REVERSES OF SECOND HERMITE-HADAMARD INEQUALITY FOR THE SQUARE NORM IN HILBERT SPACES

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Abstract. Let $(H;\langle\cdot,\cdot\rangle)$ be a complex Hilbert space and $f:[0,\infty)\to\mathbb{R}$ be convex on $[0,\infty)$. If $x,\ y\in H$ and there exists the constants $0\le m<M$ such that $m\le\|(1-t)\,x+ty\|^2\le M$ for all $t\in[0,1]$, then

$$\begin{aligned}
0 &\le \frac{f\left(\|x\|^2\right)+f\left(\operatorname{Re}\langle x,y\rangle\right)+f\left(\|y\|^2\right)}{3}-\int_0^1 f\left(\|(1-t)\,x+ty\|^2\right)dt\\
&\le \frac{2}{M-m}\left[\frac{f(m)+f(M)}{2}-f\left(\frac{m+M}{2}\right)\right]\\
&\times\left(\frac{1}{2}(M-m)+\int_0^1\left|\|(1-t)\,x+ty\|^2-\frac{m+M}{2}\right|dt\right)\\
&\le 2\left[\frac{f(m)+f(M)}{2}-f\left(\frac{m+M}{2}\right)\right].
\end{aligned}$$

Some examples for power functions and logarithm are also provided.

1. Introduction

Let $\mathbb{R}$ be the set of real numbers and $I\subseteq\mathbb{R}$ be an interval. A function $f:I\subseteq\mathbb{R}\to\mathbb{R}$ is said to be *convex* in the classical sense if it satisfies the following inequality

$$f\left(tx+(1-t)\,t\right)\le tf(x)+(1-t)\,f(y)$$

for all $x,\ y\in I$ and $t\in[0,1]$.

Let $f:I\subseteq\mathbb{R}\to\mathbb{R}$ and $a,b\in I$ with $a<b$. Then the inequality

$$f\left(\frac{a+b}{2}\right)\le\frac{1}{b-a}\int_a^b f(x)dx\le\frac{f(a)+f(b)}{2} \tag{1.1}$$

holds if f is convex, and is known in the literature as *Hermite-Hadamard inequality*, after the name of C. Hermite and J. Hadamard (see [14]). The inequalities in (1.1) hold in reversed direction if f is a concave function.

A vast literature related to (1.1) have been produced by a large number of mathematicians [12] since it is considered to be one of the most famous inequality for convex functions due to its usefulness and many applications in various branches of Pure and Applied Mathematics, such as Numerical Analysis [2], Information Theory [1], Operator Theory [6], [7] and others.

1991 *Mathematics Subject Classification.* 47A63, 26D15, 46C05.

Key words and phrases. Convex functions, Hermite-Hadamard inequality, Midpoint inequality, Power and exponential functions.

RGMIA Res. Rep. Coll. 24 (2021), Art. 142, 16 pp. Received 12/11/21

Let X be a vector space over the real or complex number field $\mathbb{K}$ and $x, y \in X,\ x \neq y$. Define the segment

$$[x, y] := \{(1-t)x + ty,\ t \in [0,1]\}.$$

We consider the function $f : [x, y] \to \mathbb{R}$ and the associated function

$$g(x, y) : [0,1] \to \mathbb{R},\ g(x, y)(t) := f[(1-t)x + ty],\ t \in [0,1].$$

Note that f is convex on $[x, y]$ if and only if $g(x, y)$ is convex on $[0,1]$.

For any convex function defined on a segment $[x, y] \subset X$, we have the *Hermite-Hadamard integral inequality* (see [3, p. 2], [4, p. 2])

$$f\left(\frac{x+y}{2}\right) \leq \int_0^1 f[(1-t)x + ty]dt \leq \frac{f(x) + f(y)}{2}, \tag{1.2}$$

which can be derived from the classical Hermite-Hadamard inequality (1.1) for the convex function $g(x, y) : [0,1] \to \mathbb{R}$.

Since $f(x) = \|x\|^p$ ($x \in X$ and $1 \leq p < \infty$) is a convex function, then for any $x, y \in X$ we have the following norm inequality from (1.2) (see [13, p. 106])

$$\left\|\frac{x+y}{2}\right\|^p \leq \int_0^1 \|(1-t)x + ty\|^p dt \leq \frac{\|x\|^p + \|y\|^p}{2}. \tag{1.3}$$

Assume that $(H; \langle \cdot, \cdot \rangle)$ is a complex Hilbert space that generates the norm $\|\cdot\|$.

In the recent paper [10] we proved the following Hermite-Hadamard type inequalities:

Theorem 1. *Let $f : [0, \infty) \to \mathbb{R}$ be convex (concave) on $[0, \infty)$. If $x,\ y \in H$ with $\operatorname{Re}\langle x, y\rangle \geq 0$, then*

$$\begin{aligned} &f\left(\frac{\|x\|^2 + \operatorname{Re}\langle x, y\rangle + \|y\|^2}{3}\right) \\ &\leq (\geq) \int_0^1 f\left(\|(1-t)\,x + ty\|^2\right) dt \\ &\leq (\geq) \frac{1}{3}\left[f\left(\|x\|^2\right) + f\left[\operatorname{Re}\langle x, y\rangle\right] + f\left(\|y\|^2\right)\right]. \end{aligned} \tag{1.4}$$

Motivated by the above results we establish in this paper several upper bounds for the nonnegative difference

$$\frac{f\left(\|x\|^2\right) + f\left(\operatorname{Re}\langle x, y\rangle\right) + f\left(\|y\|^2\right)}{3} - \int_0^1 f\left(\|(1-t)\,x + ty\|^2\right) dt$$

where $f : [0, \infty) \to \mathbb{R}$ is convex on $[0, \infty)$ and $x,\ y \in H$ with $\operatorname{Re}\langle x, y\rangle \geq 0$. Some examples for power functions and logarithm are also provided.

