DETERMINANT INEQUALITIES FOR POSITIVE DEFINITE MATRICES VIA JENSEN'S INEQUALITY FOR EXPONENTIAL FUNCTION

SILVESTRU SEVER DRAGOMIR[1,2]

Abstract. In this paper we prove among others that, if $(A_j)_{j=1,...,m}$ are positive definite matrices of order n and $p_j \geq 0$, $j = 1, ..., m$ with $\sum_{j=1}^{m} p_j = 1$, then

$$\begin{aligned} 0 &\leq \min_{j\in\{1,...,m\}} \{p_j\} \left[\sum_{j=1}^{m} [\det (A_j)]^{-1} - m^{n+1} \left[\det \left(\sum_{j=1}^{m} A_j \right) \right]^{-1} \right] \\ &\leq \sum_{j=1}^{m} p_j [\det (A_j)]^{-1} - \left[\det \left(\sum_{j=1}^{m} p_j A_j \right) \right]^{-1} \\ &\leq \max_{j\in\{1,...,m\}} \{p_j\} \left[\sum_{j=1}^{m} [\det (A_j)]^{-1} - m^{n+1} \left[\det \left(\sum_{j=1}^{m} A_j \right) \right]^{-1} \right]. \end{aligned}$$

1. Introduction

A real square matrix $A = (a_{ij})$, $i, j = 1, ..., n$ is *symmetric* provided $a_{ij} = a_{ji}$ for all $i, j = 1, ..., n$. A real symmetric matrix is said to be *positive definite* provided the quadratic form $Q(x) = \sum_{i,j=1}^{n} a_{ij} x_i x_j$ is positive for all $x = (x_1, ..., x_n) \in \mathbb{R}^n \setminus \{0\}$. It is well known that a necessary and sufficient condition for the symmetric matrix A to be positive definite, and we write $A > 0$, is that all determinants

$$\det(A_k) = \det (a_{ij}), \; i, \; j = 1, ..., k; \; k = 1, ..., n$$

are positive.

It is know that the following integral representation is valid, see [1, pp. 61-62] or [9, pp. 211-212]

$$\begin{aligned} J_n(A) &:= \int_{\mathbb{R}^n} \exp(-\langle Ax, x\rangle) dx := \int_{-\infty}^{\infty} ... \int_{-\infty}^{\infty} \exp(-\langle Ax, x\rangle) dx \\ &= \frac{\pi^{n/2}}{[\det (A)]^{1/2}}, \end{aligned} \tag{1.1}$$

where A is a positive definite matrix of order n and $\langle \cdot, \cdot \rangle$ is the usual inner product on $\mathbb{R}^n$.

By utilizing the representation (1.1) and Hölder's integral inequality for multiple integrals one can prove the *logarithmic concavity* of the determinant that is due to

1991 *Mathematics Subject Classification.* 47A63, 26D15, 46C05.
Key words and phrases. Positive definite matrices, Determinants, Inequalities.

RGMIA Res. Rep. Coll. 24 (2021), Art. 145, 14 pp. Received 23/11/21

Ky Fan ([1, p. 63] or [9, p. 212]), namely

$$\det((1-\lambda)A+\lambda B)\geq[\det(A)]^{1-\lambda}[\det(B)]^{\lambda} \tag{1.2}$$

for any positive definite matrices A, B and $\lambda\in[0,1]$.

By mathematical induction we can get a generalization of (1.2) which was obtained by L. Mirsky in [8], see also [9, p. 212]

$$\det\left(\sum_{j=1}^{m}\lambda_j A_j\right)\geq\prod_{j=1}^{m}[\det(A_j)]^{\lambda_j},\ m\geq 2, \tag{1.3}$$

where $\lambda_j>0$, $j=1,...,m$ with $\sum_{j=1}^{m}\lambda_j=1$ and $A_j>0$, $j=1,...,m$.

