DETERMINANT INEQUALITIES FOR POSITIVE DEFINITE MATRICES VIA JENSEN'S INEQUALITY FOR POWER FUNCTION

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Abstract. In this paper we prove among others that, if A_j, $j = 1, ..., m$ are positive definite matrices, $\lambda_j > 0$, $j = 1, ..., m$ with $\sum_{j=1}^{m} \lambda_j = 1$ and $p \geq 1$, then

$$\sum_{j=1}^{m} \lambda_j \left[\det \left(pA_j + I_n\right)\right]^{-1} \geq \left(\sum_{j=1}^{m} \lambda_j \left[\det \left(A_j + I_n\right)\right]^{-1}\right)^p .$$

If $p \in (0, 1)$, then the inequality above reverses.

1. Introduction

A real square matrix $A = (a_{ij})$, $i, j = 1, ..., n$ is *symmetric* provided $a_{ij} = a_{ji}$ for all $i, j = 1, ..., n$. A real symmetric matrix is said to be *positive definite* provided the quadratic form $Q(x) = \sum_{i,j=1}^{n} a_{ij} x_i x_j$ is positive for all $x = (x_1, ..., x_n) \in \mathbb{R}^n \setminus \{0\}$. It is well known that a necessary and sufficient condition for the symmetric matrix A to be positive definite, and we write $A > 0$, is that all determinants

$$\det(A_k) = \det(a_{ij}),\ i,\ j = 1, ..., k;\ k = 1, ..., n$$

are positive.

It is know that the following integral representation is valid, see [1, pp. 61-62] or [8, pp. 211-212]

$$J_n(A) := \int_{\mathbb{R}^n} \exp(-\langle Ax, x\rangle) dx := \int_{-\infty}^{\infty} ... \int_{-\infty}^{\infty} \exp(-\langle Ax, x\rangle) dx = \frac{\pi^{n/2}}{[\det(A)]^{1/2}}, \tag{1.1}$$

where A is a positive definite matrix of order n and $\langle \cdot, \cdot \rangle$ is the usual inner product on $\mathbb{R}^n$.

By utilizing the representation (1.1) and Hölder's integral inequality for multiple integrals one can prove the *logarithmic concavity* of the determinant that is due to Ky Fan ([1, p. 63] or [8, p. 212]), namely

$$\det((1 - \lambda) A + \lambda B) \geq [\det(A)]^{1-\lambda} [\det(B)]^{\lambda} \tag{1.2}$$

for any positive definite matrices A, B and $\lambda \in [0, 1]$.

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By mathematical induction we can get a generalization of (1.2) which was obtained by L. Mirsky in [7], see also [8, p. 212]

$$\det\left(\sum_{j=1}^{m}\lambda_j A_j\right)\geq\prod_{j=1}^{m}\left[\det\left(A_j\right)\right]^{\lambda_j},\ m\geq 2, \tag{1.3}$$

where $\lambda_j>0$, $j=1,...,m$ with $\sum_{j=1}^{m}\lambda_j=1$ and $A_j>0$, $j=1,...,m$.

If we write (1.3) for $A_j=B_j^{-1}$ we get

$$\det\left(\sum_{j=1}^{m}\lambda_j B_j^{-1}\right)\geq\prod_{j=1}^{m}\left[\det\left(B_j^{-1}\right)\right]^{\lambda_j}=\left(\prod_{j=1}^{m}\left[\det\left(B_j\right)\right]^{\lambda_j}\right)^{-1},$$

which also gives

$$\prod_{j=1}^{m}\left[\det\left(A_j\right)\right]^{\lambda_j}\geq\det\left[\left(\sum_{j=1}^{m}\lambda_j A_j^{-1}\right)^{-1}\right], \tag{1.4}$$

where $\lambda_j>0$, $j=1,...,m$ with $\sum_{j=1}^{m}\lambda_j=1$ and $A_j>0$, $j=1,...,m$.

Using the representation (1.1) one can also prove the result, see [8, p. 212],

$$\det\left(A\right)=\det\left(A_{1n}\right)\leq\det\left(A_{1k}\right)\det\left(A_{(k+1)n}\right),\ k=1,...,n; \tag{1.5}$$

where the determinant $\det\left(A_{rs}\right)$ is defined by

$$\det\left(A_{rs}\right)=\det\left(a_{ij}\right),i,\ j=r,...,s.$$

In particular,

$$\det\left(A\right)\leq a_{11}a_{22}...a_{nn}. \tag{1.6}$$

We recall also the Minkowski's type inequality,

$$\left[\det\left(A+B\right)\right]^{1/n}\geq\left[\det\left(A\right)\right]^{1/n}+\left[\det\left(B\right)\right]^{1/n} \tag{1.7}$$

for A, B positive definite matrices of order n. For other determinant inequalities see Chapter VIII of the classic book [8]. For some recent results see [2]-[6].

