

# Multivariate Ostrowski type inequalities for several Banach algebra valued functions

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## Abstract

Here we are dealing with several smooth functions from a compact convex set of  $\mathbb{R}^k$ ,  $k \geq 2$  to a Banach algebra. For these we prove general multivariate Ostrowski type inequalities with estimates in norms  $\|\cdot\|_p$ , for all  $1 \leq p \leq \infty$ . We provide also interesting applications.

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## 1 Introduction

In 1938, A Ostrowski [5] proved the following famous inequality:

**Theorem 1** (1938, Ostrowski [6]) *Let  $f : [a, b] \rightarrow \mathbb{R}$  be continuous on  $[a, b]$  and differentiable on  $(a, b)$  whose derivative  $f' : (a, b) \rightarrow \mathbb{R}$  is bounded on  $(a, b)$ , i.e.,  $\|f'\|_\infty^{\text{sup}} := \sup_{t \in (a, b)} |f'(t)| < +\infty$ . Then*

$$\left| \frac{1}{b-a} \int_a^b f(t) dt - f(x) \right| \leq \left[ \frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right] (b-a) \|f'\|_\infty^{\text{sup}}, \quad (1)$$

for any  $x \in [a, b]$ . The constant  $\frac{1}{4}$  is the best possible.

Since then there has been a lot of activity around these inequalities with important applications to Numerical Analysis and Probability.

This article is also greatly motivated by the following result:

**Theorem 2** (see [1]) Let  $f \in C^1 \left( \prod_{i=1}^k [a_i, b_i] \right)$ , where  $a_i < b_i$ ;  $a_i, b_i \in \mathbb{R}$ ,  $i = 1, \dots, k$ , and let  $\vec{x}_0 := (x_{01}, \dots, x_{0k}) \in \prod_{i=1}^k [a_i, b_i]$  be fixed. Then

$$\left| \frac{1}{\prod_{i=1}^k (b_i - a_i)} \int_{a_1}^{b_1} \dots \int_{a_i}^{b_i} \dots \int_{a_k}^{b_k} f(z_1, \dots, z_k) dz_1 \dots dz_k - f(\vec{x}_0) \right| \leq \quad (2)$$

$$\sum_{i=1}^k \left( \frac{(x_{0i} - a_i)^2 + (b_i - x_{0i})^2}{2(b_i - a_i)} \right) \left\| \frac{\partial f}{\partial z_i} \right\|_{\infty}.$$

Inequality (2) is sharp, here the optimal function is

$$f^*(z_1, \dots, z_k) := \sum_{i=1}^k |z_i - x_{0i}|^{\alpha_i}, \quad \alpha_i > 1.$$

Clearly inequality (2) generalizes inequality (1) to multidimension.

We are inspired also by [2].

In this article we establish multivariate Ostrowski type inequalities for several smooth functions from a compact convex subset of  $\mathbb{R}^k$ ,  $k \geq 2$ , to a Banach algebra. These involve the norms  $\|\cdot\|_p$ ,  $1 \leq p \leq \infty$ .

## 2 About Banach Algebras

All here come from [6].

We need

**Definition 3** ([6], p. 245) A complex algebra is a vector space  $A$  over the complex field  $\mathbb{C}$  in which a multiplication is defined that satisfies

$$x(yz) = (xy)z, \quad (3)$$

$$(x+y)z = xz + yz, \quad x(y+z) = xy + xz, \quad (4)$$

and

$$\alpha(xy) = (\alpha x)y = x(\alpha y), \quad (5)$$

for all  $x, y$  and  $z$  in  $A$  and for all scalars  $\alpha$ .

Additionally if  $A$  is a Banach space with respect to a norm that satisfies the multiplicative inequality

$$\|xy\| \leq \|x\| \|y\| \quad (x \in A, y \in A) \quad (6)$$

and if  $A$  contains a unit element  $e$  such that

$$xe = ex = x \quad (x \in A) \tag{7}$$

and

$$\|e\| = 1, \tag{8}$$

then  $A$  is called a Banach algebra.

$A$  is commutative iff  $xy = yx$  for all  $x, y \in A$ .