2. Quadratic Hermite-Hadamard Inequalities

We have the following reverses of Hermite-Hadamard type inequalities:

Theorem 2. *Let $f : [0, \infty) \to \mathbb{R}$ be convex on $[0, \infty)$. If $x,\ y \in H$ with $\operatorname{Re}\langle x, y\rangle \geq 0$ and there exists the constants $m,\ M$ so that $0 \leq m \leq \|(1-t)\,x + ty\|^2 \leq M$ for all*

$t\in[0,1]$, *then*

$$0\leq(1-t)^{2}f\left(\|x\|^{2}\right)+2t(1-t)f(\operatorname{Re}\langle x,y\rangle)+t^{2}f\left(\|y\|^{2}\right) \tag{2.1}$$
$$-f\left(\|(1-t)x+ty\|^{2}\right)$$
$$\leq\frac{2}{M-m}\left[\frac{f(m)+f(M)}{2}-f\left(\frac{m+M}{2}\right)\right]$$
$$\times\left(\frac{1}{2}(M-m)1_{H}+\left|\|(1-t)x+ty\|^{2}-\frac{m+M}{2}\right|\right)$$
$$\leq 2\left[\frac{f(m)+f(M)}{2}-f\left(\frac{m+M}{2}\right)\right].$$

Also, we have the integral inequalities

$$0\leq\frac{f\left(\|x\|^{2}\right)+f(\operatorname{Re}\langle x,y\rangle)+f\left(\|y\|^{2}\right)}{3}-\int_{0}^{1}f\left(\|(1-t)x+ty\|^{2}\right)dt \tag{2.2}$$
$$\leq\frac{2}{M-m}\left[\frac{f(m)+f(M)}{2}-f\left(\frac{m+M}{2}\right)\right]$$
$$\times\left(\frac{1}{2}(M-m)+\int_{0}^{1}\left|\|(1-t)x+ty\|^{2}-\frac{m+M}{2}\right|dt\right)$$
$$\leq 2\left[\frac{f(m)+f(M)}{2}-f\left(\frac{m+M}{2}\right)\right].$$

Proof. Observe that, by the properties of Hilbertian norm, we have for $t\in[0,1]$ that

$$\|(1-t)x+ty\|^{2}=(1-t)^{2}\|x\|^{2}+2t(1-t)\operatorname{Re}\langle x,y\rangle+t^{2}\|y\|^{2}.$$

Now, consider $p_1:=(1-t)^2$, $p_2:=2t(1-t)$ and $p_3:=t^2$. We observe that p_1, p_2, $p_3\geq 0$ and

$$p_1+p_2+p_3=(1-t)^2+2t(1-t)+t^2=(1-t+t)^2=1.$$

Also, put $x_1:=\|x\|^2$, $x_2:=\operatorname{Re}\langle x,y\rangle$ and $x_3:=\|y\|^2$. Then

$$p_1x_1+p_2x_3+p_3x_3=\|(1-t)x+ty\|^2$$

and

$$m\leq p_1x_1+p_2x_3+p_3x_3\leq M.$$

Recall the reverse of Jensen's inequality, see for instance [8]

$$(0\leq)\frac{1}{P_n}\sum_{j=1}^{n}p_jf(x_j)-f\left(\frac{1}{P_n}\sum_{j=1}^{n}p_jx_j\right) \tag{2.3}$$
$$\leq\frac{2}{M-m}\left[\frac{f(m)+f(M)}{2}-f\left(\frac{m+M}{2}\right)\right]$$
$$\times\left(\frac{1}{2}(M-m)1_{H}+\left|\frac{1}{P_n}\sum_{j=1}^{n}x_jA_j-\frac{m+M}{2}\right|\right)$$
$$\leq 2\left[\frac{f(m)+f(M)}{2}-f\left(\frac{m+M}{2}\right)\right]$$

where $p_j \geq 0$ $(j = 1, ...n)$ with $P_n := \sum_{j=1}^{n} p_j$ and $x_j \geq 0$ $(j = 1, ...n)$ with

$$M \geq \frac{1}{P_n} \sum_{j=1}^{n} p_j x_j \geq m \geq 0 \ \ (j = 1, ...n), \tag{2.4}$$

for some constants $M > m \geq 0$.