Motivated by the above results, in this paper we prove among others that, if A_j, $j=1,...,m$ are positive definite matrices, $\lambda_j>0$, $j=1,...,m$ with $\sum_{j=1}^{m}\lambda_j=1$ and $p\geq 1$, then

$$\sum_{j=1}^{m}\lambda_j\left[\det\left(pA_j+I_n\right)\right]^{-1}\geq\left(\sum_{j=1}^{m}\lambda_j\left[\det\left(A_j+I_n\right)\right]^{-1}\right)^{p}. \tag{1.8}$$

If $p\in(0,1)$, then the inequality above (1.8) reverses.

2. Main Results

By the convexity of power function of positive numbers we have

$$\sum_{j=1}^{m}\lambda_j u_j^p\geq\left(\sum_{j=1}^{m}\lambda_j u_j\right)^{p} \tag{2.1}$$

for $u_j>0$, $\lambda_j>0$, $j=1,...,m$ with $\sum_{j=1}^{m}\lambda_j=1$ and $p\in(-\infty,0)\cup[1,\infty)$.

If $r\in(0,1)$, then we have the reverse inequality

$$\sum_{j=1}^{m}\lambda_j u_j^r \le \left(\sum_{j=1}^{m}\lambda_j u_j\right)^r \tag{2.2}$$

under the same assumptions of the weights λ_j and u_j.

Theorem 1. *Let $\lambda_j>0$, $j=1,...,m$ with $\sum_{j=1}^{m}\lambda_j=1$ and $A_j>0$, $j=1,...,m$. If $p\ge 1$, then*

$$\begin{aligned}\sum_{j=1}^{m}\lambda_j\left[\det\left(A_j\right)\right]^{-1/2} \\ \ge p^{n(p+1)/2}\left(\sum_{j=1}^{m}\lambda_j\left[\det\left(pA_j+(p-1)I_n\right)\right]^{-1/2}\right)^p.\end{aligned} \tag{2.3}$$

Proof. From (2.1) we have for $u_j=\exp\left(-\langle A_jx,x\rangle\right)$ that

$$\sum_{j=1}^{m}\lambda_j\left[\exp\left(-\langle A_jx,x\rangle\right)\right]^p \ge \left(\sum_{j=1}^{m}\lambda_j\exp\left(-\langle A_jx,x\rangle\right)\right)^p \tag{2.4}$$

for $x\in\mathbb{R}^n$.

If we take the integral, then we get

$$\sum_{j=1}^{m}\lambda_j\int_{\mathbb{R}^n}\exp\left(-p\langle A_jx,x\rangle\right)dx \ge \int_{\mathbb{R}^n}\left(\sum_{j=1}^{m}\lambda_j\exp\left(-\langle A_jx,x\rangle\right)\right)^p dx. \tag{2.5}$$

Now, observe that

$$\begin{aligned}&\int_{\mathbb{R}^n}\left(\sum_{j=1}^{m}\lambda_j\exp\left(-\langle A_jx,x\rangle\right)\right)^p dx\\
&=\int_{\mathbb{R}^n}\frac{\exp\left(-\|x\|^2\right)}{\exp\left(-\|x\|^2\right)}\left(\sum_{j=1}^{m}\lambda_j\exp\left(-\langle A_jx,x\rangle\right)\right)^p dx\\
&=\int_{\mathbb{R}^n}\frac{\exp\left(-\|x\|^2\right)}{\left[\exp\left(-\frac{\|x\|^2}{p}\right)\right]^p}\left(\sum_{j=1}^{m}\lambda_j\exp\left(-\langle A_jx,x\rangle\right)\right)^p dx\\
&=\int_{\mathbb{R}^n}\exp\left(-\|x\|^2\right)\left(\sum_{j=1}^{m}\lambda_j\exp\left(-\langle A_jx,x\rangle+\frac{\|x\|^2}{p}\right)\right)^p dx\\
&=\int_{\mathbb{R}^n}\exp\left(-\|x\|^2\right)\left(\sum_{j=1}^{m}\lambda_j\exp\left(\frac{-\langle\left(pA_j-I_n\right)x,x\rangle}{p}\right)\right)^p dx\\
&=\int_{\mathbb{R}^n}\exp\left(-\|x\|^2\right)\left(\sum_{j=1}^{m}\lambda_j\exp\left(-\left\langle\left(A_j-\frac{1}{p}I_n\right)x,x\right\rangle\right)\right)^p dx.\end{aligned}$$

