \int_\gamma |f(\xi)| [(1-t)(|\xi|-\|x\|)+t(|\xi|-\|y\|)]^{-2} |d\xi| \\
& \le \frac{1}{2\pi} \|y-x\| \int_\gamma |f(\xi)| \left[(1-t)(|\xi|-\|x\|)^{-2}+t(|\xi|-\|y\|)^{-2}\right] |d\xi| \\
& \le \frac{1}{2\pi} \|y-x\| \int_\gamma \frac{|f(\xi)|}{\min\left\{(|\xi|-\|x\|)^2, (|\xi|-\|y\|)^2\right\}} |d\xi|
\end{aligned}
$$

for all $t \in [0,1]$.

Proof. By taking the norm in (3.5) we get

$$
\begin{aligned}
(3.9)\qquad & \left\| f_{x,y}^{\prime}(t) \right\| \\
& \le \frac{1}{2\pi} \int_\gamma |f(\xi)| \left\| (\xi-(1-t)x-ty)^{-1}(y-x)(\xi-(1-t)x-ty)^{-1} \right\| |d\xi| \\
& \le \frac{1}{2\pi} \|y-x\| \int_\gamma |f(\xi)| \left\| (\xi-(1-t)x-ty)^{-1} \right\|^2 |d\xi| \\
& = \frac{1}{2\pi} \|y-x\| \int_\gamma |f(\xi)| |\xi|^{-2} \left\| \left(1-(1-t)\frac{x}{\xi}-t\frac{y}{\xi}\right)^{-1} \right\|^2 |d\xi|
\end{aligned}
$$

for all $t \in [0,1]$, which proves the first inequality in (3.8).

Since

$$
\left\| (1-t)\frac{x}{\xi}+t\frac{y}{\xi} \right\| \le (1-t)\left\|\frac{x}{\xi}\right\| + t\left\|\frac{y}{\xi}\right\| < 1-t+t = 1
$$

for $\xi \in \gamma$, hence

$$
\left(1-(1-t)\frac{x}{\xi}-t\frac{y}{\xi}\right)^{-1} = \sum_{k=0}^{\infty}\left[(1-t)\frac{x}{\xi}+t\frac{y}{\xi}\right]^k .
$$

Therefore

$$
\begin{aligned}
\left\| \left(1-(1-t)\frac{x}{\xi}-t\frac{y}{\xi}\right)^{-1} \right\| & \le \sum_{k=0}^{\infty}\left\|(1-t)\frac{x}{\xi}+t\frac{y}{\xi}\right\|^k \\
& = \left(1-\left\|(1-t)\frac{x}{\xi}+t\frac{y}{\xi}\right\|\right)^{-1} \\
& = \left(\frac{|\xi|}{|\xi|}-\left\|(1-t)\frac{x}{\xi}+t\frac{y}{\xi}\right\|\right)^{-1} \\
& = |\xi|\,(|\xi|-\|(1-t)x+ty\|)^{-1}
\end{aligned}
$$

for $\xi \in \gamma$, which implies that

$$\left\| \left(1-(1-t)\frac{x}{\xi}-t\frac{y}{\xi}\right)^{-1} \right\|^2 \leq |\xi|^2 \left(|\xi| - \|(1-t)x+ty\| \right)^{-2}$$

for $\xi \in \gamma$.

Therefore

$$\begin{aligned} &\int_\gamma |f(\xi)| |\xi|^{-2} \left\| \left(1-(1-t)\frac{x}{\xi}-t\frac{y}{\xi}\right)^{-1} \right\|^2 |d\xi| \\ &\leq \int_\gamma |f(\xi)| \left(|\xi| - \|(1-t)x+ty\| \right)^{-2} |d\xi| \end{aligned}$$

and we derive the second inequality in (3.8).

By the triangle inequality we have

$$\begin{aligned} |\xi| - \|(1-t)x+ty\| &\geq |\xi| - (1-t)\|x\| - t\|y\| \\ &= (1-t)(|\xi|-\|x\|) + t(|\xi|-\|y\|) > 0 \end{aligned}$$

for $\xi \in \gamma$.