For $n = 3$ we get by (2.3) that

$$\begin{aligned} 0 &\leq (1-t)^2 f\left(\|x\|^2\right) + 2t(1-t) f(\operatorname{Re}\langle x, y\rangle) + t^2 f\left(\|y\|^2\right) \\ &- f\left(\|(1-t)x + ty\|^2\right) \\ &\leq \frac{2}{M-m}\left[\frac{f(m) + f(M)}{2} - f\left(\frac{m+M}{2}\right)\right] \\ &\times \left(\frac{1}{2}(M-m) 1_H + \left|\|(1-t)x + ty\|^2 - \frac{m+M}{2}\right|\right) \\ &\leq 2\left[\frac{f(m) + f(M)}{2} - f\left(\frac{m+M}{2}\right)\right], \end{aligned}$$

which proves (2.1).

If we take the integral and observe that

$$\int_0^1 (1-t)^2 dt = 2\int_0^1 t(1-t)\, dt = \int_0^1 t^2 dt = \frac{1}{3},$$

then we also obtain (2.2). □

Theorem 3. *With the assumptions of Theorem 2 we have*

$$\begin{aligned} 0 &\leq (1-t)^2 f\left(\|x\|^2\right) + 2t(1-t) f(\operatorname{Re}\langle x, y\rangle) + t^2 f\left(\|y\|^2\right) \\ &- f\left(\|(1-t)x + ty\|^2\right) \\ &\leq \frac{\left[f'_-(M) - f'_+(m)\right]}{M-m} \\ &\times \left(\|(1-t)x + ty\|^2 - m\right)\left(M - \|(1-t)x + ty\|^2\right) \\ &\leq \frac{1}{4}(M-m)\left[f'_-(M) - f'_+(m)\right] \end{aligned} \tag{2.5}$$

for all $t \in [0,1]$ *and the integral inequality*

$$0 \leq \frac{f\left(\|x\|^2\right) + f\left(\operatorname{Re}\langle x,y\rangle\right) + f\left(\|y\|^2\right)}{3} - \int_0^1 f\left(\|(1-t)x+ty\|^2\right) dt \tag{2.6}$$
$$\leq \frac{\left[f'_-(M) - f'_+(m)\right]}{M-m}$$
$$\times \left[\frac{1}{4}(M-m)^2 1_H - \int_0^1 \left(\|(1-t)x+ty\|^2 - \frac{m+M}{2}\right)^2 dt\right]$$
$$\leq \frac{\left[f'_-(M) - f'_+(m)\right]}{M-m}$$
$$\times \left[\frac{1}{4}(M-m)^2 - \left(\frac{\|x\|^2 + \operatorname{Re}\langle x,y\rangle + \|y\|^2}{3} - \frac{m+M}{2}\right)^2\right]$$
$$\leq \frac{1}{4}(M-m)\left[f'_-(M) - f'_+(m)\right].$$

Proof. If $f:[0,\infty) \to \mathbb{R}$ is convex and the n-tuple of nonnegative numbers $(x_1,...,x_n)$ has the property (2.4) for $p_j \geq 0$ with $j \in \{1,...,n\}$ and $P_n := \sum_{j=1}^n p_j > 0$, then we have, see for instance [8], that

$$(0 \leq) \frac{1}{P_n}\sum_{j=1}^n p_j f(x_j) - f\left(\frac{1}{P_n}\sum_{j=1}^n p_j x_j\right) \tag{2.7}$$
$$\leq \frac{\left[f'_-(M) - f'_+(m)\right]}{M-m} \frac{1}{P_n}\sum_{j=1}^n p_j (x_j - m) \frac{1}{P_n}\sum_{j=1}^n p_j (M - x_j)$$
$$\leq \frac{1}{4}(M-m)\left[f'_-(M) - f'_+(m)\right].$$

The inequality (2.5) follows by (2.7) by taking $p_1 := (1-t)^2$, $p_2 := 2t(1-t)$ and $p_3 := t^2$ and $x_1 := \|x\|^2$, $x_2 := \operatorname{Re}\langle x,y\rangle$ and $x_3 := \|y\|^2$.

Now, observe that, by using the elementary identity

$$(x-m)(M-x) = \frac{1}{4}(M-m)^2 - \left(x - \frac{m+M}{2}\right)^2$$

we get

$$\int_0^1 \left(\|(1-t)x+ty\|^2 - m\right)\left(M - \|(1-t)x+ty\|^2\right) dt$$
$$= \frac{1}{4}(M-m)^2 - \int_0^1 \left(\|(1-t)x+ty\|^2 - \frac{m+M}{2}\right)^2 dt.$$

By the convexity of the square function and Jensen's inequality we have

$$\begin{aligned}
&\int_0^1 \left(\|(1-t)\,x+ty\|^2 - \frac{m+M}{2} \right)^2 \\
&\geq \left(\int_0^1 \|(1-t)\,x+ty\|^2 - \frac{m+M}{2} \right)^2 \\
&= \left(\int_0^1 \left[(1-t)^2 \|x\|^2 + 2t\,(1-t)\operatorname{Re}\langle x,y\rangle + t^2 \|y\|^2 \right] dt - \frac{m+M}{2} \right)^2 \\
&= \left(\frac{\|x\|^2 + \operatorname{Re}\langle x,y\rangle + \|y\|^2}{3} - \frac{m+M}{2} \right)^2,
\end{aligned}$$

which gives that

$$\begin{aligned}
&\frac{1}{4}(M-m)^2 - \int_0^1 \left(\|(1-t)\,x+ty\|^2 - \frac{m+M}{2} \right)^2 \\
&\leq \frac{1}{4}(M-m)^2 - \left(\frac{\|x\|^2 + \operatorname{Re}\langle x,y\rangle + \|y\|^2}{3} - \frac{m+M}{2} \right)^2 \\
&\leq \frac{1}{4}(M-m)^2,
\end{aligned}$$

which proves the last part of (2.6). □

We also have the following result [9]:

Lemma 1. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a twice differentiable function on the interval $\mathring{I}$, the interior of I. If there exists the constants d, D such that*

$$d \leq f''(t) \leq D \text{ for any } t \in \mathring{I}, \tag{2.8}$$

then

$$\begin{aligned}
\frac{1}{2}\nu(1-\nu)\,d\,(b-a)^2 &\leq (1-\nu)\,f(a) + \nu f(b) - f((1-\nu)\,a + \nu b) \\
&\leq \frac{1}{2}\nu(1-\nu)\,D\,(b-a)^2
\end{aligned} \tag{2.9}$$

for any $a, b \in \mathring{I}$ and $\nu \in [0,1]$.

In particular, we have

$$\frac{1}{8}(b-a)^2 d \leq \frac{f(a)+f(b)}{2} - f\left(\frac{a+b}{2}\right) \leq \frac{1}{8}(b-a)^2 D, \tag{2.10}$$

for any $a, b \in \mathring{I}$.

The constant $\frac{1}{8}$ is best possible in both inequalities in (2.10).

Theorem 4. *Assume that the function $f : [0,\infty) \to \mathbb{R}$ is twice differentiable convex with $f''(x) \leq D$ for all $x \in (m, M)$. If $x, y \in H$ with $\operatorname{Re}\langle x,y\rangle \geq 0$ and*

$0 \le m \le \|(1-t)x + ty\|^2 \le M$ *for all* $t \in [0,1]$, *then*

$$\begin{aligned} (2.11) \qquad 0 &\le (1-t)^2 f\left(\|x\|^2\right) + 2t(1-t) f(\operatorname{Re}\langle x,y\rangle) + t^2 f\left(\|y\|^2\right) \\ &\quad - f\left(\|(1-t)x+ty\|^2\right) \\ &\le \frac{1}{2} D \left(\|(1-t)x+ty\|^2 - m\right)\left(M - \|(1-t)x+ty\|^2\right) \\ &\le \frac{1}{8}(M-m)^2 D \end{aligned}$$

for all $t \in [0,1]$ *and the integral inequality*

$$\begin{aligned} (2.12) \qquad 0 &\le \frac{f\left(\|x\|^2\right) + f(\operatorname{Re}\langle x,y\rangle) + f\left(\|y\|^2\right)}{3} - \int_0^1 f\left(\|(1-t)x+ty\|^2\right) dt \\ &\le \frac{1}{2} D \left[\frac{1}{4}(M-m)^2 - \int_0^1 \left(\|(1-t)x+ty\|^2 - \frac{m+M}{2}\right)^2 dt\right] \\ &\le \frac{1}{2} D \left[\frac{1}{4}(M-m)^2 - \left(\frac{\|x\|^2 + \operatorname{Re}\langle x,y\rangle + \|y\|^2}{3} - \frac{m+M}{2}\right)^2\right] \\ &\le \frac{1}{8}(M-m)^2 D. \end{aligned}$$

Proof. For $0 \le t \le 1$ we have from (2.9) that

$$\begin{aligned} (2.13) \qquad 0 &\le \frac{1}{2} t(1-t)(M-m)^2 d \\ &\le f(m)(1-t) + f(M)t - f(m(1-t)+Mt) \\ &\le \frac{1}{2} t(1-t)(M-m)^2 D. \end{aligned}$$

Writing the inequality (2.13) for

$$0 \le t = \frac{\frac{1}{P_n}\sum_{j=1}^n p_j x_j - m}{M-m} \le 1,$$

we derive

$$\begin{aligned} (2.14) \qquad 0 &\le \frac{1}{2} d \frac{1}{P_n} \sum_{j=1}^n p_j (x_j - m) \frac{1}{P_n} \sum_{j=1}^n p_j (M - x_j) \\ &\le \frac{1}{P_n} \sum_{j=1}^n p_j f(x_j) - f\left(\frac{1}{P_n}\sum_{j=1}^n p_j x_j\right) \\ &\le \frac{1}{2} D \frac{1}{P_n} \sum_{j=1}^n p_j (A_j - m) \frac{1}{P_n} \sum_{j=1}^n p_j (M - A_j) \le \frac{1}{8}(M-m)^2 D. \end{aligned}$$

This double inequality is of interest in itself.

The proof follows now as above and we omit the details. □

Remark 1. *We observe that, if* $0 \leq m \leq \|x\|^2, \operatorname{Re}\langle x, y\rangle, \|y\|^2 \leq M$ *then for all* $t \in [0,1]$

$$\begin{aligned} 0 &\leq (1-t)^2 m + 2t(1-t)m + t^2 m \\ &\leq (1-t)^2 \|x\|^2 + 2t(1-t)\operatorname{Re}\langle x, y\rangle + t^2 \|y\|^2 \\ &\leq (1-t)^2 M + 2t(1-t)\operatorname{Re} M + t^2 M \end{aligned}$$

that is equivalent to

$$0 \leq m \leq \|(1-t)x + ty\|^2 \leq M,$$

which gives a sufficient condition for the assumption in the above theorems to hold.