Using Jensen's integral inequality for power function, we get

$$\begin{aligned}
&\frac{\int_{\mathbb{R}^n} \exp\left(-\|x\|^2\right)\left(\sum_{j=1}^m \lambda_j \exp\left(-\left\langle\left(A_j-\frac{1}{p}I_n\right)x,x\right\rangle\right)\right)^p dx}{\int_{\mathbb{R}^n}\exp\left(-\|x\|^2\right)dx}\\
&\geq \left(\frac{\int_{\mathbb{R}^n} \exp\left(-\|x\|^2\right)\sum_{j=1}^m \lambda_j \exp\left(-\left\langle\left(A_j-\frac{1}{p}I_n\right)x,x\right\rangle\right) dx}{\int_{\mathbb{R}^n}\exp\left(-\|x\|^2\right)dx}\right)^p\\
&= \left(\frac{\int_{\mathbb{R}^n} \sum_{j=1}^m \lambda_j \exp\left(-\left\langle\left(A_j-\left(\frac{1}{p}-1\right)I_n\right)x,x\right\rangle\right) dx}{\int_{\mathbb{R}^n}\exp\left(-\|x\|^2\right)dx}\right)^p\\
&= \left(\frac{\sum_{j=1}^m \lambda_j \int_{\mathbb{R}^n} \exp\left(-\left\langle\left(A_j-\left(\frac{1}{p}-1\right)I_n\right)x,x\right\rangle\right) dx}{\int_{\mathbb{R}^n}\exp\left(-\|x\|^2\right)dx}\right)^p.
\end{aligned}$$

This gives

$$\begin{aligned}
(2.6)\qquad &\int_{\mathbb{R}^n} \exp\left(-\|x\|^2\right)\left(\sum_{j=1}^m \lambda_j \exp\left(-\left\langle\left(A_j-\frac{1}{p}I_n\right)x,x\right\rangle\right)\right)^p dx\\
&\geq \left(\int_{\mathbb{R}^n}\exp\left(-\|x\|^2\right)dx\right)^{1-p}\\
&\times\left(\sum_{j=1}^m \lambda_j \int_{\mathbb{R}^n} \exp\left(-\left\langle\left(A_j-\left(\frac{1}{p}-1\right)I_n\right)x,x\right\rangle\right) dx\right)^p.
\end{aligned}$$

By utilizing (2.5) and (2.6) we derive

$$\begin{aligned}
(2.7)\qquad &\sum_{j=1}^m \lambda_j \int_{\mathbb{R}^n}\exp\left(-\left\langle pA_j x,x\right\rangle\right)dx\\
&\geq \left(\int_{\mathbb{R}^n}\exp\left(-\|x\|^2\right)dx\right)^{1-p}\\
&\times\left(\sum_{j=1}^m \lambda_j \int_{\mathbb{R}^n} \exp\left(-\left\langle\left(A_j-\left(\frac{1}{p}-1\right)I_n\right)x,x\right\rangle\right) dx\right)^p.
\end{aligned}$$

This inequality becomes, by (1.1), that

$$\begin{aligned}
(2.8)\qquad \sum_{j=1}^m \lambda_j J_n\left(pA_j\right) &\geq \left[J_n\left(I_n\right)\right]^{1-p}\left(\sum_{j=1}^m \lambda_j J_n\left(A_j+\left(1-\frac{1}{p}\right)I_n\right)\right)^p\\
&= \left[J_n\left(I_n\right)\right]^{1-p}\left(\sum_{j=1}^m \lambda_j J_n\left(\frac{pA_j+(p-1)I_n}{p}\right)\right)^p.
\end{aligned}$$