This implies that

$$\left(|\xi| - \|(1-t)x+ty\| \right)^{-1} \leq \left[(1-t)(|\xi|-\|x\|) + t(|\xi|-\|y\|) \right]^{-1}$$

and thus

$$\left(|\xi| - \|(1-t)x+ty\| \right)^{-2} \leq \left[(1-t)(|\xi|-\|x\|) + t(|\xi|-\|y\|) \right]^{-2}$$

for $\xi \in \gamma$ and $t \in [0,1]$. This proves the third inequality in (3.8).

By the convexity of the power function $(\cdot)^{-2}$ we also have

$$\begin{aligned} &\left[(1-t)(|\xi|-\|x\|) + t(|\xi|-\|y\|) \right]^{-2} \\ &\leq (1-t)(|\xi|-\|x\|)^{-2} + t(|\xi|-\|y\|)^{-2} \end{aligned}$$

for $t \in [0,1]$, which proves the fourth inequality in (3.8).

Finally, observe that

$$(1-t)(|\xi|-\|x\|)^{-2} + t(|\xi|-\|y\|)^{-2} \leq \max\left\{ (|\xi|-\|x\|)^{-2}, (|\xi|-\|y\|)^{-2} \right\}$$

and the last part of (3.8) is thus proved. □

We have the following bounds for the p-norm of $f'_{x,y}$.

Proposition 1. *Assume that $f: G \to \mathbb{C}$ is analytic on the convex domain G and $x, y \in \mathcal{B}$, $x \neq y$ with $\sigma(x), \sigma(y) \subset G$ while $\gamma \subset G$ is taken to be close rectifiable curve in G and such that $\sigma(x), \sigma(y) \subset \operatorname{ins}(\gamma)$. Then*

$$\sup_{t\in[0,1]} \left\| f'_{x,y}(t) \right\| \leq \frac{1}{2\pi} \|y-x\| \int_\gamma \frac{|f(\xi)|}{\min\left\{ (|\xi|-\|x\|)^2, (|\xi|-\|y\|)^2 \right\}} |d\xi|, \tag{3.10}$$

$$\int_0^1 \left\| f'_{x,y}(t) \right\| dt \leq \frac{1}{2\pi} \|y-x\| \int_\gamma \frac{|f(\xi)|}{(|\xi|-\|y\|)(|\xi|-\|x\|)} |d\xi| \tag{3.11}$$

and

$$\left(\int_0^1 \left\| f'_{x,y}(t)\right\|^p dt\right)^{1/p} \leq \frac{1}{2\pi}\|y-x\| \left(\int_\gamma |f(\xi)|^q |d\xi|\right)^{1/q} \times \left(\frac{1}{(2p-1)(\|y\|-\|x\|)}\int_\gamma \frac{(|\xi|-\|x\|)^{2p-1}-(|\xi|-\|y\|)^{2p-1}}{(|\xi|-\|x\|)^{2p-1}(|\xi|-\|y\|)^{2p-1}}|d\xi|\right)^{1/p}, \tag{3.12}$$

for $p,\ q>1$ *with* $\frac{1}{p}+\frac{1}{q}=1$.

Proof. The inequality (3.10) is obvious by (3.8).

From (3.8) we get, by taking the integral and by using Fubini's theorem, that

$$\int_0^1 \left\| f'_{x,y}(t)\right\| dt \leq \frac{1}{2\pi}\|y-x\| \int_\gamma |f(\xi)| \left(\int_0^1 \left[(1-t)(|\xi|-\|x\|)+t(|\xi|-\|y\|)\right]^{-2} dt\right)|d\xi|. \tag{3.13}$$