If we write (1.3) for $A_j=B_j^{-1}$ we get

$$\det\left(\sum_{j=1}^{m}\lambda_j B_j^{-1}\right)\geq\prod_{j=1}^{m}\left[\det\left(B_j^{-1}\right)\right]^{\lambda_j}=\left(\prod_{j=1}^{m}[\det(B_j)]^{\lambda_j}\right)^{-1},$$

which also gives

$$\prod_{j=1}^{m}[\det(A_j)]^{\lambda_j}\geq\det\left[\left(\sum_{j=1}^{m}\lambda_j A_j^{-1}\right)^{-1}\right], \tag{1.4}$$

where $\lambda_j>0$, $j=1,...,m$ with $\sum_{j=1}^{m}\lambda_j=1$ and $A_j>0$, $j=1,...,m$.

Using the representation (1.1) one can also prove the result, see [9, p. 212],

$$\det(A)=\det(A_{1n})\leq\det(A_{1k})\det\left(A_{(k+1)n}\right),\ k=1,...,n; \tag{1.5}$$

where the determinant $\det(A_{rs})$ is defined by

$$\det(A_{rs})=\det(a_{ij}),i,\ j=r,...,s.$$

In particular,

$$\det(A)\leq a_{11}a_{22}...a_{nn}. \tag{1.6}$$

We recall also the Minkowski's type inequality,

$$[\det(A+B)]^{1/n}\geq[\det(A)]^{1/n}+[\det(B)]^{1/n} \tag{1.7}$$

for A, B positive definite matrices of order n. For other determinant inequalities see Chapter VIII of the classic book [9]. For some recent results see [3]-[7].

Motivated by the above results, in this paper we prove among others that, if $(A_j)_{j=1,...,m}$ are positive definite matrices and $p_j\geq 0$, $j=1,...,m$ with $\sum_{j=1}^{m}p_j=$

1, then

$$0 \leq \min_{j\in\{1,...,m\}}\{p_j\}\left[\sum_{j=1}^{m}[\det(A_j)]^{-1} - m^{n+1}\left[\det\left(\sum_{j=1}^{m}A_j\right)\right]^{-1}\right]$$
$$\leq \sum_{j=1}^{m}p_j[\det(A_j)]^{-1} - \left[\det\left(\sum_{j=1}^{m}p_jA_j\right)\right]^{-1}$$
$$\leq \max_{j\in\{1,...,m\}}\{p_j\}\left[\sum_{j=1}^{m}[\det(A_j)]^{-1} - m^{n+1}\left[\det\left(\sum_{j=1}^{m}A_j\right)\right]^{-1}\right].$$

2. Inequalities Via Single Integral Representation

We have the following result:

Theorem 1. *Assume that* $(A_j)_{j=1,...,m}$ *are positive definite and* $p_j \geq 0$, $j = 1, ..., m$ *with* $\sum_{j=1}^{m} p_j = 1$. *Then*

$$0 \leq \min_{j\in\{1,...,m\}}\{p_j\}\left[\sum_{j=1}^{m}[\det(A_j)]^{-1/2} - m^{n/2+1}\left[\det\left(\sum_{j=1}^{m}A_j\right)\right]^{-1/2}\right] \tag{2.1}$$
$$\leq \sum_{j=1}^{m}p_j[\det(A_j)]^{-1/2} - \left[\det\left(\sum_{j=1}^{m}p_jA_j\right)\right]^{-1/2}$$
$$\leq \max_{j\in\{1,...,m\}}\{p_j\}\left[\sum_{j=1}^{m}[\det(A_j)]^{-1/2} - m^{n/2+1}\left[\det\left(\sum_{j=1}^{m}A_j\right)\right]^{-1/2}\right].$$

In particular, for $m = 2$ *and* $t \in (0, 1)$, *we have*

$$0 \leq \min\{t, 1-t\} \tag{2.2}$$
$$\times\left[[\det(A_1)]^{-1/2} + [\det(A_2)]^{-1/2} - 2^{n/2+1}[\det(A_1 + A_2)]^{-1/2}\right]$$
$$\leq t[\det(A_1)]^{-1/2} + (1-t)[\det(A_2)]^{-1/2} - [\det(tA_1 + (1-t)A_2)]^{-1/2}$$
$$\leq \max\{t, 1-t\}$$
$$\times\left[[\det(A_1)]^{-1/2} + [\det(A_2)]^{-1/2} - 2^{n/2+1}[\det(A_1 + A_2)]^{-1/2}\right].$$