We make

**Remark 4** *Commutativity of  $A$  will be explicitly stated when needed.*

*There exists at most one  $e \in A$  that satisfies (7).*

*Inequality (6) makes multiplication to be continuous, more precisely left and right continuous, see [6], p. 246.*

*Multiplication in  $A$  is not necessarily the numerical multiplication, it is something more general and it is defined abstractly, that is for  $x, y \in A$  we have  $xy \in A$ , e.g. composition or convolution, etc.*

*For nice examples about Banach algebras see [6], p. 247-248, § 10.3.*

We also make

**Remark 5** *Next we mention about integration of  $A$ -valued functions, see [6], p. 259, § 10.22:*

*If  $A$  is a Banach algebra and  $f$  is a continuous  $A$ -valued function on some compact Hausdorff space  $Q$  on which a complex Borel measure  $\mu$  is defined, then  $\int f d\mu$  exists and has all the properties that were discussed in Chapter 3 of [6], simply because  $A$  is a Banach space. However, an additional property can be added to these, namely: If  $x \in A$ , then*

$$x \int_Q f \, d\mu = \int_Q xf(p) \, d\mu(p) \tag{9}$$

and

$$\left( \int_Q f \, d\mu \right) x = \int_Q f(p)x \, d\mu(p). \tag{10}$$

*The vector integrals we will involve in our article follow (9) and (10).*

### 3 Vector Analysis Background

(see [8], pp. 83-94)

Let  $f(t)$  be a function defined on  $[a, b] \subseteq \mathbb{R}$  taking values in a real or complex normed linear space  $(X, \|\cdot\|)$ , Then  $f(t)$  is said to be differentiable at a point  $t_0 \in [a, b]$  if the limit

$$f'(t_0) = \lim_{h \rightarrow 0} \frac{f(t_0 + h) - f(t_0)}{h} \quad (11)$$

exists in  $X$ , the convergence is in  $\|\cdot\|$ . This is called the derivative of  $f(t)$  at  $t = t_0$ .

We call  $f(t)$  differentiable on  $[a, b]$ , iff there exists  $f'(t) \in X$  for all  $t \in [a, b]$ .

Similarly and inductively are defined higher order derivatives of  $f$ , denoted  $f'', f^{(3)}, \dots, f^{(k)}$ ,  $k \in \mathbb{N}$ , just as for numerical functions.

For all the properties of derivatives see [8], pp. 83-86.

Let now  $(X, \|\cdot\|)$  be a Banach space, and  $f : [a, b] \rightarrow X$ .

We define the vector valued Riemann integral  $\int_a^b f(t) dt \in X$  as the limit of the vector valued Riemann sums in  $X$ , convergence is in  $\|\cdot\|$ . The definition is as for the numerical valued functions.

If  $\int_a^b f(t) dt \in X$  we call  $f$  integrable on  $[a, b]$ . If  $f \in C([a, b], X)$ , then  $f$  is integrable, [8], p. 87.

For all the properties of vector valued Riemann integrals see [8], pp. 86-91.

We define the space  $C^n([a, b], X)$ ,  $n \in \mathbb{N}$ , of  $n$ -times continuously differentiable functions from  $[a, b]$  into  $X$ ; here continuity is with respect to  $\|\cdot\|$  and defined in the usual way as for numerical functions.

Let  $(X, \|\cdot\|)$  be a Banach space and  $f \in C^n([a, b], X)$ , then we have the vector valued Taylor's formula, see [8], pp. 93-94, and also [7], (IV, 9; 47).

It holds

$$\begin{aligned} f(y) - f(x) - f'(x)(y-x) - \frac{1}{2}f''(x)(y-x)^2 - \dots - \frac{1}{(n-1)!}f^{(n-1)}(x)(y-x)^{n-1} \\ = \frac{1}{(n-1)!} \int_x^y (y-t)^{n-1} f^{(n)}(t) dt, \quad \forall x, y \in [a, b]. \end{aligned} \quad (12)$$

In particular (12) is true when  $X = \mathbb{R}^m, \mathbb{C}^m$ ,  $m \in \mathbb{N}$ , etc.

A function  $f(t)$  with values in a normed linear space  $X$  is said to be piecewise continuous (see [8], p. 85) on the interval  $a \leq t \leq b$  if there exists a partition  $a = t_0 < t_1 < t_2 < \dots < t_n = b$  such that  $f(t)$  is continuous on every open interval  $t_k < t < t_{k+1}$  and has finite limits  $f(t_0 + 0)$ ,  $f(t_1 - 0)$ ,  $f(t_1 + 0)$ ,  $f(t_2 - 0)$ ,  $f(t_2 + 0)$ , ...,  $f(t_n - 0)$ .