Since, by (1.1),

$$J_n\left(I_n\right):=\frac{\pi^{n/2}}{\left[\det\left(I_n\right)\right]^{1/2}}=\pi^{n/2},$$

$$J_n(pA_j) = \frac{\pi^{n/2}}{[\det(pA_j)]^{1/2}} = \frac{\pi^{n/2}}{p^{n/2}[\det(A_j)]^{1/2}} = \frac{\pi^{n/2}}{p^{n/2}}[\det(A_j)]^{-1/2}$$

and

$$\begin{aligned} J_n\left(\frac{pA_j + (p-1)I_n}{p}\right) &= \frac{\pi^{n/2}}{\left[\det\left(\frac{pA_j+(p-1)I_n}{p}\right)\right]^{1/2}} \\ &= \frac{\pi^{n/2}}{\frac{1}{p^{n/2}}[\det(pA_j + (p-1)I_n)]^{1/2}} \\ &= p^{n/2}\pi^{n/2}[\det(pA_j + (p-1)I_n)]^{-1/2}, \end{aligned}$$

hence by (2.8) we derive

$$\begin{aligned} &\frac{\pi^{n/2}}{p^{n/2}}\sum_{j=1}^{m}\lambda_j[\det(A_j)]^{-1/2} \\ &\geq \left[\pi^{n/2}\right]^{1-p}\left(p^{n/2}\pi^{n/2}\sum_{j=1}^{m}\lambda_j[\det(pA_j + (p-1)I_n)]^{-1/2}\right)^p \\ &= \left[\pi^{n/2}\right]^{1-p}\left(p^{n/2}\pi^{n/2}\right)^p\left(\sum_{j=1}^{m}\lambda_j[\det(pA_j + (p-1)I_n)]^{-1/2}\right)^p, \end{aligned}$$

which is equivalent to (2.3). □

Remark 1. *Observe that*

$$\det(pA_j + (p-1)I_n) = p^n\left[\det\left(A_j + \left(1 - \frac{1}{p}\right)I_n\right)\right].$$

Then by (2.3) we get

$$\begin{aligned} &\sum_{j=1}^{m}\lambda_j[\det(A_j)]^{-1/2} \\ &\geq p^{n(p+1)/2}\left(\sum_{j=1}^{m}\lambda_j\left[p^n\left[\det\left(A_j + \left(1 - \frac{1}{p}\right)I_n\right)\right]\right]^{-1/2}\right)^p \\ &= p^{n(p+1)/2}p^{-np/2}\left(\sum_{j=1}^{m}\lambda_j\left[\det\left(A_j + \left(1 - \frac{1}{p}\right)I_n\right)\right]^{-1/2}\right)^p \\ &= p^{n/2}\left(\sum_{j=1}^{m}\lambda_j\left[\det\left(A_j + \left(1 - \frac{1}{p}\right)I_n\right)\right]^{-1/2}\right)^p. \end{aligned}$$

If we take $A_j = B_j/p$ *then we get*

$$\sum_{j=1}^{m} \lambda_j \left[\det\left(B_j/p\right)\right]^{-1/2} \geq p^{n/2} \left(\sum_{j=1}^{m} \lambda_j \left[\det\left(B_j/p + \left(1-\frac{1}{p}\right) I_n\right)\right]^{-1/2}\right)^p,$$

which gives

$$\frac{1}{p^{-n/2}} \sum_{j=1}^{m} \lambda_j \left[\det\left(B_j\right)\right]^{-1/2} \geq p^{n/2} \left(\sum_{j=1}^{m} \lambda_j \left[\det\left(\frac{1}{p}B_j + \left(1-\frac{1}{p}\right) I_n\right)\right]^{-1/2}\right)^p,$$

namely

$$\sum_{j=1}^{m} \lambda_j \left[\det\left(B_j\right)\right]^{-1/2} \geq \left(\sum_{j=1}^{m} \lambda_j \left[\det\left(\frac{1}{p}B_j + \left(1-\frac{1}{p}\right) I_n\right)\right]^{-1/2}\right)^p \tag{2.9}$$

for $p \geq 1$ *and* $B_j > 0,\ j = 1, ..., m.$

Taking the power $\frac{1}{p}$ *we get*

$$\left(\sum_{j=1}^{m} \lambda_j \left[\det\left(B_j\right)\right]^{-1/2}\right)^{1/p} \geq \sum_{j=1}^{m} \lambda_j \left[\det\left(\frac{1}{p}B_j + \left(1-\frac{1}{p}\right) I_n\right)\right]^{-1/2}.$$

