Generalized Ostrowski, Opial and Hilbert-Pachpatte type inequalities for Banach algebra valued functions involving integer vectorial derivatives

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Abstract

Using a generalized vectorial Taylor formula involving ordinary vector derivatives we establish mixed Ostrowski, Opial and Hilbert-Pachpatte type inequalities for several Banach algebra valued functions. The estimates are with respect to all norms $\|\cdot\|_p$, $1 \leq p \leq \infty$. We finish with applications.

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1 Introduction

The following result motivates our work.

Theorem 1 *(1938, Ostrowski [6]) Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b) whose derivative $f' : (a, b) \to \mathbb{R}$ is bounded on (a, b), i.e., $\|f'\|_\infty^{\sup} := \sup_{t \in (a,b)} |f'(t)| < +\infty$. Then*

$$\left| \frac{1}{b-a} \int_a^b f(t)\, dt - f(x) \right| \leq \left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right] (b-a) \|f'\|_\infty^{\sup}, \qquad (1)$$

for any $x \in [a, b]$. The constant $\frac{1}{4}$ is the best possible.

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Ostrowski type inequalities have great applications to integral approximations in Numerical Analysis.

We present ([1], Ch. 8,9) mixed fractional Ostrowski inequalities for several functions for various norms.

In this article we generalize [1], Ch. 8,9 for several Banach algebra valued functions by using ordinary vector valued derivatives and our integrals here are of Bochner type [4]. Motivation comes also from [3].

We are also inspired by Z. Opial [5], 1960, famous inequality.

Theorem 2 *Let $x(t) \in C^1([0,h])$ be such that $x(0) = x(h) = 0$, and $x(t) > 0$ in $(0,h)$. Then*

$$\int_0^h |x(t)\, x'(t)|\, dt \leq \frac{h}{4} \int_0^h (x'(t))^2\, dt. \tag{2}$$

In (2), the constant $\frac{h}{4}$ is the best possible. Inequality (2) holds as equality for the optimal function

$$x(t) = \begin{cases} ct, & 0 \leq t \leq \frac{h}{2}, \\ c(h-t), & \frac{h}{2} \leq t \leq h, \end{cases} \tag{3}$$

where $c > 0$ is an arbitrary constant.

Opial-type inequalities are used a lot in proving uniqueness of solutions to differential equations and also to give upper bounds to their solutions.

In this work we also derive Opial type inequalities for Banach algebra valued functions with respect to ordinary vector valued derivatives.

Additionally we include in this article related Hilbert-Pachpatte type inequalities, [7]. We finish with selective applications to Ostrowski, Opial and Hilbert-Pachpatte inequalities.

2 About Banach Algebras

All here come from [8].

We need

Definition 3 *([8], p. 245) A complex algebra is a vector space A over the complex field $\mathbb{C}$ in which a multiplication is defined that satisfies*

$$x(yz) = (xy)z, \tag{4}$$

$$(x+y)z = xz + yz, \quad x(y+z) = xy + xz, \tag{5}$$

and

$$\alpha(xy) = (\alpha x)y = x(\alpha y), \tag{6}$$

for all x, y and z in A and for all scalars α.

Additionally if A is a Banach space with respect to a norm that satisfies the multiplicative inequality

$$\|xy\| \leq \|x\| \, \|y\| \quad (x \in A, \, y \in A) \tag{7}$$

and if A contains a unit element e such that

$$xe = ex = x \quad (x \in A) \tag{8}$$

and

$$\|e\| = 1, \tag{9}$$

then A is called a Banach algebra.

A is commutative iff $xy = yx$ for all $x, y \in A$.

We make

Remark 4 *Commutativity of A will be explicited stated when needed.*

There exists at most one $e \in A$ that satisfies (8).

Inequality (7) makes multiplication to be continuous, more precisely left and right continuous, see [8], p. 246.

Multiplication in A is not necessarily the numerical multiplication, it is something more general and it is defined abstractly, that is for $x, y \in A$ we have $xy \in A$, e.g. composition or convolution, etc.

For nice examples about Banach algebras see [8], p. 247-248, § 10.3.

We also make

Remark 5 *Next we mention about integration of A-valued functions, see [8], p. 259, § 10.22:*

If A is a Banach algebra and f is a continuous A-valued function on some compact Hausdorff space Q on which a complex Borel measure μ is defined, then $\int f d\mu$ exists and has all the properties that were discussed in Chapter 3 of [8], simply because A is a Banach space. However, an additional property can be added to these, namely: If $x \in A$, then

$$x \int_Q f \; d\mu = \int_Q x f(p) \; d\mu(p) \tag{10}$$

and

$$\left(\int_Q f \; d\mu \right) x = \int_Q f(p) \, x \; d\mu(p) \,. \tag{11}$$

The Bochner integrals we will involve in our article follow (10) and (11).

3 Background

We use the following generalized vector Taylor's formula:

Theorem 6 *([2], p. 97) Let $n \in \mathbb{N}$ and $f \in C^n([a,b], X)$, where $[a,b] \subset \mathbb{R}$ and $(X, \|\cdot\|)$ is a Banach space. Let $g \in C^1([a,b])$, strictly increasing, such that $g^{-1} \in C^n([g(a), g(b)])$. Let any $x, y \in [a,b]$. Then*

$$f(x) = f(y) + \sum_{i=1}^{n-1} \frac{(g(x) - g(y))^i}{i!} \left(f \circ g^{-1}\right)^{(i)} (g(y)) \tag{12}$$

$$+ \frac{1}{(n-1)!} \int_{g(y)}^{g(x)} (g(x) - z)^{n-1} \left(f \circ g^{-1}\right)^{(n)} (z)\, dz.$$

The derivatives here are defined similarly to the numerical ones, see [9], pp. 83-86.

The above integral is of Bochner type [4], and so are the integrals in this work. By [2], p. 3, if $f \in C([a,b], X)$ then f is Bochner integrable.

4 Main Results

We start with mixed generalized Ostrowski type inequalities for several functions that are Banach algebra valued. A uniform estimate follows.