Here  $f(t_k - 0) = \lim_{t \uparrow t_k} f(t)$ ,  $f(t_k + 0) = \lim_{t \downarrow t_k} f(t)$ .

The values of  $f(t)$  at the points  $t_k$  can be arbitrary or even undefined.

A function  $f(t)$  with values in normed linear space  $X$  is said to be piecewise smooth on  $[a, b]$ , if it is continuous on  $[a, b]$  and has a derivative  $f'(t)$  at all but a finite number of points of  $[a, b]$ , and if  $f'(t)$  is piecewise continuous on  $[a, b]$  (see [8], p. 85).

Let  $u(t)$  and  $v(t)$  be two piecewise smooth functions on  $[a, b]$ , one a numerical function and the other a vector function with values in Banach space  $X$ . Then we have the following integration by parts formula

$$\int_a^b u(t) dv(t) = u(t)v(t) \Big|_a^b - \int_a^b v(t) du(t), \quad (13)$$

see [8], p. 93.

We mention also the mean value theorem for Banach space valued functions.

**Theorem 6** (see [4], p. 3) *Let  $f \in C([a, b], X)$ , where  $X$  is a Banach space. Assume  $f'$  exists on  $[a, b]$  and  $\|f'(t)\| \leq K$ ,  $a < t < b$ , then*

$$\|f(b) - f(a)\| \leq K(b - a). \quad (14)$$

Here the multiple Riemann integral of a function from a real box or a real compact and convex subset to a Banach space is defined similarly to numerical one however convergence is with respect to  $\|\cdot\|$ . Similarly are defined the vector valued partial derivatives as in the numerical case.

We mention the equality of vector valued mixed partial derivatives.

**Proposition 7** (see Proposition 4.11 of [3], p. 90) *Let  $Q = (a, b) \times (c, d) \subseteq \mathbb{R}^2$  and  $f \in C(Q, X)$ , where  $(X, \|\cdot\|)$  is a Banach space. Assume that  $\frac{\partial}{\partial t}f(s, t)$ ,  $\frac{\partial}{\partial s}f(s, t)$  and  $\frac{\partial^2}{\partial t \partial s}f(s, t)$  exist and are continuous for  $(s, t) \in Q$ , then  $\frac{\partial^2}{\partial s \partial t}f(s, t)$  exists for  $(s, t) \in Q$  and*

$$\frac{\partial^2}{\partial s \partial t}f(s, t) = \frac{\partial^2}{\partial t \partial s}f(s, t), \text{ for } (s, t) \in Q. \quad (15)$$

## 4 Main Results

We present general Ostrowski type inequalities results regarding several Banach algebra valued functions.

**Theorem 8** *Let  $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$ ;  $(A, \|\cdot\|)$  a Banach algebra and  $f_i \in C^{n+1}(Q, A)$ ,  $i = 1, \dots, r$ ;  $r \in \mathbb{N}$ ,  $n \in \mathbb{Z}_+$ , and fixed  $\vec{x}_0 \in Q \subset \mathbb{R}^k$ ,  $k \geq 2$ , where  $Q$  is a compact and convex subset. Here all vector partial derivatives  $f_{i\alpha} := \frac{\partial^\alpha f_i}{\partial z^\alpha}$ , where  $\alpha = (\alpha_1, \dots, \alpha_k)$ ,  $\alpha_\lambda \in \mathbb{Z}^+$ ,  $\lambda = 1, \dots, k$ ,  $|\alpha| = \sum_{\lambda=1}^k \alpha_\lambda = j$ ,  $j = 1, \dots, n$ , fulfill  $f_{i\alpha}(\vec{x}_0) = 0$ ,  $i = 1, \dots, r$ .*

Denote

$$D_{n+1}(f_i) := \max_{\alpha: |\alpha|=n+1} \| \|f_{i\alpha}\| \|_{\infty, Q}, \quad (16)$$

$i = 1, \dots, r$ , and

$$\|\vec{z} - \vec{x}_0\|_{l_1} := \sum_{\lambda=1}^k |z_\lambda - x_{0\lambda}|. \quad (17)$$

