# Generalized Canavati Fractional Hilbert-Pachpatte type inequalities for Banach algebra valued functions

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**Abstract**

Using generalized Canavati fractional left and right vectorial Taylor formulae we prove corresponding left and right fractional Hilbert-Pachpatte type inequalities for Banach algebra valued functions. We cover also the sequential fractional case. We finish with applications.

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## 1 Introduction

Motivation follows:

We need

**Definition 1** *(see [5]) A definition of the Hausdorff measure $h_\alpha$ goes as follows: if $(T,d)$ is a metric space, $A \subseteq T$ and $\delta > 0$, let $\Lambda(A,\delta)$ be the set of all arbitrary collections $(C)_i$ of subsets of $T$, such that $A \subseteq \cup_i C_i$ and $diam(C_i) \leq \delta$ (diam =diameter) for every $i$. Now, for every $\alpha > 0$ define*

$$h_\alpha^\delta(A) := \inf\left\{\sum (diamC_i)^\alpha \mid (C_i)_i \in \Lambda(A,\delta)\right\}. \tag{1}$$

*Then there exists $\lim_{\delta\to 0} h_\alpha^\delta(A) = \sup_{\delta>0} h_\alpha^\delta(A)$, and $h_\alpha(A) := \lim_{\delta\to 0} h_\alpha^\delta(A)$ gives an outer measure on the power set $\mathcal{P}(T)$, which is countably additive on the $\sigma$-field*



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*of all Borel subsets of $T$. If $T=\mathbb{R}^n$, then the Hausdorff measure $h_n$, restricted to the $\sigma$-field of the Borel subsets of $\mathbb{R}^n$, equals the Lebesgue measure on $\mathbb{R}^n$ up to a constant multiple. In particular, $h_1(C)=\mu(C)$ for every Borel set $C\subseteq\mathbb{R}$, where $\mu$ is the Lebesgue measure.*

We also need

**Definition 2** *([2], Ch. 1) Let $[a,b]\subset\mathbb{R}$, $X$ be a Banach space, $\nu>0$; $n:=\lceil\nu\rceil\in\mathbb{N}$, $\lceil\cdot\rceil$ is the ceiling of the number, $f:[a,b]\to X$. We assume that $f^{(n)}\in L_1([a,b],X)$. We call the Caputo-Bochner left fractional derivative of order $\nu$:*

$$\left(D_{*a}^{\nu}f\right)(x):=\frac{1}{\Gamma(n-\nu)}\int_a^x(x-t)^{n-\nu-1}f^{(n)}(t)\,dt,\quad\forall\ x\in[a,b]. \tag{2}$$

*If $\nu\in\mathbb{N}$, we set $D_{*a}^{\nu}f:=f^{(\nu)}$ the ordinary $X$-valued derivative, and also set $D_{*a}^{0}f:=f$. Here $\Gamma$ is the gamma function and integrals are of Bochner type [3].*

By [2], Ch. 1, $\left(D_{*a}^{\nu}f\right)(x)$ exists almost everywhere in $x\in[a,b]$ and $D_{*a}^{\nu}f\in L_1([a,b],X)$.

If $\left\|f^{(n)}\right\|_{L_\infty([a,b],X)}<\infty$, then by [2], Ch. 1, $D_{*a}^{\nu}f\in C([a,b],X)$.

We are motivated by a Hilbert-Pachpatte left fractional inequality:

**Theorem 3** *([2], Ch. 1) Let $p,q>1:\frac{1}{p}+\frac{1}{q}=1$, and $\nu_1>\frac{1}{q}$, $\nu_2>\frac{1}{p}$, $n_i:=\lceil\nu_i\rceil$, $i=1,2$. Here $[a_i,b_i]\subset\mathbb{R}$, $i=1,2$; $X$ is a Banach space. Let $f_i\in C^{n_i-1}([a_i,b_i],X)$, $i=1,2$. Set*

$$F_{x_i}(t_i):=\sum_{j_i=0}^{n_i-1}\frac{(x_i-t_i)^{j_i}}{j_i!}f_i^{(j_i)}(t_i), \tag{3}$$

*$\forall\ t_i\in[a_i,x_i]$, where $x_i\in[a_i,b_i]$; $i=1,2$. Assume that $f_i^{(n_i)}$ exists outside a $\mu$-null Borel set $B_{x_i}\subseteq[a_i,x_i]$, such that*

$$h_1\left(F_{x_i}\left(B_{x_i}\right)\right)=0,\ \ \forall\ x_i\in[a_i,b_i];\ i=1,2. \tag{4}$$

*We also assume that $f_i^{(n_i)}\in L_1([a_i,b_i],X)$, and*

$$f_i^{(k_i)}(a_i)=0,\ \ k_i=0,1,...,n_i-1;\ i=1,2, \tag{5}$$

*and*

$$\left(D_{*a_1}^{\nu_1}f_1\right)\in L_q([a_1,b_1],X),\quad\left(D_{*a_2}^{\nu_2}f_2\right)\in L_p([a_2,b_2],X). \tag{6}$$

*Then*

$$\int_{a_1}^{b_1}\int_{a_2}^{b_2}\frac{\|f_1(x_1)\|\,\|f_2(x_2)\|\,dx_1dx_2}{\left(\frac{(x_1-a_1)^{p(\nu_1-1)+1}}{p(p(\nu_1-1)+1)}+\frac{(x_2-a_2)^{q(\nu_2-1)+1}}{q(q(\nu_2-1)+1)}\right)}\leq$$

$$\frac{(b_1-a_1)(b_2-a_2)}{\Gamma(\nu_1)\Gamma(\nu_2)}\left\|D_{*a_1}^{\nu_1}f_1\right\|_{L_q([a_1,b_1],X)}\left\|D_{*a_2}^{\nu_2}f_2\right\|_{L_p([a_2,b_2],X)}. \tag{7}$$