Theorem 7 *Let $n \in \mathbb{N}$ and $f_i \in C^n([a,b], A)$, $i = 1, ..., r \in \mathbb{N} - \{1\}$; where $[a,b] \subset \mathbb{R}$ and $(A, \|\cdot\|)$ is a Banach algebra. Let $g \in C^1([a,b])$, strictly increasing, such that $g^{-1} \in C^n([g(a), g(b)])$. We assume that $\left(f_i \circ g^{-1}\right)^{(j)} (g(x_0)) = 0$, $j = 1, ..., n-1$; $i = 1, ..., r$; where $x_0 \in [a,b]$ be fixed. Denote by*

$$E(f_1, ..., f_r)(x_0) :=$$

$$\sum_{i=1}^{r} \left[\int_a^b \left(\prod_{\substack{j=1 \\ j \neq i}}^{r} f_j(x) \right) f_i(x)\, dx - \left(\int_a^b \left(\prod_{\substack{j=1 \\ j \neq i}}^{r} f_j(x) \right) dx \right) f_i(x_0) \right]. \tag{13}$$

Then

1)

$$E(f_1, ..., f_r)(x_0) = \frac{1}{(n-1)!}$$

$$\sum_{i=1}^{r} \left[(-1)^n \left[\int_a^{x_0} \left(\prod_{\substack{j=1 \\ j \neq i}}^{r} f_j(x) \right) \left(\int_{g(x)}^{g(x_0)} (z - g(x))^{n-1} \left(f_i \circ g^{-1}\right)^{(n)} (z)\, dz \right) dx \right] + \right. \tag{14}$$

$$\left[\int_{x_0}^{b}\left(\prod_{\substack{j=1\\j\neq i}}^{r} f_j(x)\right)\left(\int_{g(x_0)}^{g(x)}(g(x)-z)^{n-1}\left(f_i\circ g^{-1}\right)^{(n)}(z)\,dz\right)dx\right]\Bigg],$$

and

2)

$$\|E(f_1,...,f_r)(x_0)\| \leq \frac{1}{n!}$$

$$\left\{\sum_{i=1}^{r}\left[\left[\left\|\left\|\left(f_i\circ g^{-1}\right)^{(n)}\right\|\right\|_{\infty,[g(a),g(x_0)]}(g(x_0)-g(a))^n\left(\int_a^{x_0}\left(\prod_{\substack{j=1\\j\neq i}}^{r}\|f_j(x)\|\right)dx\right)\right]+\right.\right.$$
$$\left.\left.\left[\left\|\left\|\left(f_i\circ g^{-1}\right)^{(n)}\right\|\right\|_{\infty,[g(x_0),g(b)]}(g(b)-g(x_0))^n\left(\int_{x_0}^{b}\left(\prod_{\substack{j=1\\j\neq i}}^{r}\|f_j(x)\|\right)dx\right)\right]\right]\right\}. \tag{15}$$

Proof. Let $x_0\in[a,b]$ such that $\left(f_i\circ g^{-1}\right)^{(j)}(g(x_0))=0$, $j=1,...,n-1$; $i=1,...,r$. Let $x\in[a,x_0]$, then by Theorem 6 we have

$$f_i(x)-f_i(x_0)=\frac{1}{(n-1)!}\int_{g(x_0)}^{g(x)}(g(x)-z)^{n-1}\left(f_i\circ g^{-1}\right)^{(n)}(z)\,dz \tag{16}$$
$$=\frac{(-1)^n}{(n-1)!}\int_{g(x)}^{g(x_0)}(z-g(x))^{n-1}\left(f_i\circ g^{-1}\right)^{(n)}(z)\,dz,$$

for $i=1,...,r$.

And for $x\in[x_0,b]$, then again by Theorem 6 we get

$$f_i(x)-f_i(x_0)=\frac{1}{(n-1)!}\int_{g(x_0)}^{g(x)}(g(x)-z)^{n-1}\left(f_i\circ g^{-1}\right)^{(n)}(z)\,dz, \tag{17}$$

for $i=1,...,r$.

We multiply (16) by $\left(\prod_{\substack{j=1\\j\neq i}}^{r} f_j(x)\right)$ to get:

$$\left(\prod_{\substack{j=1\\j\neq i}}^{r} f_j(x)\right)f_i(x)-\left(\prod_{\substack{j=1\\j\neq i}}^{r} f_j(x)\right)f_i(x_0)=$$

$$\frac{\left(\prod_{\substack{j=1\\j\neq i}}^{r} f_j(x)\right)(-1)^n}{(n-1)!}\int_{g(x)}^{g(x_0)}(z-g(x))^{n-1}\left(f_i\circ g^{-1}\right)^{(n)}(z)\,dz, \tag{18}$$

$\forall\ x \in [a, x_0]$; for $i = 1, ..., r$.

Similarly, we get (by (17))

$$\left(\prod_{\substack{j=1\\j\neq i}}^{r} f_j(x)\right) f_i(x) - \left(\prod_{\substack{j=1\\j\neq i}}^{r} f_j(x)\right) f_i(x_0) =$$

$$\frac{\left(\prod_{\substack{j=1\\j\neq i}}^{r} f_j(x)\right)}{(n-1)!} \int_{g(x_0)}^{g(x)} (g(x) - z)^{n-1} \left(f_i \circ g^{-1}\right)^{(n)}(z)\, dz, \tag{19}$$

$\forall\ x \in [x_0, b]$; for $i = 1, ..., r$.

Adding (18) and (19) as separate groups, we obtain

$$\sum_{i=1}^{r} \left(\prod_{\substack{j=1\\j\neq i}}^{r} f_j(x)\right) f_i(x) - \sum_{i=1}^{r} \left(\prod_{\substack{j=1\\j\neq i}}^{r} f_j(x)\right) f_i(x_0) =$$

$$\frac{(-1)^n}{(n-1)!} \sum_{i=1}^{r} \left(\prod_{\substack{j=1\\j\neq i}}^{r} f_j(x)\right) \int_{g(x)}^{g(x_0)} (z - g(x))^{n-1} \left(f_i \circ g^{-1}\right)^{(n)}(z)\, dz, \tag{20}$$

$\forall\ x \in [a, x_0]$,

and

$$\sum_{i=1}^{r} \left(\prod_{\substack{j=1\\j\neq i}}^{r} f_j(x)\right) f_i(x) - \sum_{i=1}^{r} \left(\prod_{\substack{j=1\\j\neq i}}^{r} f_j(x)\right) f_i(x_0) =$$

$$\frac{1}{(n-1)!} \sum_{i=1}^{r} \left(\prod_{\substack{j=1\\j\neq i}}^{r} f_j(x)\right) \int_{g(x_0)}^{g(x)} (g(x) - z)^{n-1} \left(f_i \circ g^{-1}\right)^{(n)}(z)\, dz, \tag{21}$$

$\forall\ x \in [x_0, b]$.