Then

$$\begin{aligned}
& \left\| \sum_{i=1}^r \int_Q \left( \prod_{\substack{\rho=1 \\ \rho \neq i}}^r f_\rho(\vec{z}) \right) f_i(\vec{z}) d\vec{z} - \sum_{i=1}^r \left( \int_Q \left( \prod_{\substack{\rho=1 \\ \rho \neq i}}^r f_\rho(\vec{z}) \right) d\vec{z} \right) f_i(\vec{x}_0) \right\| \leq \\
& \frac{\max_{i \in \{1, \dots, r\}} D_{n+1}(f_i)}{(n+1)!} \sum_{i=1}^r \left( \int_Q \left( \prod_{\substack{\rho=1 \\ \rho \neq i}}^r \|f_\rho(\vec{z})\| \right) \|\vec{z} - \vec{x}_0\|_{l_1}^{n+1} d\vec{z} \right) \leq \\
& \frac{\max_{i \in \{1, \dots, r\}} D_{n+1}(f_i)}{(n+1)!} \min \left\{ \left( \int_Q \|\vec{z} - \vec{x}_0\|_{l_1}^{n+1} d\vec{z} \right) \left[ \sum_{i=1}^r \left( \prod_{\substack{\rho=1 \\ \rho \neq i}}^r \| \|f_\rho\| \| \right) \right], \right. \\
& \left. \left\| \|\cdot - \vec{x}_0\|_{l_1}^{n+1} \right\|_{\infty, Q} \left[ \sum_{i=1}^r \left\| \left( \prod_{\substack{\rho=1 \\ \rho \neq i}}^r \|f_\rho\| \right) \right\|_{L_1(Q, A)} \right], \right. \\
& \left. \left\| \|\cdot - \vec{x}_0\|_{l_1}^{n+1} \right\|_{L_p(Q, A)} \left[ \sum_{i=1}^r \left[ \left\| \left( \prod_{\substack{\rho=1 \\ \rho \neq i}}^r \|f_\rho\| \right) \right\|_{L_q(Q, A)} \right] \right] \right\}. \quad (19)
\end{aligned}$$

**Proof.** Take  $g_{i\vec{z}}(t) := f_i(\vec{x}_0 + t(\vec{z} - \vec{x}_0))$ ,  $0 \leq t \leq 1$ ;  $i = 1, \dots, r$ . Notice that  $g_{i\vec{z}}(0) = f_i(\vec{x}_0)$  and  $g_{i\vec{z}}(1) = f_i(\vec{z})$ . The  $j$ th derivative of  $g_{i\vec{z}}(t)$ , based on Proposition 7, is given by

$$g_{i\vec{z}}^{(j)}(t) = \left[ \left( \sum_{\lambda=1}^k (z_\lambda - x_{0\lambda}) \frac{\partial}{\partial z_\lambda} \right)^j f_i \right] (x_{01} + t(z_1 - x_{01}), \dots, x_{0k} + t(z_k - x_{0k})) \quad (20)$$

and

$$g_{i\vec{z}}^{(j)}(0) = \left[ \left( \sum_{\lambda=1}^k (z_\lambda - x_{0\lambda}) \frac{\partial}{\partial z_\lambda} \right)^j f_i \right] (\vec{x}_0), \quad (21)$$

for  $j = 1, \dots, n+1$ ;  $i = 1, \dots, r$ .

Let  $f_{i\alpha}$  be a partial derivative of  $f_i \in C^{n+1}(Q, A)$ . Because by assumption of the theorem we have  $f_{i\alpha}(\vec{x}_0) = 0$  for all  $\alpha : |\alpha| = j$ ,  $j = 1, \dots, n$ , we find that

$$g_{i\vec{z}}^{(j)}(0) = 0, \quad j = 1, \dots, n; \quad i = 1, \dots, r.$$

Hence by vector Taylor's theorem (12) we see that

$$f_i(\vec{z}) - f_i(\vec{x}_0) = \sum_{j=1}^n \frac{g_{i\vec{z}}^{(j)}(0)}{j!} + R_{in}(\vec{z}, 0) = R_{in}(\vec{z}, 0), \quad (22)$$

where

$$R_{in}(\vec{z}, 0) := \int_0^1 \left( \int_0^{t_1} \dots \left( \int_0^{t_{n-1}} \left( g_{i\vec{z}}^{(n)}(t_n) - g_{i\vec{z}}^{(n)}(0) \right) dt_n \right) \dots \right) dt_1, \quad (23)$$

$i = 1, \dots, r$ .