By denoting $t = \frac{1}{p} \in (0,1]$ *we derive*

$$\left(\sum_{j=1}^{m} \lambda_j \left[\det\left(B_j\right)\right]^{-1/2}\right)^{t} \geq \sum_{j=1}^{m} \lambda_j \left[\det\left(tB_j + (1-t) I_n\right)\right]^{-1/2}. \tag{2.10}$$

Let $\lambda_j > 0,\ j = 1, ..., m$ *with* $\sum_{j=1}^{m} \lambda_j = 1$ *and* $B_j > 0,\ j = 1, ..., m.$ *Then for* $p = 2$ *we get, by (2.9), that*

$$\sum_{j=1}^{m} \lambda_j \left[\det\left(B_j\right)\right]^{-1/2} \geq \left(\sum_{j=1}^{m} \lambda_j \left[\det\left(\frac{B_j + I_n}{2}\right)\right]^{-1/2}\right)^2. \tag{2.11}$$

Also we have:

Theorem 2. *Let* $\lambda_j > 0,\ j = 1, ..., m$ *with* $\sum_{j=1}^{m} \lambda_j = 1$ *and* $A_j > 0,\ j = 1, ..., m.$ *If* $p \geq 1,$ *then*

$$\sum_{j=1}^{m} \lambda_j \left[\det\left(pA_j + I_n\right)\right]^{-1/2} \geq \left(\sum_{j=1}^{m} \lambda_j \left[\det\left(A_j + I_n\right)\right]^{-1/2}\right)^p. \tag{2.12}$$

If $p \in (0,1)\,,$ *then the inequality in (2.12) reverses.*

Proof. If we multiply (2.4) by $\exp\left(-\|x\|^2\right)$, then we get

$$\sum_{j=1}^{m} \lambda_j \exp\left(-\left\langle \left(pA_j + I_n\right) x, x\right\rangle\right) \geq \exp\left(-\|x\|^2\right) \left(\sum_{j=1}^{m} \lambda_j \exp\left(-\left\langle A_j x, x\right\rangle\right)\right)^p$$

for all $x \in \mathbb{R}^n$.

If we integrate on $\mathbb{R}^n$, then we get

$$\sum_{j=1}^{m} \lambda_j \int_{\mathbb{R}^n} \exp\left(-\left\langle \left(pA_j + I_n\right) x, x\right\rangle\right) dx \geq \int_{\mathbb{R}^n} \exp\left(-\|x\|^2\right) \left(\sum_{j=1}^{m} \lambda_j \exp\left(-\left\langle A_j x, x\right\rangle\right)\right)^p dx. \tag{2.13}$$

By Jensen's inequality for the power function we get

$$\begin{aligned} &\frac{\int_{\mathbb{R}^n} \exp\left(-\|x\|^2\right) \left(\sum_{j=1}^{m} \lambda_j \exp\left(-\left\langle A_j x, x\right\rangle\right)\right)^p dx}{\int_{\mathbb{R}^n} \exp\left(-\|x\|^2\right) dx} \\ &\geq \left(\frac{\int_{\mathbb{R}^n} \exp\left(-\|x\|^2\right) \sum_{j=1}^{m} \lambda_j \exp\left(-\left\langle A_j x, x\right\rangle\right)}{\int_{\mathbb{R}^n} \exp\left(-\|x\|^2\right) dx}\right)^p \\ &= \left(\frac{\sum_{j=1}^{m} \lambda_j \int_{\mathbb{R}^n} \exp\left(-\left\langle \left(A_j + I_n\right) x, x\right\rangle\right)}{\int_{\mathbb{R}^n} \exp\left(-\|x\|^2\right) dx}\right)^p, \end{aligned}$$

which gives that

$$\begin{aligned} &\int_{\mathbb{R}^n} \exp\left(-\|x\|^2\right) \left(\sum_{j=1}^{m} \lambda_j \exp\left(-\left\langle A_j x, x\right\rangle\right)\right)^p dx \\ &\geq \left(\int_{\mathbb{R}^n} \exp\left(-\|x\|^2\right) dx\right)^{1-p} \left(\sum_{j=1}^{m} \lambda_j \int_{\mathbb{R}^n} \exp\left(-\left\langle \left(A_j + I_n\right) x, x\right\rangle\right)\right)^p. \end{aligned} \tag{2.14}$$