Next, we integrate (20) and (21) with respect to $x \in [a, b]$. We have

$$\sum_{i=1}^{r} \int_{a}^{x_0} \left(\prod_{\substack{j=1\\j\neq i}}^{r} f_j(x)\right) f_i(x)\, dx - \sum_{i=1}^{r} \left(\int_{a}^{x_0} \left(\prod_{\substack{j=1\\j\neq i}}^{r} f_j(x)\right) dx\right) f_i(x_0) = \tag{22}$$

$$\frac{(-1)^n}{(n-1)!} \sum_{i=1}^{r} \left[\int_{a}^{x_0} \left(\prod_{\substack{j=1\\j\neq i}}^{r} f_j(x)\right) \left(\int_{g(x)}^{g(x_0)} (z - g(x))^{n-1} \left(f_i \circ g^{-1}\right)^{(n)}(z)\, dz\right) dx\right],$$

and

$$\sum_{i=1}^{r}\int_{x_0}^{b}\left(\prod_{\substack{j=1\\ j\neq i}}^{r} f_j(x)\right) f_i(x)\,dx-\sum_{i=1}^{r}\left(\int_{x_0}^{b}\left(\prod_{\substack{j=1\\ j\neq i}}^{r} f_j(x)\right)dx\right) f_i(x_0)= \tag{23}$$

$$\frac{1}{(n-1)!}\sum_{i=1}^{r}\left[\int_{x_0}^{b}\left(\prod_{\substack{j=1\\ j\neq i}}^{r} f_j(x)\right)\left(\int_{g(x_0)}^{g(x)}(g(x)-z)^{n-1}\left(f_i\circ g^{-1}\right)^{(n)}(z)\,dz\right)dx\right].$$

Finally, adding (22) and (23) we obtain the useful identity

$$E(f_1,...,f_r)(x_0):=$$

$$\sum_{i=1}^{r}\left[\int_{a}^{b}\left(\prod_{\substack{j=1\\ j\neq i}}^{r} f_j(x)\right) f_i(x)\,dx-\left(\int_{a}^{b}\left(\prod_{\substack{j=1\\ j\neq i}}^{r} f_j(x)\right)dx\right) f_i(x_0)\right]=\frac{1}{(n-1)!}$$

$$\sum_{i=1}^{r}\left[(-1)^n\left[\int_{a}^{x_0}\left(\prod_{\substack{j=1\\ j\neq i}}^{r} f_j(x)\right)\left(\int_{g(x)}^{g(x_0)}(z-g(x))^{n-1}\left(f_i\circ g^{-1}\right)^{(n)}(z)\,dz\right)dx\right]+\right.$$

$$\left.\left[\int_{x_0}^{b}\left(\prod_{\substack{j=1\\ j\neq i}}^{r} f_j(x)\right)\left(\int_{g(x_0)}^{g(x)}(g(x)-z)^{n-1}\left(f_i\circ g^{-1}\right)^{(n)}(z)\,dz\right)dx\right]\right], \tag{24}$$

proving (14).

Therefore, we get that

$$\|E(f_1,...,f_r)(x_0)\|=$$

$$\left\|\sum_{i=1}^{r}\left[\int_{a}^{b}\left(\prod_{\substack{j=1\\ j\neq i}}^{r} f_j(x)\right) f_i(x)\,dx-\left(\int_{a}^{b}\left(\prod_{\substack{j=1\\ j\neq i}}^{r} f_j(x)\right)dx\right) f_i(x_0)\right]\right\|\leq\frac{1}{(n-1)!}$$

$$\left\{\sum_{i=1}^{r}\left[\left\|\left[\int_{a}^{x_0}\left(\prod_{\substack{j=1\\ j\neq i}}^{r} f_j(x)\right)\left(\int_{g(x)}^{g(x_0)}(z-g(x))^{n-1}\left(f_i\circ g^{-1}\right)^{(n)}(z)\,dz\right)dx\right]\right\|\right.\right.$$

$$\left.\left.+\left\|\left[\int_{x_0}^{b}\left(\prod_{\substack{j=1\\ j\neq i}}^{r} f_j(x)\right)\left(\int_{g(x_0)}^{g(x)}(g(x)-z)^{n-1}\left(f_i\circ g^{-1}\right)^{(n)}(z)\,dz\right)dx\right]\right\|\right]\right\}\leq$$

$$\frac{1}{(n-1)!}\left\{\sum_{i=1}^{r}\left[\left[\int_{a}^{x_0}\left\|\left(\prod_{\substack{j=1\\ j\neq i}}^{r} f_j(x)\right)\left(\int_{g(x)}^{g(x_0)}(z-g(x))^{n-1}\left(f_i\circ g^{-1}\right)^{(n)}(z)\,dz\right)\right\|dx\right]\right.\right.$$

$$\left.\left.+\left[\int_{x_0}^{b}\left\|\left(\prod_{\substack{j=1\\ j\neq i}}^{r} f_j(x)\right)\left(\int_{g(x_0)}^{g(x)}(g(x)-z)^{n-1}\left(f_i\circ g^{-1}\right)^{(n)}(z)\,dz\right)\right\|dx\right]\right]\right\}\leq \tag{25}$$

$$\frac{1}{(n-1)!}\left\{\sum_{i=1}^{r}\left[\left[\int_{a}^{x_0}\left(\prod_{\substack{j=1\\ j\neq i}}^{r}\|f_j(x)\|\right)\left(\int_{g(x)}^{g(x_0)}(z-g(x))^{n-1}\left\|\left(f_i\circ g^{-1}\right)^{(n)}(z)\right\|dz\right)dx\right]\right.\right.$$

$$\left.\left.+\left[\int_{x_0}^{b}\left(\prod_{\substack{j=1\\ j\neq i}}^{r}\|f_j(x)\|\right)\left(\int_{g(x_0)}^{g(x)}(g(x)-z)^{n-1}\left\|\left(f_i\circ g^{-1}\right)^{(n)}(z)\right\|dz\right)dx\right]\right]\right\}=:(\xi). \tag{26}$$