Therefore,

$$\|R_{in}(\vec{z}, 0)\| \leq \int_0^1 \left( \int_0^{t_1} \dots \left( \int_0^{t_{n-1}} \left\| \left\| g_{i\vec{z}}^{(n+1)}(\xi(t_n)) \right\| \right\|_{\infty} t_n dt_n \right) \dots \right) dt_1, \quad (24)$$

by the vector mean value Theorem 6 applied on  $g_{i\vec{z}}^{(n)}$  over  $(0, t_n)$ . Moreover, we get

$$\begin{aligned} \|R_{in}(\vec{z}, 0)\| &\leq \left\| \left\| g_{i\vec{z}}^{(n+1)} \right\| \right\|_{\infty, [0,1]} \int_0^1 \int_0^{t_1} \dots \left( \int_0^{t_{n-1}} t_n dt_n \right) \dots dt_1 \\ &= \frac{\left\| \left\| g_{i\vec{z}}^{(n+1)} \right\| \right\|_{\infty, [0,1]}}{(n+1)!}. \end{aligned} \quad (25)$$

However, there exists a  $t_{i0} \in [0, 1]$  such that  $\left\| \left\| g_{i\vec{z}}^{(n+1)} \right\| \right\|_{\infty, [0,1]} = \left\| g_{i\vec{z}}^{(n+1)}(t_{i0}) \right\|$ .

That is

$$\begin{aligned} \left\| \left\| g_{i\vec{z}}^{(n+1)} \right\| \right\|_{\infty, [0,1]} &= \left\| \left[ \left( \sum_{\lambda=1}^k (z_{\lambda} - x_{0\lambda}) \frac{\partial}{\partial z_{\lambda}} \right)^{n+1} f_i \right] (\vec{x}_0 + t_{i0}(\vec{z} - \vec{z}_{0i})) \right\| \\ &\leq \left[ \left( \sum_{\lambda=1}^k |z_{\lambda} - x_{0\lambda}| \left\| \frac{\partial}{\partial z_{\lambda}} \right\| \right)^{n+1} \left\| f_i \right\| (\vec{x}_0 + t_{i0}(\vec{z} - \vec{z}_{0i})) \right]. \end{aligned}$$

I.e.,

$$\left\| \left\| g_{i\vec{z}}^{(n+1)} \right\| \right\|_{\infty, [0,1]} \leq \left[ \left( \sum_{\lambda=1}^k |z_{\lambda} - x_{0\lambda}| \left\| \frac{\partial}{\partial z_{\lambda}} \right\| \right)^{n+1} \left\| f_i \right\| \right], \quad (26)$$

$i = 1, \dots, r$ .

Hence by (26) we get

$$\begin{aligned} \|R_{in}(\vec{z}, 0)\| &\leq \frac{\left[ \left( \sum_{\lambda=1}^k |z_{\lambda} - x_{0\lambda}| \left\| \frac{\partial}{\partial z_{\lambda}} \right\| \right)^{n+1} \left\| f_i \right\| \right]}{(n+1)!} \leq \\ &\frac{D_{n+1}(f_i)}{(n+1)!} \left( \sum_{\lambda=1}^k |z_{\lambda} - x_{0\lambda}| \right)^{n+1} = \frac{D_{n+1}(f_i)}{(n+1)!} \|\vec{z} - \vec{x}_0\|_{l_1}^{n+1}, \end{aligned} \quad (27)$$

$i = 1, \dots, r$ .

Therefore it holds

$$\|R_{in}(\vec{z}, 0)\| \leq \frac{\max_{i \in \{1, \dots, r\}} D_{n+1}(f_i)}{(n+1)!} \|\vec{z} - \vec{x}_0\|_{l_1}^{n+1}, \quad (28)$$

for  $i = 1, \dots, r$ .

By (22) we get that

$$\left( \prod_{\substack{\rho=1 \\ \rho \neq i}}^r f_\rho(\vec{z}) \right) f_i(\vec{z}) - \left( \prod_{\substack{\rho=1 \\ \rho \neq i}}^r f_\rho(\vec{z}) \right) f_i(\vec{x}_0) = \left( \prod_{\substack{\rho=1 \\ \rho \neq i}}^r f_\rho(\vec{z}) \right) R_{in}(\vec{z}, 0), \quad (29)$$

for all  $i = 1, \dots, r$ .