So, if $\|x\|^2 \leq M_1$ *and* $\|y\|^2 \leq M_2$*, then by the convexity of* $\|\cdot\|^2$ *we have*

$$\|(1-\alpha)x + \alpha y\|^2 \leq (1-\alpha)\|x\|^2 + \alpha\|y\|^2 \leq (1-\alpha)M_1 + \alpha M_2.$$

So, if we take $m = 0$ *above and* $M = (1-\alpha)M_1 + \alpha M_2$ *in the previous theorems, we can get several reverse inequalities as well, of course, for functions for which the upper bounds are finite on* $(0, (1-\alpha)M_1 + \alpha M_2)$.

3. Some Examples

By writing the inequalities from Theorems 2-4 for the operator convex function $f(t) = t^r$ for $r \in [1, \infty)$ we get for $x, y \in H$ with $\operatorname{Re}\langle x, y\rangle \geq 0$ and $0 \leq m \leq \|(1-t)x + ty\|^2 \leq M$ for all $t \in [0,1]$, the following integral inequalities

$$\begin{aligned} 0 &\leq \frac{\|x\|^{2r} + [\operatorname{Re}\langle x, y\rangle]^r + \|y\|^{2r}}{3} - \int_0^1 \|(1-t)x + ty\|^{2r} dt \\ &\leq 2\left[\frac{m^r + M^r}{2} - \left(\frac{m+M}{2}\right)^r\right], \end{aligned} \tag{3.1}$$

$$\begin{aligned} 0 &\leq \frac{\|x\|^{2r} + [\operatorname{Re}\langle x, y\rangle]^r + \|y\|^{2r}}{3} - \int_0^1 \|(1-t)x + ty\|^{2r} dt \\ &\leq r\frac{M^{r-1} - m^{r-1}}{M - m} \\ &\times \left[\frac{1}{4}(M-m)^2 - \left(\frac{\|x\|^2 + \operatorname{Re}\langle x, y\rangle + \|y\|^2}{3} - \frac{m+M}{2}\right)^2\right] \\ &\leq \frac{1}{4} r (M-m)\left(M^{r-1} - m^{r-1}\right) \end{aligned} \tag{3.2}$$

and

$$\begin{aligned} 0 &\leq \frac{f\left(\|x\|^2\right) + f(\operatorname{Re}\langle x, y\rangle) + f\left(\|y\|^2\right)}{3} - \int_0^1 f\left(\|(1-t)x + ty\|^2\right) dt \\ &\leq \frac{1}{2m^{2-r}} r(r-1) \\ &\times \left[\frac{1}{4}(M-m)^2 - \left(\frac{\|x\|^2 + \operatorname{Re}\langle x, y\rangle + \|y\|^2}{3} - \frac{m+M}{2}\right)^2\right] \\ &\leq \frac{1}{8m^{2-r}} r(r-1)(M-m)^2. \end{aligned} \tag{3.3}$$

By writing the inequalities from Theorems 2-4 for the concave function $f(t)=t^p$ for $p\in(0,1]$ we get for $x,y\in H$ with $\operatorname{Re}\langle x,y\rangle>0$ and $0<m\leq\|(1-t)x+ty\|^2\leq M$ for all $t\in[0,1]$, the following integral inequalities

$$
\begin{aligned}
0 &\leq \int_0^1 \|(1-t)x+ty\|^{2p}\,dt-\frac{\|x\|^{2p}+[\operatorname{Re}\langle x,y\rangle]^p+\|y\|^{2p}}{3} \\
&\leq 2\left[\left(\frac{m+M}{2}\right)^p-\frac{m^p+M^p}{2}\right],
\end{aligned} \tag{3.4}
$$

$$
\begin{aligned}
0 &\leq \int_0^1 \|(1-t)x+ty\|^{2p}\,dt-\frac{\|x\|^{2p}+[\operatorname{Re}\langle x,y\rangle]^p+\|y\|^{2p}}{3} \\
&\leq \frac{p\left(M^{1-p}-m^{1-p}\right)}{m^{1-p}M^{1-p}(M-m)} \\
&\quad\times\left[\frac{1}{4}(M-m)^2-\left(\frac{\|x\|^2+\operatorname{Re}\langle x,y\rangle+\|y\|^2}{3}-\frac{m+M}{2}\right)^2\right] \\
&\leq \frac{1}{4}p\frac{M^{1-p}-m^{1-p}}{m^{1-p}M^{1-p}}(M-m)
\end{aligned} \tag{3.5}
$$

and

$$
\begin{aligned}
0 &\leq \int_0^1 \|(1-t)x+ty\|^{2p}\,dt-\frac{\|x\|^{2p}+[\operatorname{Re}\langle x,y\rangle]^p+\|y\|^{2p}}{3} \\
&\leq \frac{1}{2m^{2-p}}p(1-p) \\
&\quad\times\left[\frac{1}{4}(M-m)^2-\left(\frac{\|x\|^2+\operatorname{Re}\langle x,y\rangle+\|y\|^2}{3}-\frac{m+M}{2}\right)^2\right] \\
&\leq \frac{1}{8m^{2-p}}p(1-p)(M-m)^2.
\end{aligned} \tag{3.6}
$$