Hence it holds

$$\|E(f_1,...,f_r)(x_0)\|\leq(\xi). \tag{27}$$

We have that

$$(\xi)\leq\frac{1}{n!}\left\{\sum_{i=1}^{r}\left[\left[\left\|\left\|\left(f_i\circ g^{-1}\right)^{(n)}\right\|\right\|_{\infty,[g(a),g(x_0)]}\int_{a}^{x_0}\left(\prod_{\substack{j=1\\ j\neq i}}^{r}\|f_j(x)\|\right)(g(x_0)-g(x))^n\,dx\right]\right.\right.$$

$$\left.\left.+\left[\left\|\left\|\left(f_i\circ g^{-1}\right)^{(n)}\right\|\right\|_{\infty,[g(x_0),g(b)]}\int_{x_0}^{b}\left(\prod_{\substack{j=1\\ j\neq i}}^{r}\|f_j(x)\|\right)(g(x)-g(x_0))^n\,dx\right]\right]\right\}\leq \tag{28}$$

$$\frac{1}{n!}\left\{\sum_{i=1}^{r}\left[\left[\left\|\left\|\left(f_i\circ g^{-1}\right)^{(n)}\right\|\right\|_{\infty,[g(a),g(x_0)]}(g(x_0)-g(a))^n\left(\int_{a}^{x_0}\left(\prod_{\substack{j=1\\ j\neq i}}^{r}\|f_j(x)\|\right)dx\right)\right]\right.\right.$$

$$\left.\left.+\left[\left\|\left\|\left(f_i\circ g^{-1}\right)^{(n)}\right\|\right\|_{\infty,[g(x_0),g(b)]}(g(b)-g(x_0))^n\left(\int_{x_0}^{b}\left(\prod_{\substack{j=1\\ j\neq i}}^{r}\|f_j(x)\|\right)dx\right)\right]\right]\right\}, \tag{29}$$

proving (15). ■

Next comes an L_1 estimate.

Theorem 8 *All as in Theorem 7. Then*

$$\left\|E\left(f_1,...,f_r\right)(x_0)\right\| \leq \frac{1}{(n-1)!}$$

$$\left\{\sum_{i=1}^{r}\left[\left[\left\|\left\|\left(f_i\circ g^{-1}\right)^{(n)}\right\|\right\|_{L_1([g(a),g(x_0)])}\int_a^{x_0}\left(\prod_{\substack{j=1\\ j\neq i}}^{r}\left\|f_j(x)\right\|\right)\left(g(x_0)-g(x)\right)^{n-1}dx\right]\right.\right.$$

$$\left.\left.+\left[\left\|\left\|\left(f_i\circ g^{-1}\right)^{(n)}\right\|\right\|_{L_1([g(x_0),g(b)])}\int_{x_0}^{b}\left(\prod_{\substack{j=1\\ j\neq i}}^{r}\left\|f_j(x)\right\|\right)\left(g(x)-g(x_0)\right)^{n-1}dx\right]\right]\right\}. \tag{30}$$

Proof. By (26), (27), we get that

$$\left\|E\left(f_1,...,f_r\right)(x_0)\right\| \leq (\xi) \leq \frac{1}{(n-1)!}$$

$$\left\{\sum_{i=1}^{r}\left[\left[\left\|\left\|\left(f_i\circ g^{-1}\right)^{(n)}\right\|\right\|_{L_1([g(a),g(x_0)])}\int_a^{x_0}\left(\prod_{\substack{j=1\\ j\neq i}}^{r}\left\|f_j(x)\right\|\right)\left(g(x_0)-g(x)\right)^{n-1}dx\right]\right.\right.$$

$$\left.\left.+\left[\left\|\left\|\left(f_i\circ g^{-1}\right)^{(n)}\right\|\right\|_{L_1([g(x_0),g(b)])}\int_{x_0}^{b}\left(\prod_{\substack{j=1\\ j\neq i}}^{r}\left\|f_j(x)\right\|\right)\left(g(x)-g(x_0)\right)^{n-1}dx\right]\right]\right\}, \tag{31}$$

proving (30). ■

An L_p estimate follows.

Theorem 9 *All as in Theorem 7, and let* $p,q>1:\frac{1}{p}+\frac{1}{q}=1$. *Then*

$$\left\|E\left(f_1,...,f_r\right)(x_0)\right\| \leq \frac{1}{(n-1)!\left(p(n-1)+1\right)^{\frac{1}{p}}}$$

$$\sum_{i=1}^{r}\left[\left\|\left\|\left(f_i\circ g^{-1}\right)^{(n)}\right\|\right\|_{L_q([g(a),g(x_0)])}\left(\int_a^{x_0}\left(g(x_0)-g(x)\right)^{n-\frac{1}{q}}\left(\prod_{\substack{j=1\\ j\neq i}}^{r}\left\|f_j(x)\right\|\right)dx\right)\right.$$

$$\left.+\left\|\left\|\left(f_i\circ g^{-1}\right)^{(n)}\right\|\right\|_{L_q([g(x_0),g(b)])}\left(\int_{x_0}^{b}\left(g(x)-g(x_0)\right)^{n-\frac{1}{q}}\left(\prod_{\substack{j=1\\ j\neq i}}^{r}\left\|f_j(x)\right\|\right)dx\right)\right]. \tag{32}$$

Proof. By (26), (27), we get that

$$\|E(f_1,...,f_r)(x_0)\| \leq (\xi) \leq \frac{1}{(n-1)!}$$

$$\Bigg\{\sum_{i=1}^{r}\Bigg[\Bigg[\int_a^{x_0}\Bigg(\prod_{\substack{j=1\\j\neq i}}^{r}\|f_j(x)\|\Bigg)\left(\int_{g(x)}^{g(x_0)}(z-g(x))^{p(n-1)}dz\right)^{\frac{1}{p}}$$

$$\left(\int_{g(x)}^{g(x_0)}\left\|\left(f_i\circ g^{-1}\right)^{(n)}(z)\right\|^q dz\right)^{\frac{1}{q}}dx\Bigg]+$$

$$\Bigg[\int_{x_0}^{b}\Bigg(\prod_{\substack{j=1\\j\neq i}}^{r}\|f_j(x)\|\Bigg)\left(\int_{g(x_0)}^{g(x)}(g(x)-z)^{p(n-1)}dz\right)^{\frac{1}{p}} \tag{33}$$

$$\left(\int_{g(x_0)}^{g(x)}\left\|\left(f_i\circ g^{-1}\right)^{(n)}(z)\right\|^q dz\right)^{\frac{1}{q}}dx\Bigg]\Bigg]\Bigg\} = \frac{1}{(n-1)!}$$