Proof. First of all, we recall the following result obtained by the author in [2] that provides a refinement and a reverse for the weighted Jensen's discrete inequality:

$$0 \leq m \min_{j\in\{1,\dots,m\}}\{p_j\}\left[\frac{1}{m}\sum_{j=1}^{m}\Phi(x_j)-\Phi\left(\frac{1}{m}\sum_{j=1}^{m}x_j\right)\right] \leq \sum_{j=1}^{m}p_j\Phi(x_j)-\Phi\left(\sum_{j=1}^{m}p_jx_j\right) \leq m \max_{j\in\{1,\dots,m\}}\{p_j\}\left[\frac{1}{m}\sum_{j=1}^{m}\Phi(x_j)-\Phi\left(\frac{1}{m}\sum_{j=1}^{m}x_j\right)\right], \tag{2.3}$$

where $\Phi : C \to \mathbb{R}$ is a convex function defined on the convex subset C of the linear space X, $\{x_j\}_{j\in\{1,\dots,m\}} \subset C$ are vectors and $\{p_j\}_{j\in\{1,\dots,m\}}$ are nonnegative numbers with $\sum_{j=1}^{m} p_j = 1$.

For $m = 2$ we deduce from (2.3) that

$$0 \leq 2\min\{t,1-t\}\left[\frac{\Phi(x)+\Phi(y)}{2}-\Phi\left(\frac{x+y}{2}\right)\right] \leq t\Phi(x)+(1-t)\Phi(y)-\Phi(tx+(1-t)y) \leq 2\max\{t,1-t\}\left[\frac{\Phi(x)+\Phi(y)}{2}-\Phi\left(\frac{x+y}{2}\right)\right] \tag{2.4}$$

for any x, $y \in C$ and $t \in [0,1]$.

If we take $\Phi(t) = \exp(t)$, $x \in \mathbb{R}$ and $x_j \in \mathbb{R}$, $j \in \{1,...,m\}$, then we get

$$0 \leq m \min_{j\in\{1,\dots,m\}}\{p_j\}\left[\frac{1}{m}\sum_{j=1}^{m}\exp(x_j)-\exp\left(\frac{1}{m}\sum_{j=1}^{m}x_j\right)\right] \leq \sum_{j=1}^{m}p_j\exp(x_j)-\exp\left(\sum_{j=1}^{m}p_jx_j\right) \leq m \max_{j\in\{1,\dots,m\}}\{p_j\}\left[\frac{1}{m}\sum_{j=1}^{m}\exp(x_j)-\Phi\left(\frac{1}{m}\sum_{j=1}^{m}\exp_j\right)\right]. \tag{2.5}$$

Now, by taking $x_j = -\langle A_jx,x\rangle$, $j \in \{1,...,m\}$ with $x \in \mathbb{R}^n$ in (2.5), we get

$$0 \leq m \min_{j\in\{1,\dots,m\}}\{p_j\} \times\left[\frac{1}{m}\sum_{j=1}^{m}\exp(-\langle A_jx,x\rangle)-\exp\left(-\left\langle\frac{1}{m}\sum_{j=1}^{m}A_jx,x\right\rangle\right)\right] \tag{2.6}$$

$$
\begin{aligned}
&\leq \sum_{j=1}^{m} p_j \exp\left(-\left\langle A_j x, x\right\rangle\right) - \exp\left(-\left\langle \sum_{j=1}^{m} p_j A_j x, x\right\rangle\right) \\
&\leq m \max_{j\in\{1,\dots,m\}}\{p_j\} \\
&\times \left[\frac{1}{m}\sum_{j=1}^{m} \exp\left(-\left\langle A_j x, x\right\rangle\right) - \exp\left(-\left\langle \frac{1}{m}\sum_{j=1}^{m} A_j x, x\right\rangle\right)\right].
\end{aligned}
$$