Observe that

$$\begin{aligned}
&\int_0^1 \left[(1-t)(|\xi|-\|x\|)+t(|\xi|-\|y\|)\right]^{-2} dt \\
&= -\frac{1}{\|x\|-\|y\|}\int_0^1 \frac{d}{dt}\left[(1-t)(|\xi|-\|x\|)+t(|\xi|-\|y\|)\right]^{-1} dt \\
&= -\frac{1}{\|x\|-\|y\|}\left[(1-t)(|\xi|-\|x\|)+t(|\xi|-\|y\|)\right]^{-1}\Big|_0^1 \\
&= \frac{1}{\|y\|-\|x\|}\left[(|\xi|-\|y\|)^{-1}-(|\xi|-\|x\|)^{-1}\right] \\
&= \frac{1}{\|y\|-\|x\|}\frac{\|y\|-\|x\|}{(|\xi|-\|y\|)(|\xi|-\|x\|)} = \frac{1}{(|\xi|-\|y\|)(|\xi|-\|x\|)},
\end{aligned}$$

for $\|y\| \neq \|x\|$, which, by (3.13), proves (3.11).

If $\|y\| = \|x\|$, then (3.11) also holds.

From (3.8) we also have for $p,\ q>1$ with $\frac{1}{p}+\frac{1}{q}=1$,

$$\begin{aligned}
&\left\| f'_{x,y}(t)\right\| \\
&\leq \frac{1}{2\pi}\|y-x\| \int_\gamma |f(\xi)| \left[(1-t)(|\xi|-\|x\|)+t(|\xi|-\|y\|)\right]^{-2} |d\xi| \\
&\leq \frac{1}{2\pi}\|y-x\| \left(\int_\gamma |f(\xi)|^q |d\xi|\right)^{1/q} \\
&\times \left(\int_\gamma \left[(1-t)(|\xi|-\|x\|)+t(|\xi|-\|y\|)\right]^{-2p}\right)^{1/p}
\end{aligned}$$

and by taking the power p we get

$$\left\|f_{x,y}^{\prime}(t)\right\|^{p} \leq \left[\frac{1}{2\pi}\|y-x\|\left(\int_{\gamma}|f(\xi)|^{q}|d\xi|\right)^{1/q}\right]^{p} \times \left(\int_{\gamma}[(1-t)(|\xi|-\|x\|)+t(|\xi|-\|y\|)]^{-2p}\right)$$

for $t \in [0,1]$.

Integrating this inequality on $[0,1]$, we get by Fubini's theorem that

$$\begin{aligned}
\int_{0}^{1}\left\|f_{x,y}^{\prime}(t)\right\|^{p} dt &\leq \left[\frac{1}{2\pi}\|y-x\|\left(\int_{\gamma}|f(\xi)|^{q}|d\xi|\right)^{1/q}\right]^{p} \\
&\quad\times \int_{0}^{1}\left(\int_{\gamma}[(1-t)(|\xi|-\|x\|)+t(|\xi|-\|y\|)]^{-2p}|d\xi|\right) dt \\
&= \left[\frac{1}{2\pi}\|y-x\|\left(\int_{\gamma}|f(\xi)|^{q}|d\xi|\right)^{1/q}\right]^{p} \\
&\quad\times \int_{\gamma}\left(\int_{0}^{1}[(1-t)(|\xi|-\|x\|)+t(|\xi|-\|y\|)]^{-2p} dt\right)|d\xi| \\
&= \left[\frac{1}{2\pi}\|y-x\|\left(\int_{\gamma}|f(\xi)|^{q}|d\xi|\right)^{1/q}\right]^{p} \\
&\quad\times \int_{\gamma}\left(\int_{0}^{1}[(1-t)(|\xi|-\|x\|)+t(|\xi|-\|y\|)]^{-2p} dt\right)|d\xi|.
\end{aligned} \tag{3.14}$$

Observe that

$$\begin{aligned}
&\int_{\gamma}\left(\int_{0}^{1}[(1-t)(|\xi|-\|x\|)+t(|\xi|-\|y\|)]^{-2p} dt\right)|d\xi| \\
&= \int_{\gamma}\left(\left.\frac{[(1-t)(|\xi|-\|x\|)+t(|\xi|-\|y\|)]^{-2p+1}}{(1-2p)(\|x\|-\|y\|)}\right|_{0}^{1}\right)|d\xi|
\end{aligned}$$