$$\Bigg\{\sum_{i=1}^{r}\Bigg[\Bigg[\int_a^{x_0}\Bigg(\prod_{\substack{j=1\\j\neq i}}^{r}\|f_j(x)\|\Bigg)\frac{(g(x_0)-g(x))^{\frac{p(n-1)|+1}{p}}}{(p(n-1)+1)^{\frac{1}{p}}}\left\|\left\|\left(f_i\circ g^{-1}\right)^{(n)}\right\|\right\|_{L_q([g(a),g(x_0)])}dx\Bigg]$$

$$+\Bigg[\int_{x_0}^{b}\Bigg(\prod_{\substack{j=1\\j\neq i}}^{r}\|f_j(x)\|\Bigg)\frac{(g(x)-g(x_0))^{\frac{p(n-1)|+1}{p}}}{(p(n-1)+1)^{\frac{1}{p}}}\left\|\left\|\left(f_i\circ g^{-1}\right)^{(n)}\right\|\right\|_{L_q([g(x_0),g(b)])}dx\Bigg]\Bigg]\Bigg\}$$

$$=\frac{1}{(n-1)!\,(p(n-1)+1)^{\frac{1}{p}}}$$

$$\Bigg\{\sum_{i=1}^{r}\Bigg[\left\|\left\|\left(f_i\circ g^{-1}\right)^{(n)}\right\|\right\|_{L_q([g(a),g(x_0)])}\Bigg(\int_a^{x_0}(g(x_0)-g(x))^{n-\frac{1}{q}}\Bigg(\prod_{\substack{j=1\\j\neq i}}^{r}\|f_j(x)\|\Bigg)dx\Bigg)$$

$$+\left\|\left\|\left(f_i\circ g^{-1}\right)^{(n)}\right\|\right\|_{L_q([g(x_0),g(b)])}\Bigg(\int_{x_0}^{b}(g(x)-g(x_0))^{n-\frac{1}{q}}\Bigg(\prod_{\substack{j=1\\j\neq i}}^{r}\|f_j(x)\|\Bigg)dx\Bigg)\Bigg]\Bigg\}, \tag{34}$$

proving (32). ■

Next we present a left generalized Opial type inequality for ordinary derivatives:

Theorem 10 *Let* $p,q>1:\frac{1}{p}+\frac{1}{q}=1$*, and* $n\in\mathbb{N}$*,* $f\in C^{n}([a,b],A)$*; where* $[a,b]\subset\mathbb{R}$ *and* $(A,\|\cdot\|)$ *is a Banach algebra. Let* $g\in C^{1}([a,b])$*, strictly increasing, such that* $g^{-1}\in C^{n}([g(a),g(b)])$*. We assume that* $\left(f\circ g^{-1}\right)^{(j)}(g(x_0))=0$*,* $j=0,1,...,n-1$*; where* $x_0\in[a,b]$ *be fixed. Then*

$$\int_{g(x_0)}^{g(x)}\left\|\left(f\circ g^{-1}\right)(z)\left(f\circ g^{-1}\right)^{(n)}(z)\right\|dz\leq$$

$$\frac{(g(x)-g(x_0))^{n+\frac{1}{p}-\frac{1}{q}}}{2^{\frac{1}{q}}(n-1)!\left[(p(n-1)+1)(p(n-1)+2)\right]^{\frac{1}{p}}}\left(\int_{g(x_0)}^{g(x)}\left\|\left(f\circ g^{-1}\right)^{(n)}(z)\right\|^{q}dz\right)^{\frac{2}{q}},\tag{35}$$

for all $x_0\leq x\leq b$.

Proof. Let $x_0\in[a,b]$ such that $\left(f\circ g^{-1}\right)^{(j)}(g(x_0))=0$, $j=0,1,...,n-1$. For $x\in[x_0,b]$ by Theorem 6 we have

$$\left(f\circ g^{-1}\right)(g(z))=\frac{1}{(n-1)!}\int_{g(x_0)}^{g(x)}(g(x)-z)^{n-1}\left(f\circ g^{-1}\right)^{(n)}(z)\,dz.\tag{36}$$

By Hölder's inequality we obtain

$$\left\|\left(f\circ g^{-1}\right)(g(x))\right\|\leq\frac{1}{(n-1)!}\int_{g(x_0)}^{g(x)}(g(x)-z)^{n-1}\left\|\left(f\circ g^{-1}\right)^{(n)}(z)\right\|dz\leq\tag{37}$$

$$\frac{1}{(n-1)!}\left(\int_{g(x_0)}^{g(x)}(g(x)-z)^{p(n-1)}dt\right)^{\frac{1}{p}}\left(\int_{g(x_0)}^{g(x)}\left\|\left(f\circ g^{-1}\right)^{(n)}(z)\right\|^{q}dz\right)^{\frac{1}{q}}=$$

$$\frac{1}{(n-1)!}\frac{(g(x)-g(x_0))^{\frac{p(n-1)+1}{p}}}{(p(n-1)+1)^{\frac{1}{p}}}\left(\int_{g(x_0)}^{g(x)}\left\|\left(f\circ g^{-1}\right)^{(n)}(z)\right\|^{q}dz\right)^{\frac{1}{q}}.$$

Call

$$\varphi(g(x)):=\int_{g(x_0)}^{g(x)}\left\|\left(f\circ g^{-1}\right)^{(n)}(z)\right\|^{q}dz,\tag{38}$$

$\varphi(g(x_0))=0$.

Thus

$$\frac{d\varphi(g(x))}{dg(x)}=\left\|\left(f\circ g^{-1}\right)^{(n)}(g(x))\right\|^{q}\geq0,\tag{39}$$

and

$$\left(\frac{d\varphi(g(x))}{dg(x)}\right)^{\frac{1}{q}}=\left\|\left(f\circ g^{-1}\right)^{(n)}(g(x))\right\|\geq0,\tag{40}$$

$\forall\ g(x)\in[g(x_0),g(b)]$.

Consequently, we get

$$\left\|\left(f\circ g^{-1}\right)(g(w))\right\|\left\|\left(f\circ g^{-1}\right)^{(n)}(g(w))\right\|\leq$$

$$\frac{(g(w)-g(x_0))^{\frac{p(n-1)+1}{p}}}{(n-1)!\,(p(n-1)+1)^{\frac{1}{p}}}\left(\varphi(g(w))\frac{d\varphi(g(w))}{dg(w)}\right)^{\frac{1}{q}}, \tag{41}$$

$\forall\ g(w)\in[g(x_0),g(b)]$.