By (2.13) and (2.14) we derive

$$\begin{aligned} &\sum_{j=1}^{m} \lambda_j \int_{\mathbb{R}^n} \exp\left(-\left\langle \left(pA_j + I_n\right) x, x\right\rangle\right) dx \\ &\geq \left(\int_{\mathbb{R}^n} \exp\left(-\|x\|^2\right) dx\right)^{1-p} \left(\sum_{j=1}^{m} \lambda_j \int_{\mathbb{R}^n} \exp\left(-\left\langle \left(A_j + I_n\right) x, x\right\rangle\right) dx\right)^p. \end{aligned} \tag{2.15}$$

By (1.1) we have

$$\int_{\mathbb{R}^n} \exp\left(-\left\langle \left(pA_j + I_n\right) x, x\right\rangle\right) dx = J_n\left(pA_j + I_n\right) = \frac{\pi^{n/2}}{\left[\det\left(pA_j + I_n\right)\right]^{1/2}},$$

$$\int_{\mathbb{R}^n} \exp\left(-\|x\|^2\right) dx = \frac{\pi^{n/2}}{\left[\det\left(I_n\right)\right]^{1/2}} = \pi^{n/2}$$

and

$$\int_{\mathbb{R}^n} \exp\left(-\left\langle \left(A_j + I_n\right) x, x\right\rangle\right) dx = \frac{\pi^{n/2}}{\left[\det\left(A_j + I_n\right)\right]^{1/2}}$$

for $j = 1, ..., m$.

Therefore by (2.15) we get

$$\sum_{j=1}^{m} \frac{\lambda_j \pi^{n/2}}{\left[\det\left(pA_j + I_n\right)\right]^{1/2}} \geq \left(\pi^{n/2}\right)^{1-p} \left(\sum_{j=1}^{m} \lambda_j \frac{\pi^{n/2}}{\left[\det\left(A_j + I_n\right)\right]^{1/2}}\right)^p,$$

which is equivalent to (2.12). □

Remark 2. *For $m = 1$ we get from (2.12) that*

$$\left[\det\left(pA + I_n\right)\right]^{-1/2} \geq \left[\det\left(A + I_n\right)\right]^{-p/2},$$

namely

$$\left[\det\left(A + I_n\right)\right]^p \geq \det\left(pA + I_n\right) \tag{2.16}$$

for $p \geq 1$ and $A > 0$. If $p \in (0, 1)$, then the inequality (2.16) reverses.

Remark 3. *Let $\lambda_j > 0$, $j = 1, ..., m$ with $\sum_{j=1}^m \lambda_j = 1$ and $A_j > 0$, $j = 1, ..., m$. If $p \geq 1 > r > 0$, then*

$$\begin{aligned} \left(\sum_{j=1}^{m} \lambda_j \left[\det\left(pA_j + I_n\right)\right]^{-1/2}\right)^{1/p} &\geq \sum_{j=1}^{m} \lambda_j \left[\det\left(A_j + I_n\right)\right]^{-1/2} \\ &\geq \left(\sum_{j=1}^{m} \lambda_j \left[\det\left(rA_j + I_n\right)\right]^{-1/2}\right)^{1/r}. \end{aligned} \tag{2.17}$$

3. Double Integral Representations

If we take the square in the representation (1.1), then we get

$$\left(\int_{\mathbb{R}^n} \exp(-\langle x, Ax\rangle)dx\right)^2 = \frac{\pi^n}{\det(A)}.$$

Since

$$\begin{aligned} \left(\int_{\mathbb{R}^n} \exp(-\langle Ax, x\rangle)dx\right)^2 &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \exp(-\langle Ax, x\rangle)\exp(-\langle Ay, y\rangle)dxdy \\ &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \exp(-\langle Ax, x\rangle - \langle Ay, y\rangle))dxdy, \end{aligned}$$

hence

$$K_n(A) := \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \exp(-\langle Ax, x\rangle - \langle Ay, y\rangle))dxdy = \frac{\pi^n}{\det(A)} \tag{3.1}$$

for A a positive definite matrix of order n and $\langle\cdot,\cdot\rangle$ is the usual inner product on $\mathbb{R}^n$.