Hence

$$\begin{aligned} & \sum_{i=1}^r \left( \prod_{\substack{\rho=1 \\ \rho \neq i}}^r f_\rho(\vec{z}) \right) f_i(\vec{z}) - \sum_{i=1}^r \left( \prod_{\substack{\rho=1 \\ \rho \neq i}}^r f_\rho(\vec{z}) \right) f_i(\vec{x}_0) \\ &= \sum_{i=1}^r \left( \prod_{\substack{\rho=1 \\ \rho \neq i}}^r f_\rho(\vec{z}) \right) R_{in}(\vec{z}, 0). \end{aligned} \quad (30)$$

Therefore we find

$$\begin{aligned} & E(f_1, \dots, f_r)(x_0) := \\ & \sum_{i=1}^r \int_Q \left( \prod_{\substack{\rho=1 \\ \rho \neq i}}^r f_\rho(\vec{z}) \right) f_i(\vec{z}) d\vec{z} - \sum_{i=1}^r \left( \int_Q \left( \prod_{\substack{\rho=1 \\ \rho \neq i}}^r f_\rho(\vec{z}) \right) d\vec{z} \right) f_i(\vec{x}_0) = \\ & \sum_{i=1}^r \int_Q \left( \prod_{\substack{\rho=1 \\ \rho \neq i}}^r f_\rho(\vec{z}) \right) R_{in}(\vec{z}, 0) d\vec{z}. \end{aligned} \quad (31)$$

Consequently, we have that

$$\begin{aligned} & \|E(f_1, \dots, f_r)(x_0)\| = \\ & \left\| \sum_{i=1}^r \int_Q \left( \prod_{\substack{\rho=1 \\ \rho \neq i}}^r f_\rho(\vec{z}) \right) f_i(\vec{z}) d\vec{z} - \sum_{i=1}^r \left( \int_Q \left( \prod_{\substack{\rho=1 \\ \rho \neq i}}^r f_\rho(\vec{z}) \right) d\vec{z} \right) f_i(\vec{x}_0) \right\| = \end{aligned}$$

$$\left\| \sum_{i=1}^r \int_Q \left( \prod_{\substack{\rho=1 \\ \rho \neq i}}^r f_\rho(\vec{z}) \right) R_{in}(\vec{z}, 0) d\vec{z} \right\| \leq \quad (32)$$

$$\begin{aligned} & \sum_{i=1}^r \left\| \int_Q \left( \prod_{\substack{\rho=1 \\ \rho \neq i}}^r f_\rho(\vec{z}) \right) R_{in}(\vec{z}, 0) d\vec{z} \right\| \leq \\ & \sum_{i=1}^r \left( \int_Q \left\| \left( \prod_{\substack{\rho=1 \\ \rho \neq i}}^r f_\rho(\vec{z}) \right) R_{in}(\vec{z}, 0) \right\| d\vec{z} \right) \stackrel{(6)}{\leq} \\ & \sum_{i=1}^r \left( \int_Q \left( \prod_{\substack{\rho=1 \\ \rho \neq i}}^r \|f_\rho(\vec{z})\| \right) \|R_{in}(\vec{z}, 0)\| d\vec{z} \right) \stackrel{(28)}{\leq} \end{aligned} \quad (33)$$

$$\frac{\max_{i \in \{1, \dots, r\}} D_{n+1}(f_i)}{(n+1)!} \sum_{i=1}^r \left( \int_Q \left( \prod_{\substack{\rho=1 \\ \rho \neq i}}^r \|f_\rho(\vec{z})\| \right) \|\vec{z} - \vec{x}_0\|_{l_1}^{n+1} d\vec{z} \right).$$

So far we have proved

$$\begin{aligned} & \|E(f_1, \dots, f_r)(x_0)\| \leq \\ & \frac{\max_{i \in \{1, \dots, r\}} D_{n+1}(f_i)}{(n+1)!} \sum_{i=1}^r \left( \int_Q \left( \prod_{\substack{\rho=1 \\ \rho \neq i}}^r \|f_\rho(\vec{z})\| \right) \|\vec{z} - \vec{x}_0\|_{l_1}^{n+1} d\vec{z} \right) =: (\xi). \end{aligned} \quad (34)$$

Furthermore it holds

$$(\xi) \leq \frac{\max_{i \in \{1, \dots, r\}} D_{n+1}(f_i)}{(n+1)!} \left( \int_Q \|\vec{z} - \vec{x}_0\|_{l_1}^{n+1} d\vec{z} \right) \left[ \sum_{i=1}^r \left( \prod_{\substack{\rho=1 \\ \rho \neq i}}^r \| \|f_\rho\| \|_{\infty, Q} \right) \right], \quad (35)$$