Then we observe that

$$\int_{g(x_0)}^{g(x)}\left\|\left(f\circ g^{-1}\right)(g(w))\left(f\circ g^{-1}\right)^{(n)}(g(w))\right\|dg(w)\overset{(7)}{\leq}$$

$$\int_{g(x_0)}^{g(x)}\left\|\left(f\circ g^{-1}\right)(g(w))\right\|\left\|\left(f\circ g^{-1}\right)^{(n)}(g(w))\right\|dg(w)\leq$$

$$\frac{1}{(n-1)!\,(p(n-1)+1)^{\frac{1}{p}}}$$

$$\int_{g(x_0)}^{g(x)}(g(w)-g(x_0))^{\frac{p(n-1)+1}{p}}\left(\varphi(g(w))\frac{d\varphi(g(w))}{dg(w)}\right)^{\frac{1}{q}}dg(w)\leq \tag{42}$$

$$\frac{1}{(n-1)!\,(p(n-1)+1)^{\frac{1}{p}}}$$

$$\left(\int_{g(x_0)}^{g(x)}(g(w)-g(x_0))^{p(n-1)+1}\,dg(w)\right)^{\frac{1}{p}}\left(\int_{g(x_0)}^{g(x)}\varphi(g(w))\frac{d\varphi(g(w))}{dg(w)}dg(w)\right)^{\frac{1}{q}}=$$

$$\frac{1}{(n-1)!\,(p(n-1)+1)^{\frac{1}{p}}\,(p(n-1)+2)^{\frac{1}{p}}}$$

$$(g(x)-g(x_0))^{\frac{p(n-1)+2}{p}}\left(\int_{g(x_0)}^{g(x)}\varphi(g(w))\ \,d\varphi(g(w))\right)^{\frac{1}{q}}= \tag{43}$$

$$\frac{(g(x)-g(x_0))^{n+\frac{1}{p}-\frac{1}{q}}}{(n-1)!\,(p(n-1)+1)^{\frac{1}{p}}\,(p(n-1)+2)^{\frac{1}{p}}}\left(\frac{\varphi^2(g(x))}{2}\right)^{\frac{1}{q}}=$$

$$\frac{(g(x)-g(x_0))^{n+\frac{1}{p}-\frac{1}{q}}}{2^{\frac{1}{q}}\,(n-1)!\,((p(n-1)+1)(p(n-1)+2))^{\frac{1}{p}}}\left(\int_{g(x_0)}^{g(x)}\left\|\left(f\circ g^{-1}\right)^{(n)}(z)\right\|^q dz\right)^{\frac{2}{q}}, \tag{44}$$

for all $g(x_0)\leq g(x)\leq g(b)$, proving (35). ■

The corresponding right generalized Opial type inequality follows:

Theorem 11 *All as in Theorem 10. Then*

$$\int_{g(x)}^{g(x_0)} \left\| \left(f \circ g^{-1}\right)(z) \left(f \circ g^{-1}\right)^{(n)}(z) \right\| dz \leq$$

$$\frac{\left(g\left(x_0\right)-g\left(x\right)\right)^{n+\frac{1}{p}-\frac{1}{q}}}{2^{\frac{1}{q}}\left(n-1\right)!\left(\left(p\left(n-1\right)+1\right)\left(p\left(n-1\right)+2\right)\right)^{\frac{1}{p}}} \left(\int_{g(x)}^{g(x_0)} \left\| \left(f \circ g^{-1}\right)^{(n)}(z) \right\|^q dz \right)^{\frac{2}{q}}, \tag{45}$$

for all $a \leq x \leq x_0$.

Proof. As similar to Theorem 10 is omitted. ■

Next we present a left generalized Hilbert-Pachpatte inequality for ordinary derivatives.

Theorem 12 *Let* $i = 1, 2$; $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$, *and* $n_i \in \mathbb{N}$, $f_i \in C^{n_i}\left(\left[a_i, b_i\right], A\right)$; *where* $\left[a_i, b_i\right] \subset \mathbb{R}$ *and* $\left(A, \|\cdot\|\right)$ *is a Banach algebra. Let* $g_i \in C^1\left(\left[a_i, b_i\right]\right)$, *strictly increasing, such that* $g_i^{-1} \in C^{n_i}\left(\left[g_i\left(a_i\right), g_i\left(b_i\right)\right]\right)$. *We assume that* $\left(f_i \circ g_i^{-1}\right)^{(j_i)}\left(g_i\left(x_{0i}\right)\right) = 0$, $j_i = 0, 1, ..., n_i - 1$; *where* $x_{0i} \in \left[a_i, b_i\right]$ *be fixed. Then*

$$\int_{g_1(x_{01})}^{g_1(b_1)} \int_{g_2(x_{02})}^{g_2(b_2)} \frac{\left\| \left(f_1 \circ g_1^{-1}\right)(z_1) \left(f_2 \circ g_2^{-1}\right)(z_2) \right\| dz_1 dz_2}{\left(\frac{(z_1 - g_1(x_{01}))^{p(n_1-1)+1}}{p(p(n_1-1)+1)} + \frac{(z_2 - g_2(x_{02}))^{q(n_2-1)+1}}{q(q(n_2-1)+1)} \right)} \leq$$

$$\frac{\left(g_1\left(b_1\right)-g_1\left(x_{01}\right)\right)\left(g_2\left(b_2\right)-g_2\left(x_{02}\right)\right)}{\left(n_1-1\right)!\left(n_2-1\right)!} \tag{46}$$

$$\left\| \left\| \left(f_1 \circ g_1^{-1}\right)^{(n_1)} \right\| \right\|_{L_q([g_1(x_{01}), g_1(b_1)], A)} \left\| \left\| \left(f_2 \circ g_2^{-1}\right)^{(n_2)} \right\| \right\|_{L_p([g_2(x_{02}), g_2(b_2)], A)}.$$

Proof. Let $i = 1, 2$; $x_0 \in \left[a_i, b_i\right]$, such that $\left(f_i \circ g_i^{-1}\right)^{(j_i)}\left(g_i\left(x_{0i}\right)\right) = 0$, $j_i = 0, 1, ..., n_i - 1$.