By taking the integral on $\mathbb{R}^n$, we get

$$
\begin{aligned}
0 &\leq m \min_{j\in\{1,\dots,m\}}\{p_j\} \\
&\times \left[\frac{1}{m}\sum_{j=1}^{m} \int_{\mathbb{R}^n} \exp\left(-\left\langle A_j x, x\right\rangle\right) dx - \int_{\mathbb{R}^n} \exp\left(-\left\langle \frac{1}{m}\sum_{j=1}^{m} A_j x, x\right\rangle\right) dx\right] \\
&\leq \sum_{j=1}^{m} p_j \int_{\mathbb{R}^n} \exp\left(-\left\langle A_j x, x\right\rangle\right) dx - \int_{\mathbb{R}^n} \exp\left(-\left\langle \sum_{j=1}^{m} p_j A_j x, x\right\rangle\right) dx \\
&\leq m \max_{j\in\{1,\dots,m\}}\{p_j\} \\
&\times \left[\frac{1}{m}\sum_{j=1}^{m} \int_{\mathbb{R}^n} \exp\left(-\left\langle A_j x, x\right\rangle\right) dx - \int_{\mathbb{R}^n} \exp\left(-\left\langle \frac{1}{m}\sum_{j=1}^{m} A_j x, x\right\rangle\right) dx\right],
\end{aligned}
$$

which by the representation (1.1) gives that

$$
\begin{aligned}
0 &\leq m \min_{j\in\{1,\dots,m\}}\{p_j\} \left[\frac{1}{m}\sum_{j=1}^{m} J_n\left(A_j\right) - J_n\left(\frac{1}{m}\sum_{j=1}^{m} A_j\right)\right] \\
&\leq \sum_{j=1}^{m} p_j J_n\left(A_j\right) - J_n\left(\sum_{j=1}^{m} p_j A_j\right) \\
&\leq m \max_{j\in\{1,\dots,m\}}\{p_j\} \left[\frac{1}{m}\sum_{j=1}^{m} J_n\left(A_j\right) - J_n\left(\frac{1}{m}\sum_{j=1}^{m} A_j\right)\right],
\end{aligned}
$$

namely

$$
\begin{aligned}
0 &\leq m \min_{j\in\{1,\dots,m\}}\{p_j\} \left[\frac{1}{m}\sum_{j=1}^{m} \frac{1}{\left[\det\left(A_j\right)\right]^{1/2}} - \frac{1}{\left[\det\left(\frac{1}{m}\sum_{j=1}^{m} A_j\right)\right]^{1/2}}\right] \\
&\leq \sum_{j=1}^{m} \frac{p_j}{\left[\det\left(A_j\right)\right]^{1/2}} - \frac{1}{\left[\det\left(\sum_{j=1}^{m} p_j A_j\right)\right]^{1/2}} \\
&\leq m \max_{j\in\{1,\dots,m\}}\{p_j\} \left[\frac{1}{m}\sum_{j=1}^{m} \frac{1}{\left[\det\left(A_j\right)\right]^{1/2}} - \frac{1}{\left[\det\left(\frac{1}{m}\sum_{j=1}^{m} A_j\right)\right]^{1/2}}\right].
\end{aligned}
$$

This is equivalent to

$$0 \le \min_{j\in\{1,\dots,m\}}\{p_j\}\left[\sum_{j=1}^{m}\frac{1}{[\det(A_j)]^{1/2}} - \frac{m}{\frac{1}{m^{n/2}}\left[\det\left(\sum_{j=1}^{m}A_j\right)\right]^{1/2}}\right]$$
$$\le \sum_{j=1}^{m}\frac{p_j}{[\det(A_j)]^{1/2}} - \frac{1}{\left[\det\left(\sum_{j=1}^{m}p_jA_j\right)\right]^{1/2}}$$
$$\le \max_{j\in\{1,\dots,m\}}\{p_j\}\left[\sum_{j=1}^{m}\frac{1}{[\det(A_j)]^{1/2}} - \frac{m}{\frac{1}{m^{n/2}}\left[\det\left(\sum_{j=1}^{m}A_j\right)\right]^{1/2}}\right]$$