$$\begin{aligned}
&= \int_{\gamma}\frac{(|\xi|-\|y\|)^{-2p+1}-(|\xi|-\|x\|)^{-2p+1}}{(1-2p)(\|x\|-\|y\|)}|d\xi| \\
&= \int_{\gamma}\frac{\frac{1}{(|\xi|-\|y\|)^{2p-1}}-\frac{1}{(|\xi|-\|x\|)^{2p-1}}}{(1-2p)(\|x\|-\|y\|)}|d\xi| \\
&= \frac{1}{(2p-1)(\|y\|-\|x\|)}\int_{\gamma}\frac{(|\xi|-\|x\|)^{2p-1}-(|\xi|-\|y\|)^{2p-1}}{(|\xi|-\|x\|)^{2p-1}(|\xi|-\|y\|)^{2p-1}}|d\xi|,
\end{aligned}$$

then by (3.14) we get

$$\int_0^1 \left\| f_{x,y}'(t) \right\|^p dt$$
$$\le \left[\frac{1}{2\pi} \|y-x\| \left(\int_\gamma |f(\xi)|^q |d\xi| \right)^{1/q} \right]^p$$
$$\times \frac{1}{(2p-1)(\|y\|-\|x\|)} \int_\gamma \frac{(|\xi|-\|x\|)^{2p-1} - (|\xi|-\|y\|)^{2p-1}}{(|\xi|-\|x\|)^{2p-1}(|\xi|-\|y\|)^{2p-1}} |d\xi|,$$

which proves (3.12). □

We can state now the main result of this section:

Theorem 4. *Assume that f, $g : G \to \mathbb{C}$ are analytic on the convex domain G and x, $y \in \mathcal{B}$, $x \ne y$ with $\sigma(x)$, $\sigma(y) \subset G$. Define*

$$D(f,g,x,y) := \int_0^1 f((1-t)x+ty)\, g((1-t)x+ty)\, dt - \int_0^1 f((1-t)x+ty)\, dt \int_0^1 g((1-t)x+ty)\, dt. \tag{3.15}$$

Then we have the norm inequalities

$$\|D(f,g,x,y)\| \le \frac{1}{16\pi^2} \|y-x\|^2 \int_\gamma \frac{|f(\xi)|}{(|\xi|-\|y\|)(|\xi|-\|x\|)} |d\xi| \times \int_\gamma \frac{|g(\xi)|}{(|\xi|-\|y\|)(|\xi|-\|x\|)} |d\xi|, \tag{3.16}$$

$$\|D(f,g,x,y)\| \le \frac{1}{48\pi^2} \|y-x\|^2 \int_\gamma \frac{|f(\xi)|}{\min\left\{ (|\xi|-\|x\|)^2, (|\xi|-\|y\|)^2 \right\}} |d\xi| \times \int_\gamma \frac{|g(\xi)|}{\min\left\{ (|\xi|-\|x\|)^2, (|\xi|-\|y\|)^2 \right\}} |d\xi| \tag{3.17}$$

and

$$\|D(f,g,x,y)\| \le \frac{1}{32\pi^2} \|y-x\|^2 \int_\gamma \frac{|f(\xi)|}{(|\xi|-\|y\|)(|\xi|-\|x\|)} |d\xi| \times \int_\gamma \frac{|f(\xi)|}{\min\left\{ (|\xi|-\|x\|)^2, (|\xi|-\|y\|)^2 \right\}} |d\xi|. \tag{3.18}$$

Also,

$$\|D(f,g,x,y)\| \le \frac{1}{12\pi^4} \|y-x\|^2 \left(\int_\gamma |f(\xi)|^2 |d\xi| \right)^{1/2} \left(\int_\gamma |g(\xi)|^2 |d\xi| \right)^{1/2} \times \int_\gamma \frac{(|\xi|-\|x\|)^2 + (|\xi|-\|x\|)(|\xi|-\|y\|) + (|\xi|-\|y\|)^2}{(|\xi|-\|x\|)^3 (|\xi|-\|y\|)^3} |d\xi|. \tag{3.19}$$

Proof. The proof follows by Proposition 1 and the inequalities (2.1), (2.5) and (2.6) for the functions $x(t) = f((1-t)x + ty)$, $y(t) = g((1-t)x + ty)$, $t \in [0,1]$.