For $x_i \in \left[x_{0i}, b_i\right]$ by Theorem 6 we have

$$\left(f_i \circ g_i^{-1}\right)\left(g_i\left(x_i\right)\right) = \frac{1}{\left(n_i-1\right)!} \int_{g_i(x_{0i})}^{g_i(x_i)} \left(g_i\left(x_i\right) - z_i\right)^{n_i-1} \left(f_i \circ g_i^{-1}\right)^{(n_i)}(z_i)\, dz_i. \tag{47}$$

As in (37) we have

$$\left\| \left(f_1 \circ g_1^{-1}\right)\left(g_1\left(x_1\right)\right) \right\| \leq \frac{1}{\left(n_1-1\right)!} \frac{\left(g_1\left(x_1\right)-g_1\left(x_{01}\right)\right)^{\frac{p(n_1-1)+1}{p}}}{\left(p\left(n_1-1\right)+1\right)^{\frac{1}{p}}}$$

$$\left(\int_{g_1(x_{01})}^{g_1(x_1)} \left\| \left(f_1 \circ g_1^{-1}\right)^{(n_1)}(z) \right\|^q dz \right)^{\frac{1}{q}} \leq$$

$$\frac{1}{(n_1-1)!}\frac{\left(g_1\left(x_1\right)-g_1\left(x_{01}\right)\right)^{\frac{p(n_1-1)+1}{p}}}{\left(p\left(n_1-1\right)+1\right)^{\frac{1}{p}}}\left\|\left\|\left(f_1\circ g_1^{-1}\right)^{(n_1)}\right\|\right\|_{L_q([g_1(x_{01}),g_1(b_1)])}, \tag{48}$$

for all $x_1 \in [x_{01}, b_1]$.

Similarly, we obtain that

$$\left\|\left(f_2\circ g_2^{-1}\right)\left(g_2\left(x_2\right)\right)\right\| \leq \frac{1}{(n_2-1)!}\frac{\left(g_2\left(x_2\right)-g_2\left(x_{02}\right)\right)^{\frac{q(n_2-1)+1}{q}}}{\left(q\left(n_2-1\right)+1\right)^{\frac{1}{q}}}$$

$$\left\|\left\|\left(f_2\circ g_2^{-1}\right)^{(n_2)}\right\|\right\|_{L_p([g_2(x_{02}),g_2(b_2)])}, \tag{49}$$

for all $x_2 \in [x_{02}, b_2]$.

By (48) and (49) we get

$$\left\|\left(f_1\circ g_1^{-1}\right)\left(g_1\left(x_1\right)\right)\left(f_2\circ g_2^{-1}\right)\left(g_2\left(x_2\right)\right)\right\| \leq$$

$$\left\|\left(f_1\circ g_1^{-1}\right)\left(g_1\left(x_1\right)\right)\right\|\left\|\left(f_2\circ g_2^{-1}\right)\left(g_2\left(x_2\right)\right)\right\| \leq \frac{1}{(n_1-1)!\,(n_2-1)!}$$

$$\frac{\left(g_1\left(x_1\right)-g_1\left(x_{01}\right)\right)^{\frac{p(n_1-1)+1}{p}}}{\left(p\left(n_1-1\right)+1\right)^{\frac{1}{p}}}\frac{\left(g_2\left(x_2\right)-g_2\left(x_{02}\right)\right)^{\frac{q(n_2-1)+1}{q}}}{\left(q\left(n_2-1\right)+1\right)^{\frac{1}{q}}} \tag{50}$$

$$\left\|\left\|\left(f_1\circ g_1^{-1}\right)^{(n_1)}\right\|\right\|_{L_q([g_1(x_{01}),g_1(b_1)])}\left\|\left\|\left(f_2\circ g_2^{-1}\right)^{(n_2)}\right\|\right\|_{L_p([g_2(x_{02}),g_2(b_2)])} \leq$$

(using Young's inequality for $a, b \geq 0$, $a^{\frac{1}{p}}b^{\frac{1}{q}} \leq \frac{a}{p}+\frac{b}{q}$)

$$\frac{1}{(n_1-1)!\,(n_2-1)!}\left(\frac{\left(g_1\left(x_1\right)-g_1\left(x_{01}\right)\right)^{p(n_1-1)+1}}{p\left(p\left(n_1-1\right)+1\right)}+\frac{\left(g_2\left(x_2\right)-g_2\left(x_{02}\right)\right)^{q(n_2-1)+1}}{q\left(q\left(n_2-1\right)+1\right)}\right) \tag{51}$$

$$\left\|\left\|\left(f_1\circ g_1^{-1}\right)^{(n_1)}\right\|\right\|_{L_q([g_1(x_{01}),g_1(b_1)])}\left\|\left\|\left(f_2\circ g_2^{-1}\right)^{(n_2)}\right\|\right\|_{L_p([g_2(x_{02}),g_2(b_2)])},$$

$\forall\ (x_1, x_2) \in [x_{01}, b_1] \times [x_{02}, b_2]$.

So far we have

$$\frac{\left\|\left(f_1\circ g_1^{-1}\right)\left(g_1\left(x_1\right)\right)\left(f_2\circ g_2^{-1}\right)\left(g_2\left(x_2\right)\right)\right\|}{\left(\frac{\left(g_1(x_1)-g_1(x_{01})\right)^{p(n_1-1)+1}}{p(p(n_1-1)+1)}+\frac{\left(g_2(x_2)-g_2(x_{02})\right)^{q(n_2-1)+1}}{q(q(n_2-1)+1)}\right)} \leq \tag{52}$$

$$\frac{1}{(n_1-1)!\,(n_2-1)!}\left\|\left\|\left(f_1\circ g_1^{-1}\right)^{(n_1)}\right\|\right\|_{L_q([g_1(x_{01}),g_1(b_1)],A)}$$

$$\left\|\left\|\left(f_2\circ g_2^{-1}\right)^{(n_2)}\right\|\right\|_{L_p([g_2(x_{02}),g_2(b_2)],A)},$$

$\forall\ (x_1, x_2) \in [x_{01}, b_1] \times [x_{02}, b_2]$.

The denominator in (52) can be zero, only when both $g_1(x_1) = g_1(x_{01})$ and $g_2(x_2) = g_2(x_{02})$.

Therefore we obtain (46), by integrating (52) over $[g_1(x_{01}), g_1(b_1)] \times [g_2(x_{02}), g_2(b_2)]$. ■

It follows the right generalized Hilbert-Pachpate inequality for ordinary derivatives.