We need

**Definition 4** *([2], Ch. 2) Let $[a,b] \subset \mathbb{R}$, $X$ be a Banach space, $\alpha > 0$, $m := \lceil \alpha \rceil$. We assume that $f^{(m)} \in L_1([a,b], X)$, where $f : [a,b] \to X$. We call the Caputo-Bochner right fractional derivative of order $\alpha$:*

$$\left(D_{b-}^{\alpha} f\right)(x) := \frac{(-1)^m}{\Gamma(m-\alpha)} \int_x^b (J-x)^{m-\alpha-1} f^{(m)}(J)\, dJ, \quad \forall\, x \in [a,b]. \tag{8}$$

*We observe that $D_{b-}^{m} f(x) = (-1)^m f^{(m)}(x)$, for $m \in \mathbb{N}$, and $D_{b-}^{0} f(x) = f(x)$.*

By [2], Ch. 2, $\left(D_{b-}^{\alpha} f\right)(x)$ exists almost everywhere on $[a,b]$ and $\left(D_{b-}^{\alpha} f\right) \in L_1([a,b], X)$.

If $\left\| f^{(m)} \right\|_{L_\infty([a,b],X)} < \infty$, and $\alpha \notin \mathbb{N}$, then by [2], Ch. 2, $D_{b-}^{\alpha} f \in C([a,b], X)$, hence $\left\| D_{b-}^{\alpha} f \right\| \in C([a,b])$.

We are motivated also by the following Hilbert-Pachpatte right fractional inequality:

**Theorem 5** *([2], Ch. 2) Let $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$, and $\alpha_1 > \frac{1}{q}$, $\alpha_2 > \frac{1}{p}$, $m_i := \lceil \alpha_i \rceil$, $i = 1,2$. Here $[a_i, b_i] \subset \mathbb{R}$, $i = 1,2$; $X$ is a Banach space. Let $f_i \in C^{m_i - 1}([a_i, b_i], X)$, $i = 1,2$. Set*

$$F_{x_i}(t_i) := \sum_{j_i=0}^{m_i-1} \frac{(x_i - t_i)^{j_i}}{j_i!} f_i^{(j_i)}(t_i), \tag{9}$$

*$\forall\, t_i \in [x_i, b_i]$, where $x_i \in [a_i, b_i]$; $i = 1,2$. Assume that $f_i^{(m_i)}$ exists outside a $\mu$-null Borel set $B_{x_i} \subseteq [x_i, b_i]$, such that*

$$h_1\left(F_{x_i}\left(B_{x_i}\right)\right) = 0, \quad \forall\, x_i \in [a_i, b_i]; \; i = 1,2. \tag{10}$$

*We also assume that $f_i^{(m_i)} \in L_1([a_i, b_i], X)$, and*

$$f_i^{(k_i)}(b_i) = 0, \quad k_i = 0, 1, ..., , m_i - 1; \; i = 1,2, \tag{11}$$

*and*

$$\left(D_{b_1-}^{a_1} f_1\right) \in L_q([a_1, b_1], X), \quad \left(D_{b_2-}^{\alpha_2} f_2\right) \in L_p([a_2, b_2], X). \tag{12}$$

*Then*

$$\int_{a_1}^{b_1} \int_{a_2}^{b_2} \frac{\|f_1(x_1)\| \, \|f_2(x_2)\| \, dx_1 dx_2}{\left( \frac{(b_1 - x_1)^{p(\alpha_1 - 1)+1}}{p(p(\alpha_1 - 1)+1)} + \frac{(b_2 - x_2)^{q(\alpha_2 - 1)+1}}{q(q(\alpha_2 - 1)+1)} \right)} \leq$$

$$\frac{(b_1 - a_1)(b_2 - a_2)}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \left\| D_{b_1-}^{\alpha_1} f_1 \right\|_{L_q([a_1,b_1],X)} \left\| D_{b_2-}^{\alpha_2} f_2 \right\|_{L_p([a_2,b_2],X)}. \tag{13}$$

In this work we derive Hilbert-Pachpatte inequalities for Banach algebra valued functions with respect to their Canavati type generalized left and right fractional derivatives. We cover also the sequential fractional case. We finish with applications.

## 2 Background on Vectorial generalized Canavati fractional calculus

All in this section come from [2], pp. 109-115 and [1].

Let $g : [a, b] \to \mathbb{R}$ be a strictly increasing function. such that $g \in C^1([a, b])$, and $g^{-1} \in C^n([g(a), g(b)])$, $n \in \mathbb{N}$, $(X, \|\cdot\|)$ is a Banach space. Let $f \in C^n([a, b], X)$, and call $l := f \circ g^{-1} : [g(a), g(b)] \to X$. It is clear that $l, l', ..., l^{(n)}$ are continuous functions from $[g(a), g(b)]$ into $f([a, b]) \subseteq X$.

Let $\nu \geq 1$ such that $[\nu] = n$, $n \in \mathbb{N}$ as above, where $[\cdot]$ is the integral part of the number.

Clearly when $0 < \nu < 1$, $[\nu] = 0$.

I) Let $h \in C([g(a), g(b)], X)$, we define the left Riemann-Liouville Bochner fractional integral as

$$\left(J_\nu^{z_0} h\right)(z) := \frac{1}{\Gamma(\nu)} \int_{z_0}^{z} (z - t)^{\nu - 1} h(t)\, dt, \tag{14}$$

for $g(a) \leq z_0 \leq z \leq g(b)$, where $\Gamma$ is the gamma function; $\Gamma(\nu) = \int_0^\infty e^{-t} t^{\nu-1} dt$. We set $J_0^{z_0} h = h$.