and

$$(\xi) \leq \frac{\max_{i \in \{1, \dots, r\}} D_{n+1}(f_i)}{(n+1)!} \left\| \|\cdot - \vec{x}_0\|_{l_1}^{n+1} \right\|_{\infty, Q} \left[ \sum_{i=1}^r \left\| \left( \prod_{\substack{\rho=1 \\ \rho \neq i}}^r \|f_\rho\| \right) \right\|_{L_1(Q, A)} \right], \quad (36)$$

and finally

$$(\xi) \leq \frac{\max_{i \in \{1, \dots, r\}} D_{n+1}(f_i)}{(n+1)!} \left[ \sum_{i=1}^r \left[ \left\| \left( \prod_{\substack{\rho=1 \\ \rho \neq i}}^r \|f_\rho\| \right) \right\|_{L_q(Q,A)} \right] \right] \left\| \|\cdot - \vec{x}_0\|_{l_1}^{n+1} \right\|_{L_p(Q,A)}, \quad (37)$$

proving (18), (19). ■

We give

**Corollary 9** (to Theorem 8) All as in Theorem 8, with  $f_1 = \dots = f_r = f$ ,  $r \in \mathbb{N}$ . Then

$$\begin{aligned} & \left\| \int_Q f^r(\vec{z}) d\vec{z} - \left( \int_Q f^{r-1}(\vec{z}) d\vec{z} \right) f(\vec{x}_0) \right\| \leq \\ & \frac{D_{n+1}(f)}{(n+1)!} \left( \int_Q \|f(\vec{z})\|^{r-1} \|\vec{z} - \vec{x}_0\|_{l_1}^{n+1} d\vec{z} \right) \leq \quad (38) \\ & \frac{D_{n+1}(f)}{(n+1)!} \min \left\{ \left( \int_Q \|\vec{z} - \vec{x}_0\|_{l_1}^{n+1} d\vec{z} \right) \left( \|f\|_{\infty, Q} \right)^{r-1}, \right. \\ & \left. \left\| \|\cdot - \vec{x}_0\|_{l_1}^{n+1} \right\|_{\infty, Q} \left\| \|f\|^{r-1} \right\|_{L_1(Q,A)}, \left\| \|\cdot - \vec{x}_0\|_{l_1}^{n+1} \right\|_{L_p(Q,A)} \left\| \|f\|^{r-1} \right\|_{L_q(Q,A)} \right\}. \quad (39) \end{aligned}$$

We also give

**Corollary 10** (to Theorem 8) All as in Theorem 8, with  $(A, \|\cdot\|)$  being a commutative Banach algebra. Then

$$\left\| r \int_Q \left( \prod_{\rho=1}^r f_\rho(\vec{z}) \right) d\vec{z} - \sum_{i=1}^r \left( \int_Q \left( \prod_{\substack{\rho=1 \\ \rho \neq i}}^r f_\rho(\vec{z}) \right) d\vec{z} \right) f_i(\vec{x}_0) \right\| \leq$$

Right hand side of (18)  $\leq$  Right hand side of (19). (40)

We make

**Remark 11** Of great interest are applications of Theorem 8 when  $Q = \prod_{\lambda=1}^k [a_\lambda, b_\lambda]$ ,

where  $[a_\lambda, b_\lambda] \subset \mathbb{R}$ ,  $\lambda = 1, \dots, k$ .

We observe that by the multinomial theorem we get:

$$\int_{\prod_{\lambda=1}^k [a_\lambda, b_\lambda]} \left( \sum_{\lambda=1}^k |z_\lambda - x_{0\lambda}| \right)^{n+1} dz_1 \dots dz_k = \sum_{\rho_1 + \rho_2 + \dots + \rho_k = n+1} \frac{(n+1)!}{\rho_1! \rho_2! \dots \rho_k!}$$

$$\int_{\prod_{\lambda=1}^k [a_\lambda, b_\lambda]} |z_1 - x_{01}|^{\rho_1} |z_2 - x_{02}|^{\rho_2} \dots |z_k - x_{0k}|^{\rho_k} dz_1 \dots dz_k = \quad (41)$$