Theorem 3. *Let* $\lambda_j > 0,\ j = 1, ..., m$ *with* $\sum_{j=1}^{m} \lambda_j = 1$ *and* $A_j > 0,\ j = 1, ..., m.$ *If* $p \geq 1,$ *then*

$$\sum_{j=1}^{m} \lambda_j \left[\det\left(pA_j + I_n\right)\right]^{-1} \geq \left(\sum_{j=1}^{m} \lambda_j \left[\det\left(A_j + I_n\right)\right]^{-1}\right)^{p}. \tag{3.2}$$

If $p \in (0, 1)$, *then the inequality in (3.2) reverses.*

Proof. From (2.1) we have for $v_j = \exp\left(-\langle A_j x, x\rangle - \langle A_j y, y\rangle\right)$ that

$$\begin{aligned} &\sum_{j=1}^{m} \lambda_j \exp\left(-\langle pA_j x, x\rangle - \langle pA_j y, y\rangle\right) \\ &\geq \left(\sum_{j=1}^{m} \lambda_j \exp\left(-\langle A_j x, x\rangle - \langle A_j y, y\rangle\right)\right)^{p}. \end{aligned} \tag{3.3}$$

If we multiply this inequality by $\exp\left(-\|x\|^2 - \|y\|^2\right)$, then we get

$$\begin{aligned} &\sum_{j=1}^{m} \lambda_j \exp\left(-\langle\left(pA_j + I_n\right)x, x\rangle - \langle\left(pA_j + I_n\right)y, y\rangle\right) \\ &\geq \exp\left(-\|x\|^2 - \|y\|^2\right)\left(\sum_{j=1}^{m} \lambda_j \exp\left(-\langle A_j x, x\rangle - \langle A_j y, y\rangle\right)\right)^{p}, \end{aligned}$$

for all $x,\ y \in \mathbb{R}^n$.

If we take the double integral $\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}$, then we get

$$\begin{aligned} &\sum_{j=1}^{m} \lambda_j \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \exp\left(-\langle\left(pA_j + I_n\right)x, x\rangle - \langle\left(pA_j + I_n\right)y, y\rangle\right) dxdy \\ &\geq \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \exp\left(-\|x\|^2 - \|y\|^2\right) \\ &\times \left(\sum_{j=1}^{m} \lambda_j \exp\left(-\langle A_j x, x\rangle - \langle A_j y, y\rangle\right)\right)^{p} dxdy. \end{aligned} \tag{3.4}$$

By the Jensen's integral for the power function, we have

$$\begin{aligned} &\frac{1}{\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \exp\left(-\|x\|^2 - \|y\|^2\right) dxdy} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \exp\left(-\|x\|^2 - \|y\|^2\right) \\ &\times \left(\sum_{j=1}^{m} \lambda_j \exp\left(-\langle A_j x, x\rangle - \langle A_j y, y\rangle\right)\right)^{p} dxdy \\ &\geq \left(\frac{\sum_{j=1}^{m} \lambda_j \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \exp\left(-\|x\|^2 - \|y\|^2\right)\exp\left(-\langle A_j x, x\rangle - \langle A_j y, y\rangle\right) dxdy}{\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \exp\left(-\|x\|^2 - \|y\|^2\right) dxdy}\right)^{p} \\ &= \left(\frac{\sum_{j=1}^{m} \lambda_j \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \exp\left(-\langle\left(A_j + I_n\right)x, x\rangle - \langle\left(A_j + I_n\right)y, y\rangle\right) dxdy}{\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \exp\left(-\|x\|^2 - \|y\|^2\right) dxdy}\right)^{p}, \end{aligned}$$

which gives that

$$\begin{aligned}(3.5)\quad & \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \exp\left(-\|x\|^2-\|y\|^2\right) \\ & \times \left(\sum_{j=1}^m \lambda_j \exp\left(-\langle A_j x, x\rangle - \langle A_j y, y\rangle\right)\right)^p dxdy \\ & \geq \left(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \exp\left(-\|x\|^2-\|y\|^2\right) dxdy\right)^{1-p} \\ & \times \left(\sum_{j=1}^m \lambda_j \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \exp\left(-\langle (A_j+I_n)x, x\rangle - \langle (A_j+I_n) y, y\rangle\right) dxdy\right)^p .\end{aligned}$$

Then by (3.4) and (3.5) we get

$$\begin{aligned}(3.6)\quad & \sum_{j=1}^m \lambda_j \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \exp\left(-\langle (pA_j+I_n)x, x\rangle - \langle (pA_j+I_n) y, y\rangle\right) dxdy \\ & \geq \left(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \exp\left(-\|x\|^2-\|y\|^2\right) dxdy\right)^{1-p} \\ & \times \left(\sum_{j=1}^m \lambda_j \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \exp\left(-\langle (A_j+I_n)x, x\rangle - \langle (A_j+I_n) y, y\rangle\right) dxdy\right)^p .\end{aligned}$$