Let $\alpha := \nu - [\nu]$ $(0 < \alpha < 1)$. We define the subspace $C_{g(x_0)}^{\nu}([g(a), g(b)], X)$ of $C^{[\nu]}([g(a), g(b)], X)$, where $x_0 \in [a, b]$ as:

$$C_{g(x_0)}^{\nu}([g(a), g(b)], X) =$$

$$\left\{ h \in C^{[\nu]}([g(a), g(b)], X) : J_{1-\alpha}^{g(x_0)} h^{([\nu])} \in C^1([g(x_0), g(b)], X) \right\}. \tag{15}$$

So let $h \in C_{g(x_0)}^{\nu}([g(a), g(b)], X)$, we define the left $g$-generalized $X$-valued fractional derivative of $h$ of order $\nu$, of Canavati type, over $[g(x_0), g(b)]$ as

$$D_{g(x_0)}^{\nu} h := \left( J_{1-\alpha}^{g(x_0)} h^{([\nu])} \right)'. \tag{16}$$

Clearly, for $h \in C_{g(x_0)}^{\nu}([g(a), g(b)], X)$, there exists

$$\left( D_{g(x_0)}^{\nu} h \right)(z) = \frac{1}{\Gamma(1 - \alpha)} \frac{d}{dz} \int_{g(x_0)}^{z} (z - t)^{-\alpha} h^{([\nu])}(t)\, dt, \tag{17}$$

for all $g(x_0) \leq z \leq g(b)$.

In particular, when $f \circ g^{-1} \in C_{g(x_0)}^{\nu}([g(a), g(b)], X)$, we have that

$$\left( D_{g(x_0)}^{\nu} \left( f \circ g^{-1} \right) \right)(z) = \frac{1}{\Gamma(1 - \alpha)} \frac{d}{dz} \int_{g(x_0)}^{z} (z - t)^{-\alpha} \left( f \circ g^{-1} \right)^{([\nu])}(t)\, dt, \tag{18}$$

for all $g(x_0) \leq z \leq g(b)$. We have that $D_{g(x_0)}^{n} \left( f \circ g^{-1} \right) = \left( f \circ g^{-1} \right)^{(n)}$ and $D_{g(x_0)}^{0} \left( f \circ g^{-1} \right) = f \circ g^{-1}$, see [1].

By [1], we have for $f \circ g^{-1} \in C^{\nu}_{g(x_0)}([g(a), g(b)], X)$, where $x_0 \in [a,b]$ the following left generalized $g$-fractional, of Canavati type,$X$-valued Taylor's formula:

**Theorem 6** *Let $f \circ g^{-1} \in C^{\nu}_{g(x_0)}([g(a), g(b)], X)$, where $x_0 \in [a,b]$ is fixed.*

*(i) If $\nu \geq 1$, then*

$$f(x) - f(x_0) = \sum_{k=1}^{[\nu]-1} \frac{\left(f \circ g^{-1}\right)^{(k)}(g(x_0))}{k!} (g(x) - g(x_0))^k +$$

$$\frac{1}{\Gamma(\nu)} \int_{g(x_0)}^{g(x)} (g(x) - t)^{\nu-1} \left(D^{\nu}_{g(x_0)}\left(f \circ g^{-1}\right)\right)(t)\, dt, \tag{19}$$

*for all $x_0 \leq x \leq b$.*

*(ii) If $0 < \nu < 1$, we get*

$$f(x) = \frac{1}{\Gamma(\nu)} \int_{g(x_0)}^{g(x)} (g(x) - t)^{\nu-1} \left(D^{\nu}_{g(x_0)}\left(f \circ g^{-1}\right)\right)(t)\, dt, \tag{20}$$

*for all $x_0 \leq x \leq b$.*

II) Let $h \in C([g(a), g(b)], X)$, we define the right Riemann-Liouville Bochner fractional integral as

$$\left(J^{\nu}_{z_0-}h\right)(z) := \frac{1}{\Gamma(\nu)} \int_{z}^{z_0} (t - z)^{\nu-1} h(t)\, dt, \tag{21}$$

for $g(a) \leq z \leq z_0 \leq g(b)$. We set $J^{0}_{z_0-}h = h$.

Let $\alpha := \nu - [\nu]$ $(0 < \alpha < 1)$. We define the subspace $C^{\nu}_{g(x_0)-}([g(a), g(b)], X)$ of $C^{[\nu]}([g(a), g(b)], X)$, where $x_0 \in [a,b]$ as:

$$C^{\nu}_{g(x_0)-}([g(a), g(b)], X) :=$$

$$\left\{h \in C^{[\nu]}([g(a), g(b)], X) : J^{1-\alpha}_{g(x_0)-}h^{([\nu])} \in C^{1}([g(a), g(x_0)], X)\right\}. \tag{22}$$

So let $h \in C^{\nu}_{g(x_0)-}([g(a), g(b)], X)$, we define the right $g$-generalized $X$-valued fractional derivative of $h$ of order $\nu$, of Canavati type, over $[g(a), g(x_0)]$ as

$$D^{\nu}_{g(x_0)-}h := (-1)^{n-1} \left(J^{1-\alpha}_{g(x_0)-}h^{([\nu])}\right)'. \tag{23}$$

Clearly, for $h \in C^{\nu}_{g(x_0)-}([g(a), g(b)], X)$, there exists

$$\left(D^{\nu}_{g(x_0)-}h\right)(z) = \frac{(-1)^{n-1}}{\Gamma(1-\alpha)} \frac{d}{dz} \int_{z}^{g(x_0)} (t - z)^{-\alpha} h^{([\nu])}(t)\, dt, \tag{24}$$

for all $g(a) \leq z \leq g(x_0) \leq g(b)$.