For $p = q = 2$ in (3.12) we get

$$\begin{aligned}
&\left(\int_0^1 \left\| f'_{x,y}(t) \right\|^2 dt\right)^{1/2} \\
&\leq \frac{1}{2\pi} \|y - x\| \left(\int_\gamma |f(\xi)|^2 |d\xi|\right)^{1/2} \\
&\times \left(\frac{1}{3(\|y\| - \|x\|)} \int_\gamma \frac{(|\xi| - \|x\|)^3 - (|\xi| - \|y\|)^3}{(|\xi| - \|x\|)^{23} (|\xi| - \|y\|)^3} |d\xi|\right)^{1/2} \\
&= \frac{\sqrt{3}}{6\pi} \|y - x\| \left(\int_\gamma |f(\xi)|^2 |d\xi|\right)^{1/2} \\
&\times \left(\int_\gamma \frac{(|\xi| - \|x\|)^2 + (|\xi| - \|x\|)(|\xi| - \|y\|) + (|\xi| - \|y\|)^2}{(|\xi| - \|x\|)^3 (|\xi| - \|y\|)^3} |d\xi|\right)^{1/2}.
\end{aligned}$$

By utilising this inequality and (2.7) for $x(t) = f((1-t)x + ty)$, $y(t) = g((1-t)x + ty)$, $t \in [0,1]$, we deduce (3.19). □

4. The Case of Circular Paths

We consider the circular path $\xi(s) = Re^{2\pi i s}$ where $s \in [0,1]$, then $d\xi(s) = 2\pi i Re^{2\pi i s} ds$, $|d\xi(s)| = 2\pi R ds$ and $|\xi| = R$.

Assume that $f : G \to \mathbb{C}$ is analytic on the convex domain G and $x, y \in \mathcal{B}$, $x \neq y$ with $\sigma(x), \sigma(y) \subset D(0,R) \subset G$. Then by Proposition 1 we derive the simpler inequalities

$$\sup_{t\in[0,1]} \left\| f'_{x,y}(t) \right\| \leq \frac{R\|y - x\|}{\min\left\{(R - \|x\|)^2, (R - \|y\|)^2\right\}} \int_0^1 \left| f\left(Re^{2\pi i s}\right)\right| ds, \tag{4.1}$$

$$\int_0^1 \left\| f'_{x,y}(t) \right\| dt \leq \frac{R\|y - x\|}{(R - \|y\|)(R - \|x\|)} \int_0^1 \left| f\left(Re^{2\pi i s}\right)\right| ds, \tag{4.2}$$

and

$$\begin{aligned}
&\left(\int_0^1 \left\| f'_{x,y}(t) \right\|^p dt\right)^{1/p} \\
&\leq R\|y - x\| \left(\int_0^1 \left| f\left(Re^{2\pi i s}\right)\right|^q ds\right)^{1/q} \\
&\times \left(\frac{1}{(2p-1)(\|y\| - \|x\|)} \frac{(R - \|x\|)^{2p-1} - (R - \|y\|)^{2p-1}}{(R - \|x\|)^{2p-1} (R - \|y\|)^{2p-1}}\right)^{1/p},
\end{aligned} \tag{4.3}$$

for $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$.

Assume that $f,\, g : G \to \mathbb{C}$ are analytic on the convex domain G and $x,\, y \in \mathcal{B}$, $x \neq y$ with $\sigma(x),\, \sigma(y) \subset D(0,R) \subset G$. Then by Theorem 4 we derive

$$\|D(f,g,x,y)\| \leq \frac{1}{4}R^2 \frac{\|y-x\|^2}{(R-\|y\|)^2 (R-\|x\|)^2} \times \int_0^1 \left|f\left(Re^{2\pi i s}\right)\right| ds \int_0^1 \left|g\left(Re^{2\pi i s}\right)\right| ds \tag{4.4}$$

$$\|D(f,g,x,y)\| \leq \frac{1}{12}R^2 \frac{\|y-x\|^2}{\min\left\{(R-\|x\|)^4, (R-\|y\|)^4\right\}} \times \int_0^1 \left|f\left(Re^{2\pi i s}\right)\right| ds \int_0^1 \left|g\left(Re^{2\pi i s}\right)\right| ds \tag{4.5}$$