For $p=1/2$ we derive

$$
\begin{aligned}
0 &\leq \int_0^1 \|(1-t)x+ty\|\,dt-\frac{\|x\|+[\operatorname{Re}\langle x,y\rangle]^{1/2}+\|y\|}{3} \\
&\leq 2\left[\sqrt{\frac{m+M}{2}}-\frac{\sqrt{m}+\sqrt{M}}{2}\right],
\end{aligned} \tag{3.7}
$$

$$0 \le \int_0^1 \|(1-t)x+ty\| \, dt - \frac{\|x\| + [\operatorname{Re}\langle x,y\rangle]^{1/2} + \|y\|}{3} \tag{3.8}$$
$$\le \frac{p\left(\sqrt{M}-\sqrt{m}\right)}{\sqrt{mM}(M-m)}$$
$$\times \left[\frac{1}{4}(M-m)^2 - \left(\frac{\|x\|^2 + \operatorname{Re}\langle x,y\rangle + \|y\|^2}{3} - \frac{m+M}{2}\right)^2\right]$$
$$\le \frac{1}{4}\frac{p\left(\sqrt{M}-\sqrt{m}\right)}{\sqrt{mM}}(M-m)$$

and

$$0 \le \int_0^1 \|(1-t)x+ty\| \, dt - \frac{\|x\| + [\operatorname{Re}\langle x,y\rangle]^{1/2} + \|y\|}{3} \tag{3.9}$$
$$\le \frac{1}{8m^{3/2}}$$
$$\times \left[\frac{1}{4}(M-m)^2 - \left(\frac{\|x\|^2 + \operatorname{Re}\langle x,y\rangle + \|y\|^2}{3} - \frac{m+M}{2}\right)^2\right]$$
$$\le \frac{1}{32m^{3/2}}(M-m)^2.$$

By writing the inequalities from Theorems 2-4 for the convex function $f(t) = t^{-q}$ on $(0,\infty)$ with $q \in (0,\infty)$ we get for $x, y \in H$ with $\operatorname{Re}\langle x,y\rangle > 0$ and $0 < m \le \|(1-t)x+ty\|^2 \le M$ for all $t \in [0,1]$, the following integral inequalities

$$0 \le \frac{\|x\|^{-2q} + [\operatorname{Re}\langle x,y\rangle]^{-q} + \|y\|^{-2q}}{3} - \int_0^1 \|(1-t)x+ty\|^{-2q} \, dt \tag{3.10}$$
$$\le 2\left[\frac{m^q + M^q}{2m^q M^q} - \left(\frac{2}{m+M}\right)^q\right],$$

$$0 \le \frac{\|x\|^{-2q} + [\operatorname{Re}\langle x,y\rangle]^{-q} + \|y\|^{-2q}}{3} - \int_0^1 \|(1-t)x+ty\|^{-2q} \, dt \tag{3.11}$$
$$\le q\frac{M^q - m^q}{M^q m^q (M-m)}$$
$$\times \left[\frac{1}{4}(M-m)^2 - \left(\frac{\|x\|^2 + \operatorname{Re}\langle x,y\rangle + \|y\|^2}{3} - \frac{m+M}{2}\right)^2\right]$$
$$\le \frac{1}{4}q(M-m)\frac{M^q - m^q}{M^q m^q}$$

and

$$0 \leq \frac{\|x\|^{-2q} + [\operatorname{Re}\langle x, y\rangle]^{-q} + \|y\|^{-2q}}{3} - \int_0^1 \|(1-t)x + ty\|^{-2q} dt \tag{3.12}$$
$$\leq \frac{1}{2m^{q+2}} q(q+1)$$
$$\times \left[\frac{1}{4}(M-m)^2 - \left(\frac{\|x\|^2 + \operatorname{Re}\langle x, y\rangle + \|y\|^2}{3} - \frac{m+M}{2}\right)^2\right]$$
$$\leq \frac{1}{8m^{q+2}} q(q+1)(M-m)^2.$$

For $q = 1$ we obtain

$$0 \leq \frac{\|x\|^{-2} + [\operatorname{Re}\langle x, y\rangle]^{-1} + \|y\|^{-2}}{3} - \int_0^1 \|(1-t)x + ty\|^{-2} dt \tag{3.13}$$
$$\leq \frac{M-m}{mM(m+M)},$$

$$0 \leq \frac{\|x\|^{-2} + [\operatorname{Re}\langle x, y\rangle]^{-1} + \|y\|^{-2}}{3} - \int_0^1 \|(1-t)x + ty\|^{-2} dt \tag{3.14}$$
$$\leq \frac{1}{Mm}\left[\frac{1}{4}(M-m)^2 - \left(\frac{\|x\|^2 + \operatorname{Re}\langle x, y\rangle + \|y\|^2}{3} - \frac{m+M}{2}\right)^2\right]$$
$$\leq \frac{1}{4}\frac{(M-m)^2}{Mm} 1_H.$$

and

$$0 \leq \frac{\|x\|^{-2} + [\operatorname{Re}\langle x, y\rangle]^{-1} + \|y\|^{-2}}{3} - \int_0^1 \|(1-t)x + ty\|^{-2} dt \tag{3.15}$$
$$\leq \frac{1}{m^3}\left[\frac{1}{4}(M-m)^2 1_H - \left(\frac{\|x\|^2 + \operatorname{Re}\langle x, y\rangle + \|y\|^2}{3} - \frac{m+M}{2}1_H\right)^2\right]$$
$$\leq \frac{1}{4m^3}(M-m)^2.$$