In particular, when $f \circ g^{-1} \in C^{\nu}_{g(x_0)-}([g(a), g(b)], X)$, we have that

$$\left(D^{\nu}_{g(x_0)-}\left(f \circ g^{-1}\right)\right)(z) = \frac{(-1)^{n-1}}{\Gamma(1-\alpha)} \frac{d}{dz} \int_z^{g(x_0)} (t-z)^{-\alpha} \left(f \circ g^{-1}\right)^{([\nu])}(t)\, dt, \tag{25}$$

for all $g(a) \leq z \leq g(x_0) \leq g(b)$.

We get that

$$\left(D^{n}_{g(x_0)-}\left(f \circ g^{-1}\right)\right)(z) = (-1)^n \left(f \circ g^{-1}\right)^{(n)}(z) \tag{26}$$

and $\left(D^{0}_{g(x_0)-}\left(f \circ g^{-1}\right)\right)(z) = \left(f \circ g^{-1}\right)(z)$, all $z \in [g(a), g(b)]$, see [1].

By [1], we have for $f \circ g^{-1} \in C^{\nu}_{g(x_0)-}([g(a), g(b)], X)$, where $x_0 \in [a, b]$ is fixed, the following right generalized $g$-fractional, of Canavati type, $X$-valued Taylor's formula:

**Theorem 7** *Let $f \circ g^{-1} \in C^{\nu}_{g(x_0)-}([g(a), g(b)], X)$, where $x_0 \in [a, b]$ is fixed.*

*(i) If $\nu \geq 1$, then*

$$f(x) - f(x_0) = \sum_{k=1}^{[\nu]-1} \frac{\left(f \circ g^{-1}\right)^{(k)}(g(x_0))}{k!} (g(x) - g(x_0))^k +$$

$$\frac{1}{\Gamma(\nu)} \int_{g(x)}^{g(x_0)} (t - g(x))^{\nu-1} \left(D^{\nu}_{g(x_0)-}\left(f \circ g^{-1}\right)\right)(t)\, dt, \tag{27}$$

*for all $a \leq x \leq x_0$,*

*(ii) If $0 < \nu < 1$, we get*

$$f(x) = \frac{1}{\Gamma(\nu)} \int_{g(x)}^{g(x_0)} (t - g(x))^{\nu-1} \left(D^{\nu}_{g(x_0)-}\left(f \circ g^{-1}\right)\right)(t)\, dt, \tag{28}$$

*all $a \leq x \leq x_0$.*

III) Denote by

$$D^{m\nu}_{g(x_0)} = D^{\nu}_{g(x_0)} D^{\nu}_{g(x_0)} \ldots D^{\nu}_{g(x_0)} \quad (m\text{-times}),\ m \in \mathbb{N}. \tag{29}$$

We mention the following modified and generalized left $X$-valued fractional Taylor's formula of Canavati type:

**Theorem 8** *Let $f \in C^1([a, b], X)$, $g \in C^1([a, b])$, strictly increasing: $g^{-1} \in C^1([g(a), g(b)])$. Assume that $\left(D^{i\nu}_{g(x_0)}\left(f \circ g^{-1}\right)\right) \in C^{\nu}_{g(x_0)}([g(a), g(b)], X)$, $0 < \nu < 1$, $x_0 \in [a, b]$, for $i = 0, 1, \ldots, m$. Then*

$$f(x) = \frac{1}{\Gamma((m+1)\nu)} \int_{g(x_0)}^{g(x)} (g(x) - z)^{(m+1)\nu - 1} \left(D^{(m+1)\nu}_{g(x_0)}\left(f \circ g^{-1}\right)\right)(z)\, dz, \tag{30}$$

*all $x_0 \leq x \leq b$.*

IV) Denote by

$$D^{m\nu}_{g(x_0)-} = D^{\nu}_{g(x_0)-} D^{\nu}_{g(x_0)-} ... D^{\nu}_{g(x_0)-} \quad (m \text{ times}),\ m \in \mathbb{N}. \tag{31}$$

We mention the following modified and generalized right $X$-valued fractional Taylor's formula of Canavati type:

**Theorem 9** *Let $f \in C^1([a,b], X)$, $g \in C^1([a,b])$, strictly increasing: $g^{-1} \in C^1([g(a), g(b)])$. Assume that $\left(D^{i\nu}_{g(x_0)-}\left(f \circ g^{-1}\right)\right) \in C^{\nu}_{g(x_0)-}([g(a), g(b)], X)$, $0 < \nu < 1$, $x_0 \in [a,b]$, for all $i = 0, 1, ..., m$. Then*

$$f(x) = \frac{1}{\Gamma((m+1)\nu)} \int_{g(x)}^{g(x_0)} (z - g(x))^{(m+1)\nu - 1} \left(D^{(m+1)\nu}_{g(x_0)-}\left(f \circ g^{-1}\right)\right)(z)\, dz, \tag{32}$$

*all $a \leq x \leq x_0 \leq b$.*

# 3 Banach Algebras background

All here come from [4].