and

$$\begin{aligned} &\|D(f,g,x,y)\| \\ &\leq \frac{1}{8}R^2 \frac{\|y-x\|^2}{(R-\|y\|)(R-\|x\|)\min\left\{(R-\|x\|)^2, (R-\|y\|)^2\right\}} \\ &\times \int_0^1 \left|f\left(Re^{2\pi i s}\right)\right| ds \int_0^1 \left|g\left(Re^{2\pi i s}\right)\right| ds \end{aligned} \tag{4.6}$$

Also,

$$\begin{aligned} &\|D(f,g,x,y)\| \\ &\leq \frac{1}{3\pi^2}R^2 \|y-x\|^2 \left(\int_0^1 \left|f\left(Re^{2\pi i s}\right)\right|^2 ds\right)^{1/2} \left(\int_0^1 \left|g\left(Re^{2\pi i s}\right)\right|^2 ds\right)^{1/2} \\ &\times \frac{(R-\|x\|)^2 + (R-\|x\|)(R-\|y\|) + (R-\|y\|)^2}{(R-\|x\|)^3 (R-\|y\|)^3}. \end{aligned} \tag{4.7}$$

It is natural to ask how the bounds provided by (4.4), (4.5) and (4.6) compare to each other?

If we define the functions B_1, B_2 and B_3 in the box $(0,1)\times(0,1)$ by

$$B_1(x,y) := \frac{1}{(1-|y|)^2 (1-|x|)^2},$$

$$B_2(x,y) := \frac{1}{3\min\left\{(1-|y|)^4, (1-|x|)^4\right\}}$$

and

$$B_3(x,y) := \frac{1}{2(1-|y|)(1-|x|)\min\left\{(1-|y|)^2, (1-|x|)^2\right\}},$$

then we observe that the $3d$-plots for the differences $B_1(x,y) - B_2(x,y)$, $B_2(x,y) - B_3(x,y)$ and $B_3(x,y) - B_1(x,y)$ in the box $(0,1)\times(0,1)$ take both positive and negative values, showing that neither of the bounds (4.4), (4.5) and (4.6) are always best.

The *modified Bessel function of the first kind* $I_\nu(z)$ for real number ν can be defined by the power series as [1, p. 376]

$$I_\nu(z) = \left(\frac{1}{2}z\right)^\nu \sum_{k=0}^{\infty} \frac{\left(\frac{1}{4}z^2\right)^k}{k!\Gamma(\nu+k+1)},$$

where Γ is the *gamma function*. For $n = 0$ we have $I_0(z)$ given by

$$I_0(z) = \sum_{k=0}^{\infty} \frac{\left(\frac{1}{4}z^2\right)^k}{(k!)^2}.$$

An integral formula for real number ν is

$$I_\nu(z) = \frac{1}{\pi}\int_0^\pi e^{z\cos\theta}\cos(\nu\theta)\,d\theta - \frac{\sin(\nu\pi)}{\pi}\int_0^\infty e^{-z\cosh t-\nu t}dt,$$

which simplifies for ν an integer n to

$$I_n(z) = \frac{1}{\pi}\int_0^\pi e^{z\cos\theta}\cos(n\theta)\,d\theta.$$

For $n = 0$ we have

$$I_0(z) = \frac{1}{\pi}\int_0^\pi e^{z\cos\theta}d\theta.$$

Consider the exponential function $f(a) = \exp a$, $a \in \mathcal{B}$. Assume that $x, y \in \mathcal{B}$ and $\|x\|, \|y\| < R$ for some $R > 0$. Observe that

$$\left|\exp\left(Re^{2\pi it}\right)\right| = \left|\exp\left[R\left(\cos(2\pi t) + i\sin(2\pi t)\right)\right]\right| = \exp\left[R\cos(2\pi t)\right]$$

and, by changing the variable $\theta = 2\pi t$, we get $dt = \frac{1}{2\pi}d\theta$ and

$$\begin{aligned}\int_0^1 \exp\left[R\cos(2\pi t)\right]dt &\\ &= \frac{1}{2\pi}\int_0^{2\pi}\exp\left[R\cos\theta\right]d\theta\\ &= \frac{1}{2}\left(\frac{1}{\pi}\int_0^\pi \exp\left[R\cos\theta\right]d\theta + \frac{1}{\pi}\int_\pi^{2\pi}\exp\left[R\cos\theta\right]d\theta\right)\\ &= \frac{1}{2}\left(I_0(R) + I_0(-R)\right) = I_0(R).\end{aligned} \tag{4.8}$$