Theorem 13 *All as in Theorem 12. Then*

$$\int_{g_1(a_1)}^{g_1(x_{01})} \int_{g_2(a_2)}^{g_2(x_{02})} \frac{\left\| \left(f_1 \circ g_1^{-1}\right)(z_1) \left(f_2 \circ g_2^{-1}\right)(z_2) \right\| dz_1 dz_2}{\left(\frac{(g_1(x_{01})-z_1)^{p(n_1-1)+1}}{p(p(n_1-1)+1)} + \frac{(g_2(x_{02})-z_2)^{q(n_2-1)+1}}{q(q(n_2-1)+1)} \right)} \leq$$

$$\frac{(g_1(x_{01}) - g_1(a_1))(g_2(x_{02}) - g_2(a_2))}{(n_1-1)!(n_2-1)!} \tag{53}$$

$$\left\| \left\| \left(f_1 \circ g_1^{-1}\right)^{(n_1)} \right\| \right\|_{L_q([g_1(a_1), g_1(x_{01})], A)} \left\| \left\| \left(f_2 \circ g_2^{-1}\right)^{(n_2)} \right\| \right\|_{L_p([g_2(a_2), g_2(x_{02})], A)}.$$

Proof. As similar to theorem 12 is omitted. ■

5 Applications

We make

Remark 14 *Assume next that $(A, \|\cdot\|)$ is a commutative Banach algebra. Then, we get that*

$$E(f_1, ..., f_r)(x_0) \overset{(13)}{=} r \int_a^b \left(\prod_{j=1}^{r} f_j(x) \right) dx - \sum_{i=1}^{r} \left(\int_a^b \left(\prod_{\substack{j=1 \\ j \neq i}}^{r} f_j(x) \right) dx \right) f_i(x_0), \tag{54}$$

$x_0 \in [a, b]$.

When $r = 2$, we have that

$$E(f_1, f_2)(x_0) = 2 \int_a^b f_1(x) f_2(x) dx - f_1(x_0) \int_a^b f_2(x) dx - f_2(x_0) \int_a^b f_1(x) dx, \tag{55}$$

$x_0 \in [a, b]$.

We give

Corollary 15 *(to Theorem 7) All as in Theorem 7, $(A, \|\cdot\|)$ is a commutative Banach algebra, $r = 2$. Then*

$$\|E(f_1, f_2)(x_0)\| \leq \frac{1}{n!} \sum_{i=1}^{2} \left[\left[\left\| \left\| \left(f_i \circ g^{-1}\right)^{(n)} \right\| \right\|_{\infty, [g(a), g(x_0)]} \right. \right.$$

$$(g(x_0)-g(a))^n\left(\int_a^{x_0}\left(\prod_{\substack{j=1\\j\neq i}}^{2}\|f_j(x)\|\right)dx\right)\Bigg]+$$

$$\left[\left\|\left\|\left(f_i\circ g^{-1}\right)^{(n)}\right\|\right\|_{\infty,[g(x_0),g(b)]}(g(b)-g(x_0))^n\left(\int_{x_0}^{b}\left(\prod_{\substack{j=1\\j\neq i}}^{2}\|f_j(x)\|\right)dx\right)\right]\Bigg]. \tag{56}$$

It follows

Corollary 16 *(to Corollary 15) All as in Corollary 15, with $g(t)=e^t$. Then*

$$\|E(f_1,f_2)(x_0)\|\leq\frac{1}{n!}\sum_{i=1}^{2}\Bigg[\Bigg[\left\|\left\|(f_i\circ\log)^{(n)}\right\|\right\|_{\infty,[e^a,e^{x_0}]}$$

$$(e^{x_0}-e^a)^n\left(\int_a^{x_0}\left(\prod_{\substack{j=1\\j\neq i}}^{2}\|f_j(x)\|\right)dx\right)\Bigg]+$$

$$\left[\left\|\left\|(f_i\circ\log)^{(n)}\right\|\right\|_{\infty,[e^{x_0},e^b]}\left(e^b-e^{x_0}\right)^n\left(\int_{x_0}^{b}\left(\prod_{\substack{j=1\\j\neq i}}^{2}\|f_j(x)\|\right)dx\right)\right]\Bigg]. \tag{57}$$

We continue with

Corollary 17 *(to Theorem 10) All as in Theorem 10 for $g(t)=e^t$. Then*

$$\int_{e^{x_0}}^{e^x}\left\|((f\circ\log)(z))(f\circ\log)^{(n)}(z)\right\|dz\leq$$

$$\frac{(e^x-e^{x_0})^{n+\frac{1}{p}-\frac{1}{q}}}{2^{\frac{1}{q}}(n-1)!\left[(p(n-1)+1)(p(n-1)+2)\right]^{\frac{1}{p}}}\left(\int_{e^{x_0}}^{z}\left\|(f\circ\log)^{(n)}(z)\right\|^q dz\right)^{\frac{2}{q}}, \tag{58}$$

for all $x_0\leq x\leq b$.

We finish with

Corollary 18 *(to Theorem 12) All as in Theorem 12 for $g_i(t)=e^t$, $i=1,2$. Then*

$$\int_{e^{x_{01}}}^{e^{b_1}}\int_{e^{x_{02}}}^{e^{b_2}}\frac{\|(f_1\circ\log)(z_1)(f_2\circ\log)(z_2)\|\,dz_1dz_2}{\left(\frac{(z_1-e^{x_{01}})^{p(n_1-1)+1}}{p(p(n_1-1)+1)}+\frac{(z_2-e^{x_{02}})^{q(n_2-1)+1}}{q(q(n_2-1)+1)}\right)}\leq$$

$$\frac{\left(e^{b_1}-e^{x_{01}}\right)\left(e^{b_2}-e^{x_{02}}\right)}{(n_1-1)!\,(n_2-1)!} \tag{59}$$
$$\left\|\left\|(f_1\circ\log)^{(n_1)}\right\|\right\|_{L_q\left(\left[e^{x_{01}},e^{b_1}\right],A\right)}\left\|\left\|(f_2\circ\log)^{(n_2)}\right\|\right\|_{L_p\left(\left[e^{x_{02}},e^{b_2}\right],A\right)}.$$

The simplest applications derive when $g(t)=t$ and $A=\mathbb{R}$, leading to basic known results.

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