We need

**Definition 10** *([4], p. 245) A complex algebra is a vector space $A$ over the complex filed $\mathbb{C}$ in which a multiplication is defined that satisfies*

$$x(yz) = (xy)z, \tag{33}$$

$$(x+y)z = xz + yz, \quad x(y+z) = xy + xz, \tag{34}$$

*and*

$$\alpha(xy) = (\alpha x)y = x(\alpha y), \tag{35}$$

*for all $x, y$ and $z$ in $A$ and for all scalars $\alpha$.*

*Additionally if $A$ is a Banach space with respect to a norm that satisfies the multiplicative inequality*

$$\|xy\| \leq \|x\| \|y\| \quad (x \in A,\ y \in A) \tag{36}$$

*and if $A$ contains a unit element $e$ such that*

$$xe = ex = x \quad (x \in A) \tag{37}$$

*and*

$$\|e\| = 1, \tag{38}$$

*then $A$ is called a Banach algebra.*

*$A$ is commutative iff $xy = yx$ for all $x, y \in A$.*

We make

**Remark 11** *Commutativity of $A$ will be explicited stated when needed.*

*There exists at most one $e \in A$ that satisfies (37).*

*Inequality (36) makes multiplication to be continuous, more precisely left and right continuous, see [4], p. 246.*

*Multiplication in $A$ is not necessarily the numerical multiplication, it is something more general and it is defined abstractly, that is for $x, y \in A$ we have $xy \in A$, e.g. composition or convolution, etc.*

*For nice examples about Banach algebras see [4], p. 247-248, § 10.3.*

We also make

**Remark 12** *Next we mention about integration of $A$-valued functions, see [4], p. 259, § 10.22:*

*If $A$ is a Banach algebra and $f$ is a continuous $A$-valued function on some compact Hausdorff space $Q$ on which a complex Borel measure $\mu$ is defined, then $\int f d\mu$ exists and has all the properties that were discussed in Chapter 3 of [4], simply because $A$ is a Banach space. However, an additional property can be added to these, namely: If $x \in A$, then*

$$x \int_Q f \ d\mu = \int_Q x f(p) \ d\mu(p) \tag{39}$$

*and*

$$\left( \int_Q f \ d\mu \right) x = \int_Q f(p) \, x \ d\mu(p) . \tag{40}$$

*The Bochner integrals we will involve in our article follow (39) and (40). Also, let $f \in C([a, b], X)$, where $[a, b] \subset \mathbb{R}$, $(X, \|\cdot\|)$ is a Banach space. By [2], p. 3, $f$ is Bochner integrable.*

# 4 Main Results

We start with a left generalized Canavati fractional Hilbert-Pachpatte type inequality over a Banach algebra.

**Theorem 13** *Let $p, q > 1$, such that $\frac{1}{p} + \frac{1}{q} = 1$, and $(A, \|\cdot\|)$ is a Banach algebra; and $i = 1, 2$. Let also $x_{0i} \in [a_i, b_i] \subset \mathbb{R}$, $\nu_i \geq 1$, $n_i = [\nu_i]$, $f_i \in C^{n_i}([a_i, b_i], A)$; $g_i \in C^1([a_i, b_i])$, strictly increasing, such that $g_i^{-1} \in C^{n_i}([g_i(a_i), g_i(b_i)])$, with $\left(f_i \circ g_i^{-1}\right)^{(k_i)}(g_i(x_{0i})) = 0$, $k_i = 0, 1, ..., n_i - 1$. Assume further that $f_i \circ g_i^{-1} \in C^{\nu_i}_{g_i(x_{0i})}([g_i(a_i), g_i(b_i)], A)$. Then*

$$\int_{g_1(x_{01})}^{g_1(b_1)} \int_{g_2(x_{02})}^{g_2(b_2)} \frac{\left\| \left(f_1 \circ g_1^{-1}\right)(z_1) \left(f_2 \circ g_2^{-1}\right)(z_2) \right\| dz_1 dz_2}{\left( \frac{(z_1 - g_1(x_{01}))^{p(\nu_1 - 1) + 1}}{p(p(\nu_1 - 1) + 1)} + \frac{(z_2 - g_2(x_{02}))^{q(\nu_2 - 1) + 1}}{q(q(\nu_2 - 1) + 1)} \right)} \leq$$

$$\frac{(g_1(b_1)-g_1(x_{01}))(g_2(b_2)-g_2(x_{02}))}{\Gamma(\nu_1)\Gamma(\nu_2)} \tag{41}$$

$$\left\|\left\|\left\|D_{g_1(x_{01})}^{\nu_1}\left(f_1\circ g_1^{-1}\right)\right\|\right\|\right\|_{L_q([g_1(x_{01}),g_1(b_1)],A)}\left\|\left\|\left\|D_{g_2(x_{02})}^{\nu_2}\left(f_2\circ g_2^{-1}\right)\right\|\right\|\right\|_{L_p([g_2(x_{02}),g_2(b_2)],A)}.$$

**Proof.** By (19) and assumptions we get that

$$\left(f_i\circ g_i^{-1}\right)(z_i)=\frac{1}{\Gamma(\nu_i)}\int_{g_i(x_{0i})}^{z_i}(z_i-t_i)^{\nu_i-1}\left(D_{g_i(x_{0i})}^{\nu_i}\left(f_i\circ g_i^{-1}\right)\right)(t_i)\,dt_i, \tag{42}$$

for all $g_i(x_{0i})\leq z_i\leq g_i(b_i)$; $i=1,2$.

By Hölder's inequality we obtain

$$\left\|\left(f_1\circ g_1^{-1}\right)(z_1)\right\|\leq\frac{1}{\Gamma(\nu_1)}\int_{g_1(x_{01})}^{z_1}(z_1-t_1)^{\nu_1-1}\left\|\left(D_{g_1(x_{01})}^{\nu_1}\left(f_1\circ g_1^{-1}\right)\right)(t_1)\right\|dt_1\leq$$

$$\frac{1}{\Gamma(\nu_1)}\left(\int_{g_1(x_{01})}^{z_1}(z_1-t_1)^{p(\nu_1-1)}dt_1\right)^{\frac{1}{p}}\left(\int_{g_1(x_{01})}^{z_1}\left\|\left(D_{g_1(x_{01})}^{\nu_1}\left(f_1\circ g_1^{-1}\right)\right)(t_1)\right\|^q dt_1\right)^{\frac{1}{q}}=$$

$$\frac{1}{\Gamma(\nu_1)}\frac{(z_1-g_1(x_{01}))^{\frac{p(\nu_1-1)+1}{p}}}{(p(\nu_1-1)+1)^{\frac{1}{p}}}\left(\int_{g_1(x_{01})}^{z_1}\left\|\left(D_{g_1(x_{01})}^{\nu_1}\left(f_1\circ g_1^{-1}\right)\right)(t_1)\right\|^q dt_1\right)^{\frac{1}{q}}. \tag{43}$$