If we apply the inequalities (4.4)-(4.6) to the functions $f(z) = \exp z$ and $g(z) = z^n$ with n a natural number, then we get for $x, y \in \mathcal{B}$ with $\|x\|, \|y\| < R$ that

$$\begin{aligned}&\left\|\int_0^1 \exp\left((1-t)x+ty\right)\left((1-t)x+ty\right)^n dt\right.\\ &\left.-\int_0^1 \exp\left((1-t)x+ty\right)dt\int_0^1\left((1-t)x+ty\right)^n dt\right\|\\ &\le \frac{1}{4}R^{n+2}I_0(R)\frac{\|y-x\|^2}{\left(R-\|y\|\right)^2\left(R-\|x\|\right)^2}\end{aligned} \tag{4.9}$$

$$\left\| \int_0^1 \exp\left((1-t)x+ty\right)\left((1-t)x+ty\right)^n dt - \int_0^1 \exp\left((1-t)x+ty\right)dt \int_0^1 \left((1-t)x+ty\right)^n dt \right\| \le \frac{1}{12} R^{n+2} I_0(R) \frac{\|y-x\|^2}{\min\left\{(R-\|x\|)^4, (R-\|y\|)^4\right\}} \tag{4.10}$$

and

$$\left\| \int_0^1 \exp\left((1-t)x+ty\right)\left((1-t)x+ty\right)^n dt - \int_0^1 \exp\left((1-t)x+ty\right)dt \int_0^1 \left((1-t)x+ty\right)^n dt \right\| \le \frac{1}{8} R^{n+2} I_0(R) \frac{\|y-x\|^2}{(R-\|y\|)(R-\|x\|)\min\left\{(R-\|x\|)^2, (R-\|y\|)^2\right\}}. \tag{4.11}$$

References

[1] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions*, National Bureau of Standars, Applied Mathematics Series, **55**, 1972.

[2] M. V. Boldea, S. S. Dragomir and M. Megan, New bounds for Čebyšev functional for power series in Banach algebras via a Grüss-Lupaş type inequality. *PanAmer. Math. J.* **26** (2016), no. 3, 71–88.

[3] P. Cerone and S. S. Dragomir, New bounds for the Čebyšev functional, *Appl. Math. Lett.,* **18** (2005), 603-611.

[4] P. Cerone and S. S. Dragomir, A refinement of the Grüss inequality and applications, *Tamkang J. Math.* **38**(2007), No. 1, 37-49. Preprint available at *RGMIA Res. Rep. Coll.*, **5**(2) (2002), Art. 14. [ONLINE `http://rgmia.vu.edu.au/v5n2.html`].

[5] X.-L. Cheng and J. Sun, Note on the perturbed trapezoid inequality, *J. Inequal. Pure Appl. Math.,* **3**(2) (2002), Art. 29. [ONLINE`http://jipam.vu.edu.au/article.php?sid=181`

[6] J. B. Conway, *A Course in Functional Analysis, Second Edition*, Springer-Verlag, New York, 1990.

[7] P. L. Chebyshev, Sur les expressions approximatives des intègrals dèfinis par les outres prises entre les même limites, *Proc. Math. Soc. Charkov,* **2** (1882), 93-98.

[8] R. Douglas, *Banach Algebra Techniques in Operator Theory*, Academic Press, 1972.

[9] S. S. Dragomir, An inequality improving the first Hermite-Hadamard inequality for convex functions defined on linear spaces and applications for semi-inner products, *J. Inequal. Pure Appl. Math.* **3** (2002), No. 2, Article 31.`[Online https://www.emis.de/journals/JIPAM/article183.html?sid=183]`

[10] S. S. Dragomir, Inequalities of Grüss type for the Stieltjes integral and applications, *Kragujevac J. Math.*, **26** (2004), 89-112.