and the theorem is proved. □

Corollary 1. *If A_1, A_2 are positive definite, then*

$$(2.7)\qquad 0 \le \frac{1}{4}\left[[\det(A_1)]^{-1/2} + [\det(A_2)]^{-1/2} - 2^{n/2+1}[\det(A_1+A_2)]^{-1/2}\right]$$
$$\le \frac{[\det(A_1)]^{-1/2} + [\det(A_2)]^{-1/2}}{2} - \int_0^1 [\det(tA_1+(1-t)A_2)]^{-1/2}\,dt$$
$$\le \frac{3}{4}\left[[\det(A_1)]^{-1/2} + [\det(A_2)]^{-1/2} - 2^{n/2+1}[\det(A_1+A_2)]^{-1/2}\right].$$

Proof. If we take the integral in (2.2) we get

$$(2.8)\qquad 0 \le \int_0^1 \min\{t,1-t\}\,dt$$
$$\times\left[[\det(A_1)]^{-1/2} + [\det(A_2)]^{-1/2} - 2^{n/2+1}[\det(A_1+A_2)]^{-1/2}\right]$$
$$\le \frac{[\det(A_1)]^{-1/2} + [\det(A_2)]^{-1/2}}{2} - \int_0^1 [\det(tA_1+(1-t)A_2)]^{-1/2}\,dt$$
$$\le \int_0^1 \max\{t,1-t\}\,dt$$
$$\times\left[[\det(A_1)]^{-1/2} + [\det(A_2)]^{-1/2} - 2^{n/2+1}[\det(A_1+A_2)]^{-1/2}\right].$$

Since

$$\int_0^1 \min\{t,1-t\}\,dt = \frac{1}{4} \text{ and } \int_0^1 \max\{t,1-t\}\,dt = \frac{3}{4},$$

hence by (2.8) we get (2.7). □

Remark 1. *If we take* $A_j = B_j^{-2}$, *where* $B_j > 0$, $j = 1, ..., m$, *then by (2.1) we get*

$$0 \le \min_{j\in\{1,\dots,m\}} \{p_j\} \left[\sum_{j=1}^{m} \det(B_j) - m^{n/2+1}\left[\det\left(\sum_{j=1}^{m} B_j^{-2}\right)\right]^{-1/2}\right] \tag{2.9}$$
$$\le \sum_{j=1}^{m} p_j \det(B_j) - \left[\det\left(\sum_{j=1}^{m} p_j B_j^{-2}\right)\right]^{-1/2}$$
$$\le \max_{j\in\{1,\dots,m\}} \{p_j\} \left[\sum_{j=1}^{m} \det(B_j) - m^{n/2+1}\left[\det\left(\sum_{j=1}^{m} B_j^{-2}\right)\right]^{-1/2}\right].$$

In particular, for $m = 2$ *and* $t \in (0,1)$, *we have*

$$0 \le \min\{t, 1-t\} \times \left[\det(B_1) + \det(B_2) - 2^{n/2+1}\left[\det\left(B_1^{-2} + B_2^{-2}\right)\right]^{-1/2}\right] \tag{2.10}$$
$$\le t\det(B_1) + (1-t)\det(B_2) - \left[\det\left(tB_1^{-2} + (1-t)B_2^{-2}\right)\right]^{-1/2}$$
$$\le \max\{t, 1-t\} \times \left[\det(B_1) + \det(B_2) - 2^{n/2+1}\left[\det\left(B_1^{-2} + B_2^{-2}\right)\right]^{-1/2}\right].$$