That is

$$\left\|\left(f_1\circ g_1^{-1}\right)(z_1)\right\|\leq\frac{1}{\Gamma(\nu_1)}\frac{(z_1-g_1(x_{01}))^{\frac{p(\nu_1-1)+1}{p}}}{(p(\nu_1-1)+1)^{\frac{1}{p}}}$$

$$\left(\int_{g_1(x_{01})}^{z_1}\left\|\left(D_{g_1(x_{01})}^{\nu_1}\left(f_1\circ g_1^{-1}\right)\right)(t_1)\right\|^q dt_1\right)^{\frac{1}{q}}, \tag{44}$$

for all $g_1(x_{01})\leq z_1\leq g_1(b_1)$.

Similarly, we prove that

$$\left\|\left(f_2\circ g_2^{-1}\right)(z_2)\right\|\leq\frac{1}{\Gamma(\nu_2)}\frac{(z_2-g_2(x_{02}))^{\frac{q(\nu_2-1)+1}{q}}}{(q(\nu_2-1)+1)^{\frac{1}{q}}}$$

$$\left(\int_{g_2(x_{02})}^{z_2}\left\|\left(D_{g_2(x_{02})}^{\nu_2}\left(f_2\circ g_2^{-1}\right)\right)(t_2)\right\|^p dt_2\right)^{\frac{1}{p}}, \tag{45}$$

for all $g_2(x_{02})\leq z_2\leq g_2(b_2)$.

Therefore we have

$$\left\|\left(f_1\circ g_1^{-1}\right)(z_1)\right\|\leq\frac{1}{\Gamma(\nu_1)}\frac{(z_1-g_1(x_{01}))^{\frac{p(\nu_1-1)+1}{p}}}{(p(\nu_1-1)+1)^{\frac{1}{p}}}$$

$$\left\|\left\|\left(D_{g_1(x_{01})}^{\nu_1}\left(f_1\circ g_1^{-1}\right)\right)\right\|\right\|_{q,[g_1(x_{01}),g_1(b_1)]}, \tag{46}$$

for all $g_1(x_{01}) \le z_1 \le g_1(b_1)$;

and

$$\left\|\left(f_2\circ g_2^{-1}\right)(z_2)\right\| \le \frac{1}{\Gamma(\nu_2)}\frac{(z_2-g_2(x_{02}))^{\frac{q(\nu_2-1)+1}{q}}}{(q(\nu_2-1)+1)^{\frac{1}{q}}}$$
$$\left\|\left\|\left(D_{g_2(x_{02})}^{\nu_2}\left(f_2\circ g_2^{-1}\right)\right)\right\|\right\|_{p,[g_2(x_{02}),g_2(b_2)]}, \tag{47}$$

for all $g_2(x_{02}) \le z_2 \le g_2(b_2)$.

Hence we get that

$$\left\|\left(f_1\circ g_1^{-1}\right)(z_1)\right\|\left\|\left(f_2\circ g_2^{-1}\right)(z_2)\right\| \le \frac{1}{\Gamma(\nu_1)\Gamma(\nu_2)(p(\nu_1-1)+1)^{\frac{1}{p}}(q(\nu_2-1)+1)^{\frac{1}{q}}}$$
$$(z_1-g_1(x_{01}))^{\frac{p(\nu_1-1)+1}{p}}(z_2-g_2(x_{02}))^{\frac{q(\nu_2-1)+1}{q}} \tag{48}$$
$$\left\|\left\|\left(D_{g_1(x_{01})}^{\nu_1}\left(f_1\circ g_1^{-1}\right)\right)\right\|\right\|_{q,[g_1(x_{01}),g_1(b_1)]}\left\|\left\|\left(D_{g_2(x_{02})}^{\nu_2}\left(f_2\circ g_2^{-1}\right)\right)\right\|\right\|_{p,[g_2(x_{02}),g_2(b_2)]} \le$$

(using Young's inequality for $a,b\ge 0$, $a^{\frac{1}{p}}b^{\frac{1}{q}} \le \frac{a}{p}+\frac{b}{q}$)

$$\frac{1}{\Gamma(\nu_1)\Gamma(\nu_2)}\left(\frac{(z_1-g_1(x_{01}))^{p(\nu_1-1)+1}}{p(p(\nu_1-1)+1)}+\frac{(z_2-g_2(x_{02}))^{q(\nu_2-1)+1}}{q(q(\nu_2-1)+1)}\right)$$
$$\left\|\left\|\left(D_{g_1(x_{01})}^{\nu_1}\left(f_1\circ g_1^{-1}\right)\right)\right\|\right\|_{L_q([g_1(x_{01}),g_1(b_1)],A)}\left\|\left\|\left(D_{g_2(x_{02})}^{\nu_2}\left(f_2\circ g_2^{-1}\right)\right)\right\|\right\|_{L_p([g_2(x_{02}),g_2(b_2)],A)},$$

$\forall\,(z_1,z_2)\in[g_1(x_{01}),g_1(b_1)]\times[g_2(x_{02}),g_2(b_2)]$.