For $q = 1/2$ we derive

$$0 \leq \frac{\|x\|^{-1} + [\operatorname{Re}\langle x, y\rangle]^{-1/2} + \|y\|^{-1}}{3} - \int_0^1 \|(1-t)x + ty\|^{-1} dt \tag{3.16}$$
$$\leq 2\left[\frac{m^{1/2} + M^{1/2}}{2m^{1/2}M^{1/2}} - \left(\frac{2}{m+M}\right)^{1/2}\right],$$

$$0 \leq \frac{\|x\|^{-1} + [\operatorname{Re}\langle x, y\rangle]^{-1/2} + \|y\|^{-1}}{3} - \int_0^1 \|(1-t)x + ty\|^{-1} dt \tag{3.17}$$
$$\leq \frac{M^{1/2} - m^{1/2}}{2M^{1/2}m^{1/2}(M-m)}$$
$$\times \left[\frac{1}{4}(M-m)^2 - \left(\frac{\|x\|^2 + \operatorname{Re}\langle x, y\rangle + \|y\|^2}{3} - \frac{m+M}{2}\right)^2\right]$$
$$\leq \frac{1}{8}(M-m)\frac{M^{1/2} - m^{1/2}}{M^{1/2}m^{1/2}}$$

and

$$0 \leq \frac{\|x\|^{-1} + [\operatorname{Re}\langle x, y\rangle]^{-1/2} + \|y\|^{-1}}{3} - \int_0^1 \|(1-t)x + ty\|^{-1} dt \tag{3.18}$$
$$\leq \frac{3}{8m^{3/2}} q(q+1)$$
$$\times \left[\frac{1}{4}(M-m)^2 - \left(\frac{\|x\|^2 + \operatorname{Re}\langle x, y\rangle + \|y\|^2}{3} - \frac{m+M}{2}\right)^2\right]$$
$$\leq \frac{3}{32m^{3/2}}(M-m)^2.$$

By writing the inequalities from Theorems 2-4 for the operator convex function $f(t) = -\ln t$ on $(0, \infty)$ with $q \in (0, 1]$ we get for $x, y \in H$ with $\operatorname{Re}\langle x, y\rangle > 0$ and $0 < m \leq \|(1-t)x + ty\|^2 \leq M$ for all $t \in [0, 1]$, the following integral inequalities

$$0 \leq \int_0^1 \ln\left(\|(1-t)x + ty\|^2\right) dt - \frac{\ln\left(\|x\|^2\right) + \ln(\operatorname{Re}\langle x, y\rangle) + \ln\left(\|y\|^2\right)}{3} \tag{3.19}$$
$$\leq \ln\left(\frac{m+M}{2\sqrt{mM}}\right)^2,$$

$$0 \leq \int_0^1 \ln\left(\|(1-t)x + ty\|^2\right) dt - \frac{\ln\left(\|x\|^2\right) + \ln(\operatorname{Re}\langle x, y\rangle) + \ln\left(\|y\|^2\right)}{3} \tag{3.20}$$
$$\leq \frac{1}{mM}$$
$$\times \left[\frac{1}{4}(M-m)^2 - \left(\frac{\|x\|^2 + \operatorname{Re}\langle x, y\rangle + \|y\|^2}{3} - \frac{m+M}{2}\right)^2\right]$$
$$\leq \frac{1}{4mM}(M-m)^2$$

and

$$
\begin{aligned}
(3.21)\quad 0 &\leq \int_0^1 \ln\left(\|(1-t)\,x+ty\|^2\right)dt-\frac{\ln\left(\|x\|^2\right)+\ln\left(\operatorname{Re}\langle x,y\rangle\right)+\ln\left(\|y\|^2\right)}{3}\\
&\leq \frac{1}{2m^2}\\
&\times\left[\frac{1}{4}\,(M-m)^2-\left(\frac{\|x\|^2+\operatorname{Re}\langle x,y\rangle+\|y\|^2}{3}-\frac{m+M}{2}\right)^2\right]\\
&\leq \frac{1}{8m^2}\,(M-m)^2\,.
\end{aligned}
$$

4. The Case of Real Numbers

Assume that $f:[0,\infty)\to\mathbb{R}$ is convex and $0\leq a<b$. We observe that $a\leq (1-t)\,a+tb\leq b$ for all $t\in[0,1]$. For $x=a$, $y=b$, $m=a^2$, $M=b^2$ and modulus as a norm, we get from Theorem 2 that

$$
\begin{aligned}
(4.1)\qquad 0 &\leq \frac{f\left(a^2\right)+f\left(ab\right)+f\left(b^2\right)}{3}-\int_0^1 f\left(|(1-t)\,a+tb|^2\right)dt\\
&\leq 2\left[\frac{f\left(a^2\right)+f\left(b^2\right)}{2}-f\left(\frac{a^2+b^2}{2}\right)\right].
\end{aligned}
$$