$$\begin{aligned} & \sum_{\rho_1 + \rho_2 + \dots + \rho_k = n+1} \frac{(n+1)!}{\rho_1! \rho_2! \dots \rho_k!} \prod_{\lambda=1}^k \left( \int_{a_\lambda}^{b_\lambda} |z_\lambda - x_{0\lambda}|^{\rho_\lambda} dz_\lambda \right) = \\ & \sum_{\sum_{\lambda=1}^k \rho_\lambda = n+1} \frac{(n+1)!}{\prod_{\lambda=1}^k \rho_\lambda!} \prod_{\lambda=1}^k \left( \int_{a_\lambda}^{x_{0\lambda}} (x_{0\lambda} - z_\lambda)^{\rho_\lambda} dz_\lambda + \int_{x_{0\lambda}}^{b_\lambda} (z_\lambda - x_{0\lambda})^{\rho_\lambda} dz_\lambda \right) = \\ & \sum_{\sum_{\lambda=1}^k \rho_\lambda = n+1} \frac{(n+1)!}{\prod_{\lambda=1}^k \rho_\lambda!} \prod_{\lambda=1}^k \left( \frac{(x_{0\lambda} - a_\lambda)^{\rho_\lambda + 1} + (b_\lambda - x_{0\lambda})^{\rho_\lambda + 1}}{\rho_\lambda + 1} \right). \end{aligned} \quad (42)$$

We have found that

$$\begin{aligned} & \int_{\prod_{\lambda=1}^k [a_\lambda, b_\lambda]} \|\vec{z} - \vec{x}_0\|_{l_1}^{n+1} d\vec{z} = \quad (43) \\ & \sum_{\sum_{\lambda=1}^k \rho_\lambda = n+1} \frac{(n+1)!}{\prod_{\lambda=1}^k \rho_\lambda!} \prod_{\lambda=1}^k \left( \frac{(b_\lambda - x_{0\lambda})^{\rho_\lambda + 1} + (x_{0\lambda} - a_\lambda)^{\rho_\lambda + 1}}{\rho_\lambda + 1} \right). \end{aligned}$$

Based on (18), (19) and (43) we conclude:

**Theorem 12** Let  $(A, \|\cdot\|)$  a Banach algebra and  $f_i \in C^{n+1} \left( \prod_{\lambda=1}^k [a_\lambda, b_\lambda], A \right)$ ,  $i = 1, \dots, r$ ;  $r \in \mathbb{N}$ ,  $n \in \mathbb{Z}_+$ , and fixed  $\vec{x}_0 \in \prod_{\lambda=1}^k [a_\lambda, b_\lambda] \subset \mathbb{R}^k$ ,  $k \geq 2$ . Here all vector partial derivatives  $f_{i\alpha} := \frac{\partial^\alpha f_i}{\partial z^\alpha}$ , where  $\alpha = (\alpha_1, \dots, \alpha_k)$ ,  $\alpha_\lambda \in \mathbb{Z}^+$ ,  $\lambda = 1, \dots, k$ ,  $|\alpha| = \sum_{\lambda=1}^k \alpha_\lambda = j$ ,  $j = 1, \dots, n$ , fulfill  $f_{i\alpha}(\vec{x}_0) = 0$ ,  $i = 1, \dots, r$ .

Denote

$$D_{n+1}(f_i) := \max_{\alpha: |\alpha|=n+1} \|\|f_{i\alpha}\|\|_{\infty, \prod_{\lambda=1}^k [a_\lambda, b_\lambda]}, \quad (44)$$

$i = 1, \dots, r$ .

Then

$$\begin{aligned} & \left\| \sum_{i=1}^r \int_{\prod_{\lambda=1}^k [a_\lambda, b_\lambda]} \left( \prod_{\substack{\rho=1 \\ \rho \neq i}}^r f_\rho(\vec{z}) \right) f_i(\vec{z}) d\vec{z} - \right. \\ & \left. \sum_{i=1}^r \left( \int_{\prod_{\lambda=1}^k [a_\lambda, b_\lambda]} \left( \prod_{\substack{\rho=1 \\ \rho \neq i}}^r f_\rho(\vec{z}) \right) d\vec{z} \right) f_i(\vec{x}_0) \right\| \leq \end{aligned} \quad (45)$$

$$\left( \max_{i \in \{1, \dots, r\}} D_{n+1}(f_i) \right) \left[ \sum_{i=1}^r \left( \prod_{\substack{\rho=1 \\ \rho \neq i}}^r \|f_\rho\| \right)_{\infty, \prod_{\lambda=1}^k [a_\lambda, b_\lambda]} \right] \\ \left[ \sum_{\lambda=1}^k \frac{1}{\prod_{\lambda=1}^k \rho_\lambda! \prod_{\lambda=1}^k (\rho_\lambda + 1)} \prod_{\lambda=1}^k \left( (b_\lambda - x_{0\lambda})^{\rho_\lambda+1} + (x_{0\lambda} - a_\lambda)^{\rho_\lambda+1} \right) \right].$$

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