So far we have

$$\frac{\left\|\left(f_1\circ g_1^{-1}\right)(z_1)\left(f_2\circ g_2^{-1}\right)(z_2)\right\|}{\left(\frac{(z_1-g_1(x_{01}))^{p(\nu_1-1)+1}}{p(p(\nu_1-1)+1)}+\frac{(z_2-g_2(x_{02}))^{q(\nu_2-1)+1}}{q(q(\nu_2-1)+1)}\right)} \le \tag{50}$$

$$\frac{\left\|\left(f_1\circ g_1^{-1}\right)(z_1)\right\|\left\|\left(f_2\circ g_2^{-1}\right)(z_2)\right\|}{\left(\frac{(z_1-g_1(x_{01}))^{p(\nu_1-1)+1}}{p(p(\nu_1-1)+1)}+\frac{(z_2-g_2(x_{02}))^{q(\nu_2-1)+1}}{q(q(\nu_2-1)+1)}\right)} \le \tag{51}$$
$$\frac{1}{\Gamma(\nu_1)\Gamma(\nu_2)}\left\|\left\|\left(D_{g_1(x_{01})}^{\nu_1}\left(f_1\circ g_1^{-1}\right)\right)\right\|\right\|_{L_q([g_1(x_{01}),g_1(b_1)],A)}$$
$$\left\|\left\|\left(D_{g_2(x_{02})}^{\nu_2}\left(f_2\circ g_2^{-1}\right)\right)\right\|\right\|_{L_p([g_2(x_{02}),g_2(b_2)],A)},$$

$\forall\,(z_1,z_2)\in[g_1(x_{01}),g_1(b_1)]\times[g_2(x_{02}),g_2(b_2)]$.

The denominators in (50), (51) can be zero only when both $z_1 = g_1(x_{01})$ and $z_2 = g_2(x_{02})$.

Therefore we obtain (41), by integrating (50), (51) over $[g_1(x_{01}),g_1(b_1)]\times[g_2(x_{02}),g_2(b_2)]$. ■

We continue with a right generalized Canavati fractional Hilbert-Pachpatte type inequality over a Banach algebra.

**Theorem 14** *All as in Theorem 13, however now it is* $f_i \circ g_i^{-1} \in C^{\nu_i}_{g_i(x_{0i})-}([g_i(a_i), g_i(b_i)], A)$, *for* $i = 1, 2$. *Then*

$$\int_{g_1(a_1)}^{g_1(x_{01})} \int_{g_2(a_2)}^{g_2(x_{02})} \frac{\left\| \left(f_1 \circ g_1^{-1}\right)(z_1) \left(f_2 \circ g_2^{-1}\right)(z_2) \right\| dz_1 dz_2}{\left( \frac{(g_1(x_{01}) - z_1)^{p(\nu_1 - 1)+1}}{p(p(\nu_1-1)+1)} + \frac{(g_2(x_{02}) - z_2)^{q(\nu_2-1)+1}}{q(q(\nu_2-1)+1)} \right)} \leq$$

$$\frac{(g_1(x_{01}) - g_1(a_1))(g_2(x_{02}) - g_2(a_2))}{\Gamma(\nu_1)\Gamma(\nu_2)} \tag{52}$$

$$\left\| \left\| D^{\nu_1}_{g_1(x_{01})-} \left(f_1 \circ g_1^{-1}\right) \right\| \right\|_{L_q([g_1(a_1), g_1(x_{01})], A)} \left\| \left\| D^{\nu_2}_{g_2(x_{02})-} \left(f_2 \circ g_2^{-1}\right) \right\| \right\|_{L_p([g_2(a_2), g_2(x_{02})], A)}.$$

**Proof.** Similar to Theorem 13, by using now (27). ■

Next comes a sequential left generalized Canavati fractional Hilbert-Pachpatte type inequality over a Banach algebra.

**Theorem 15** *Let* $p, q > 1$, *such that* $\frac{1}{p} + \frac{1}{q} = 1$, *and* $(A, \|\cdot\|)$ *is a Banach algebra; and* $i = 1, 2$. *Let also* $f_i \in C^1([a_i, b_i], A)$; $g_i \in C^1([a_i, b_i])$, *strictly increasing, such that* $g_i^{-1} \in C^1([g_i(a_i), g_i(b_i)])$. *Assume that* $\frac{1}{(m_i+1)q} < \nu_i < 1$, $x_{0i} \in [a_i, b_i]$, *and* $D^{j_i \nu_i}_{g_i(x_{0i})} \left(f_i \circ g_i^{-1}\right) \in C^{\nu_i}_{g_i(x_{0i})}([g_i(a_i), g_i(b_i)], A)$, *for* $j_i = 0, 1, ..., m_i \in \mathbb{N}$. *Then*

$$\int_{g_1(x_{01})}^{g_1(b_1)} \int_{g_2(x_{02})}^{g_2(b_2)} \frac{\left\| \left(f_1 \circ g_1^{-1}\right)(z_1) \left(f_2 \circ g_2^{-1}\right)(z_2) \right\| dz_1 dz_2}{\left( \frac{(z_1 - g_1(x_{01}))^{p((m_1+1)\nu_1 - 1)+1}}{p(p((m_1+1)\nu_1-1)+1)} + \frac{(z_2 - g_2(x_{02}))^{q((m_2+1)\nu_2-1)+1}}{q(q((m_2+1)\nu_2-1)+1)} \right)} \leq$$

$$\frac{(g_1(b_1) - g_1(x_{01}))(g_2(b_2) - g_2(x_{02}))}{\Gamma((m_1+1)\nu_1)\Gamma((m_2+1)\nu_2)} \tag{53}$$

$$\left\| \left\| D^{(m_1+1)\nu_1}_{g_1(x_{01})} \left(f_1 \circ g_1^{-1}\right) \right\| \right\|_{L_q([g_1(x_{01}), g_1(b_1)], A)} \left\| \left\| D^{(m_2+1)\nu_2}_{g_2(x_{02})} \left(f_2 \circ g_2^{-1}\right) \right\| \right\|_{L_p([g_2(x_{02}), g_2(b_2)], A)}.$$