Using the change of variable $(1-t)\,a+tb=s$, $t\in[0,1]$ we derive

$$
\int_0^1 f\left(((1-t)\,a+tb)^2\right)dt=\frac{1}{b-a}\int_a^b f\left(s^2\right)ds
$$

and by (4.1) we get

$$
\begin{aligned}
(4.2)\qquad 0 &\leq \frac{f\left(a^2\right)+f\left(ab\right)+f\left(b^2\right)}{3}-\frac{1}{b-a}\int_a^b f\left(s^2\right)ds\\
&\leq 2\left[\frac{f\left(a^2\right)+f\left(b^2\right)}{2}-f\left(\frac{a^2+b^2}{2}\right)\right].
\end{aligned}
$$

By Theorem 3 we get

$$
\begin{aligned}
(4.3)\qquad 0 &\leq \frac{f\left(a^2\right)+f\left(ab\right)+f\left(b^2\right)}{3}-\int_0^1 f\left(|(1-t)\,a+tb|^2\right)dt\\
&\leq \frac{\left[f'_-\left(b^2\right)-f'_+\left(a^2\right)\right]}{b^2-a^2}\\
&\times\left[\frac{1}{4}\left(b^2-a^2\right)^2-\left(\frac{a^2+ab+b^2}{3}-\frac{a^2+b^2}{2}\right)^2\right]\\
&\leq \frac{1}{4}\left(b^2-a^2\right)\left[f'_-\left(b^2\right)-f'_+\left(a^2\right)\right].
\end{aligned}
$$

Since

$$\begin{aligned}
&\frac{1}{4}\left(b^2-a^2\right)^2-\left(\frac{a^2+ab+b^2}{3}-\frac{a^2+b^2}{2}\right)^2 \\
&=\left[\frac{1}{2}\left(b^2-a^2\right)-\left(\frac{a^2+ab+b^2}{3}-\frac{a^2+b^2}{2}\right)\right] \\
&\times\left[\frac{1}{2}\left(b^2-a^2\right)+\left(\frac{a^2+ab+b^2}{3}-\frac{a^2+b^2}{2}\right)\right] \\
&=\frac{(b-a)^2(2b+a)(b+2a)}{9},
\end{aligned}$$

hence by (4.3) we get

$$\begin{aligned}
0 &\leq \frac{f\left(a^2\right)+f(ab)+f\left(b^2\right)}{3}-\frac{1}{b-a}\int_a^b f\left(s^2\right)ds \\
&\leq \frac{(b-a)(2b+a)(b+2a)}{9(b+a)}\left[f'_-\left(b^2\right)-f'_+\left(a^2\right)\right] \\
&\leq \frac{1}{4}\left(b^2-a^2\right)\left[f'_-\left(b^2\right)-f'_+\left(a^2\right)\right].
\end{aligned} \tag{4.4}$$

Assume that the function $f:[0,\infty)\to\mathbb{R}$ is twice differentiable convex with $f''(x)\leq D$ for all $x\in\left(a^2,b^2\right)$. Then by Theorem 4 we get

$$\begin{aligned}
0 &\leq \frac{f\left(a^2\right)+f(ab)+f\left(b^2\right)}{3}-\frac{1}{b-a}\int_a^b f\left(s^2\right)ds \\
&\leq \frac{(b-a)(2b+a)(b+2a)}{18(b+a)}D\leq\frac{1}{8}\left(b^2-a^2\right)^2 D.
\end{aligned} \tag{4.5}$$

Let $p\geq 1$, then $f(x)=x^p$ is convex on $[0,\infty)$ and for $a,\ b\in[0,\infty)$ with $a<b$ we get from (4.2) that

$$\begin{aligned}
0 &\leq \frac{a^{2p}+a^p b^p+b^{2p}}{3}-\frac{b^{2p+1}-a^{2p+1}}{(2p+1)(b-a)} \\
&\leq 2\left[\frac{a^{2p}+b^{2p}}{2}-\left(\frac{a^2+b^2}{2}\right)^p\right],
\end{aligned} \tag{4.6}$$

while from (4.4) we get

$$\begin{aligned}
0 &\leq \frac{a^{2p}+a^p b^p+b^{2p}}{3}-\frac{b^{2p+1}-a^{2p+1}}{(2p+1)(b-a)} \\
&\leq \frac{p(b-a)(2b+a)(b+2a)}{9(b+a)}\left[b^{2(p-1)}-a^{2(p-1)}\right] \\
&\leq \frac{1}{4}p\left(b^2-a^2\right)\left[b^{2(p-1)}-a^{2(p-1)}\right].
\end{aligned} \tag{4.7}$$

From, (4.5) we also get

$$0 \leq \frac{a^{2p}+a^pb^p+b^{2p}}{3} - \frac{b^{2p+1}-a^{2p+1}}{(2p+1)(b-a)} \tag{4.8}$$
$$\leq p(p-1)\frac{(b-a)(2b+a)(b+2a)}{18(b+a)}$$
$$\times \begin{cases} a^{2(p-2)}, & p\in[1,2] \\ b^{2(p-2)}, & p\in(2,\infty) \end{cases}$$
$$\leq \frac{1}{8}p(p-1)\left(b^2-a^2\right)^2 \times \begin{cases} a^{2(p-2)}, & p\in[1,2] \\ b^{2(p-2)}, & p\in(2,\infty) \end{cases}$$

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