By (3.1) we have

$$\begin{aligned}& \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \exp\left(-\langle (pA_j+I_n)x, x\rangle - \langle (pA_j+I_n) y, y\rangle\right) dxdy \\ & = K_n(pA_j+I_n) = \frac{\pi^n}{\det(pA_j+I_n)},\end{aligned}$$

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \exp\left(-\|x\|^2-\|y\|^2\right) dxdy = K_n(I_n) = \pi^n$$

and

$$\begin{aligned}& \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \exp\left(-\langle (A_j+I_n)x, x\rangle - \langle (A_j+I_n) y, y\rangle\right) dxdy \\ & = K_n(A_j+I_n) = \frac{\pi^n}{\det(A_j+I_n)}.\end{aligned}$$

Then by (3.6) we get

$$\sum_{j=1}^m \frac{\lambda_j \pi^n}{\det(pA_j+I_n)} \geq \pi^{n(1-p)} \left(\sum_{j=1}^m \frac{\lambda_j \pi^n}{\det(A_j+I_n)}\right)^p,$$

which is equivalent to (3.2). □

Remark 4. *Let* $\lambda_j > 0,\ j = 1, ..., m$ *with* $\sum_{j=1}^{m} \lambda_j = 1$ *and* $A_j > 0,\ j = 1, ..., m.$ *If* $p \geq 1 > r > 0,$ *then*

$$\left(\sum_{j=1}^{m} \lambda_j \left[\det\left(pA_j + I_n\right)\right]^{-1}\right)^{1/p} \geq \sum_{j=1}^{m} \lambda_j \left[\det\left(A_j + I_n\right)\right]^{-1} \geq \left(\sum_{j=1}^{m} \lambda_j \left[\det\left(rA_j + I_n\right)\right]^{-1}\right)^{1/r}. \tag{3.7}$$

4. The Case of Hermitian Matrices

A complex square matrix $H = (h_{ij})$, $i, j = 1, ..., n$ is said to be Hermitian provided $h_{ij} = \overline{h_{ji}}$ for all $i,\ j = 1, ..., n$. A Hermitian matrix is said to be positive definite if the Hermitian form $P(z) = \sum_{i,j=1}^{n} a_{ij} z_i \overline{z_j}$ is positive for all $z = (z_1, ..., z_n) \in \mathbb{C}^n \setminus \{0\}$.

It is known that, see for instance [8, p. 215], for a positive definite Hermitian matrix H, we have

$$K_n(H) := \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \exp\left(-\left\langle \overline{z}, Hz\right\rangle\right) dx dy = \frac{\pi^n}{\det(H)}, \tag{4.1}$$

where $z = x + iy$ and dx and dy denote integration over real n-dimensional space $\mathbb{R}^n$. Here the inner product $\langle x, y\rangle$ is understood in the real sense, i.e. $\langle x, y\rangle = \sum_{k=1}^{n} x_k y_k$.

On making use of a similar argument to the one in Theorem 3 we can state the following result as well:

Theorem 4. *Let* $\lambda_j > 0,\ j = 1, ..., m$ *with* $\sum_{j=1}^{m} \lambda_j = 1$ *and* $H_j > 0,\ j = 1, ..., m.$ *If* $p \geq 1,$ *then*

$$\sum_{j=1}^{m} \lambda_j \left[\det\left(pH_j + I_n\right)\right]^{-1} \geq \left(\sum_{j=1}^{m} \lambda_j \left[\det\left(H_j + I_n\right)\right]^{-1}\right)^{p}. \tag{4.2}$$

If $p \in (0, 1)$, *then the inequality in (4.2) reverses.*

Remark 5. *Let* $\lambda_j > 0,\ j = 1, ..., m$ *with* $\sum_{j=1}^{m} \lambda_j = 1$ *and* $H_j > 0,\ j = 1, ..., m.$ *If* $p \geq 1 > r > 0,$ *then*

$$\left(\sum_{j=1}^{m} \lambda_j \left[\det\left(pH_j + I_n\right)\right]^{-1}\right)^{1/p} \geq \sum_{j=1}^{m} \lambda_j \left[\det\left(H_j + I_n\right)\right]^{-1} \geq \left(\sum_{j=1}^{m} \lambda_j \left[\det\left(rH_j + I_n\right)\right]^{-1}\right)^{1/r}. \tag{4.3}$$

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