**Proof.** Using (30), as similar to Theorem 13 the proof is omitted. ■

The right side analog of Theorem 15 follows:

**Theorem 16** *Let* $p, q > 1$, *such that* $\frac{1}{p} + \frac{1}{q} = 1$, *and* $(A, \|\cdot\|)$ *is a Banach algebra; and* $i = 1, 2$. *Let also* $f_i \in C^1([a_i, b_i], A)$; $g_i \in C^1([a_i, b_i])$, *strictly increasing, such that* $g_i^{-1} \in C^1([g_i(a_i), g_i(b_i)])$. *Assume that* $\frac{1}{(m_i+1)q} < \nu_i < 1$, $x_{0i} \in [a_i, b_i]$, *and* $D^{j_i \nu_i}_{g_i(x_{0i})-} \left(f_i \circ g_i^{-1}\right) \in C^{\nu_i}_{g_i(x_{0i})-}([g_i(a_i), g_i(b_i)], A)$, *for* $j_i = 0, 1, ..., m_i \in \mathbb{N}$. *Then*

$$\int_{g_1(a_1)}^{g_1(x_{01})} \int_{g_2(a_2)}^{g_2(x_{02})} \frac{\left\| \left(f_1 \circ g_1^{-1}\right)(z_1) \left(f_2 \circ g_2^{-1}\right)(z_2) \right\| dz_1 dz_2}{\left( \frac{(g_1(x_{01}) - z_1)^{p((m_1+1)\nu_1 - 1)+1}}{p(p((m_1+1)\nu_1-1)+1)} + \frac{(g_2(x_{02}) - z_2)^{q((m_2+1)\nu_2-1)+1}}{q(q((m_2+1)\nu_2-1)+1)} \right)} \leq$$

$$\frac{(g_1(x_{01}) - g_1(a_1))(g_2(x_{02}) - g_2(a_2))}{\Gamma((m_1+1)\nu_1)\Gamma((m_2+1)\nu_2)} \tag{54}$$

$$\left\| \left\| D^{(m_1+1)\nu_1}_{g_1(x_{01})-} \left(f_1 \circ g_1^{-1}\right) \right\| \right\|_{L_q([g_1(a_1), g_1(x_{01})], A)} \left\| \left\| D^{(m_2+1)\nu_2}_{g_2(x_{02})-} \left(f_2 \circ g_2^{-1}\right) \right\| \right\|_{L_p([g_2(a_2), g_2(x_{02})], A)}.$$

**Proof.** Using (32), as similar to Theorem 13 is omitted. ■

# 5 Applications

We give

**Corollary 17** *(to Theorem 13) All as in Theorem 13 for $g_i(t) = e^t$, $i = 1, 2$. Then*

$$\int_{e^{x_{01}}}^{e^{b_1}} \int_{e^{x_{02}}}^{e^{b_2}} \frac{\left\| (f_1 \circ \log)(z_1)(f_2 \circ \log)(z_2) \right\| dz_1 dz_2}{\left( \frac{(z_1 - e^{x_{01}})^{p(\nu_1 - 1) + 1}}{p(p(\nu_1 - 1) + 1)} + \frac{(z_2 - e^{x_{02}})^{q(\nu_2 - 1) + 1}}{q(q(\nu_2 - 1) + 1)} \right)} \leq$$

$$\frac{\left(e^{b_1} - e^{x_{01}}\right)\left(e^{b_2} - e^{x_{02}}\right)}{\Gamma(\nu_1)\Gamma(\nu_2)} \tag{55}$$

$$\left\| \left\| D_{e^{x_{01}}}^{\nu_1} (f_1 \circ \log) \right\| \right\|_{L_q\left(\left[e^{x_{01}}, e^{b_1}\right], A\right)} \left\| \left\| D_{e^{x_{02}}}^{\nu_2} (f_2 \circ \log) \right\| \right\|_{L_p\left(\left[e^{x_{02}}, e^{b_2}\right], A\right)}.$$

We finish with

**Corollary 18** *(to Theorem 15) All as in Theorem 15 for $[a_1, b_1] \subset \mathbb{R}$, $[a_2, b_2] \subset (0, \infty)$, and $g_1(t) = e^t$ and $g_2(t) = \log t$. Then*

$$\int_{e^{x_{01}}}^{e^{b_1}} \int_{\log(x_{02})}^{\log(b_2)} \frac{\left\| (f_1 \circ \log)(z_1)(f_2 \circ e^t)(z_2) \right\| dz_1 dz_2}{\left( \frac{(z_1 - e^{x_{01}})^{p((m_1+1)\nu_1 - 1) + 1}}{p(p((m_1+1)\nu_1 - 1) + 1)} + \frac{(z_2 - \log(x_{02}))^{q((m_2+1)\nu_2 - 1) + 1}}{q(q((m_2+1)\nu_2 - 1) + 1)} \right)} \leq$$

$$\frac{\left(e^{b_1} - e^{x_{01}}\right) \log(b_2 / x_{02})}{\Gamma((m_1 + 1)\nu_1)\Gamma((m_2 + 1)\nu_2)} \tag{56}$$

$$\left\| \left\| D_{e^{x_{01}}}^{(m_1+1)\nu_1} (f_1 \circ \log) \right\| \right\|_{L_q\left(\left[e^{x_{01}}, e^{b_1}\right], A\right)} \left\| \left\| D_{\log(x_{02})}^{(m_2+1)\nu_2} \left(f_2 \circ e^t\right) \right\| \right\|_{L_p([\log(x_{02}), \log(b_2)], A)}.$$

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