[11] S. S. Dragomir, Inequalities for Stieltjes integrals with convex integrators and applications, *Appl. Math. Lett.,* **20** (2007), 123-130.

[12] S. S. Dragomir, Čebyšev's type inequalities for functions of selfadjoint operators in Hilbert spaces, Preprint *RGMIA Res. Rep. Coll.*, **11**(e) (2008), Art. 9. [ONLINE: http://www.staff.vu.edu.au/RGMIA/v11(E).asp]

[13] S. S. Dragomir, Grüss' type inequalities for functions of selfadjoint operators in Hilbert spaces, Preprint *RGMIA Res. Rep. Coll.*, **11**(e) (2008), Art. 11. [ONLINE: http://www.staff.vu.edu.au/RGMIA/v11(E).asp]

[14] S. S. Dragomir, Inequalities for the Čebyšev functional of two functions of selfadjoint operators in Hilbert spaces, Prerpint *RGMIA Res. Rep. Coll.*, **11**(e) (2008), Art. . [ONLINE: http://www.staff.vu.edu.au/RGMIA/v11(E).asp]

[15] S. S. Dragomir, New Grüss' type inequalities for functions of bounded variation and applications. *Appl. Math. Lett.* **25** (2012), no. 10, 1475–1479.

[16] S. S. Dragomir, Inequalities for power series in Banach algebras. *SUT J. Math.* **50** (2014), no. 1, 25–45

[17] S. S. Dragomir, Inequalities of Lipschitz type for power series in Banach algebras. *Ann. Math. Sil.* **No**. 29 (2015), 61–83.

[18] S. S. Dragomir, Ostrowski type inequalities for Lebesgue integral: a survey of recent results. *Aust. J. Math. Anal. Appl.* **14** (2017), no. 1, Art. 1, 283 pp.

[19] S. S. Dragomir, Lipschitz-type inequalities for analytic functions in Banach algebras, *Bull. Aust. Math. Soc.* **100** (2019), no. 3, 489–497.

[20] S. S. Dragomir, M. V. Boldea and M. Megan, New norm inequalities of Čebyšev type for power series in Banach algebras. *Sarajevo J. Math.* **11** (24) (2015), no. 2, 253–266.

[21] S. S. Dragomir, M. V. Boldea, C. Buşe and M. Megan, Norm inequalities of Čebyšev type for power series in Banach algebras. *J. Inequal. Appl.* **2014**, 2014:294, 19 pp.

[22] S. S. Dragomir, M. V. Boldea and M. Megan, Further bounds for Čebyšev functional for power series in Banach algebras via Grüss-Lupaş type inequalities for p-norms. *Mem. Grad. Sch. Sci. Eng. Shimane Univ. Ser. B Math.* **49** (2016), 15–34.

[23] S. S. Dragomir, M. V. Boldea and M. Megan, Inequalities for Chebyshev functional in Banach algebras. *Cubo* **19** (2017), no. 1, 53–77.

[24] G. Grüss, Über das maximum des absoluten Betrages von $\frac{1}{b-a}\int_a^b f(x)\,g(x)\,dx - \frac{1}{(b-a)^2}\int_a^b f(x)\,dx \cdot \int_a^b g(x)\,dx$, *Math. Z.,* **39** (1934), 215-226.

[25] A. Lupaş, The best constant in an integral inequality, *Mathematica* (Cluj), **15** (**38**) (1973), No. 2, 219-222.

[26] J. Mikusiński, *The Bochner Integral*, Birkhäuser Verlag, 1978.

[27] D. S. Mitrinović, J. E. Pečarić and A. M. Fink, *Classical and New Inequalities in Analysis,* Kluwer Academic Publishers, Dordrecht, 1993.

[28] A. M. Ostrowski, On an integral inequality, *Aequationes Math.,* **4** (1970), 358-373.

[29] W. Rudin, *Functional Analysis*, McGraw Hill, 1973.

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