From (2.7) we also have

$$0 \le \frac{1}{4}\left[\det(B_1) + \det(B_2) - 2^{n/2+1}\left[\det\left(B_1^{-2} + B_2^{-2}\right)\right]^{-1/2}\right] \tag{2.11}$$
$$\le \frac{\det(B_1) + \det(B_2)}{2} - \int_0^1 \left[\det\left(tB_1^{-2} + (1-t)B_2^{-2}\right)\right]^{-1/2} dt$$
$$\le \frac{3}{4}\left[\det(B_1) + \det(B_2) - 2^{n/2+1}\left[\det\left(B_1^{-2} + B_2^{-2}\right)\right]^{-1/2}\right].$$

3. Inequalities Via Double Integral Representation

If we take the square in the representation (1.1), then we get

$$\left(\int_{\mathbb{R}^n} \exp(-\langle x, Ax\rangle)dx\right)^2 = \frac{\pi^n}{\det(A)}.$$

Since

$$\left(\int_{\mathbb{R}^n} \exp(-\langle Ax, x\rangle)dx\right)^2 = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \exp(-\langle Ax, x\rangle)\exp(-\langle Ay, y\rangle)dxdy$$
$$= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \exp(-\langle Ax, x\rangle - \langle Ay, y\rangle))dxdy,$$

hence

$$K_n(A) := \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \exp(-\langle Ax,x\rangle - \langle Ay,y\rangle)dxdy = \frac{\pi^n}{\det(A)} \tag{3.1}$$

for A a positive definite matrix of order n and $\langle\cdot,\cdot\rangle$ is the usual inner product on $\mathbb{R}^n$.

Theorem 2. *Assume that $(A_j)_{j=1,\dots,m}$ are positive definite and $p_j \geq 0$, $j = 1, ..., m$ with $\sum_{j=1}^m p_j = 1$. Then*

$$\begin{aligned}
0 &\leq \min_{j\in\{1,\dots,m\}}\{p_j\}\left[\sum_{j=1}^m [\det(A_j)]^{-1} - m^{n+1}\left[\det\left(\sum_{j=1}^m A_j\right)\right]^{-1}\right] \\
&\leq \sum_{j=1}^m p_j[\det(A_j)]^{-1} - \left[\det\left(\sum_{j=1}^m p_j A_j\right)\right]^{-1} \\
&\leq \max_{j\in\{1,\dots,m\}}\{p_j\}\left[\sum_{j=1}^m [\det(A_j)]^{-1} - m^{n+1}\left[\det\left(\sum_{j=1}^m A_j\right)\right]^{-1}\right].
\end{aligned} \tag{3.2}$$

In particular, for $m = 2$ and $t \in (0,1)$, we have

$$\begin{aligned}
0 &\leq \min\{t, 1-t\} \\
&\quad\times\left[[\det(A_1)]^{-1} + [\det(A_2)]^{-1} - 2^{n+1}[\det(A_1+A_2)]^{-1}\right] \\
&\leq t[\det(A_1)]^{-1} + (1-t)[\det(A_2)]^{-1} - [\det(tA_1 + (1-t)A_2)]^{-1} \\
&\leq \max\{t, 1-t\} \\
&\quad\times\left[[\det(A_1)]^{-1} + [\det(A_2)]^{-1} - 2^{n+1}[\det(A_1+A_2)]^{-1}\right].
\end{aligned} \tag{3.3}$$

Proof. By taking $x_j = -\langle A_j x, x\rangle - \langle A_j y, y\rangle$, $j \in \{1, ..., m\}$ with $x, y \in \mathbb{R}^n$ in (2.5), we get

$$\begin{aligned}
0 \leq m \min_{j\in\{1,\dots,m\}}\{p_j\}&\left[\frac{1}{m}\sum_{j=1}^m \exp(-\langle A_j x, x\rangle - \langle A_j y, y\rangle)\right. \\
&\left. - \exp\left(-\left\langle \frac{1}{m}\sum_{j=1}^m A_j x, x\right\rangle - \left\langle \frac{1}{m}\sum_{j=1}^m A_j x, x\right\rangle\right)\right]
\end{aligned}$$

$$\leq \sum_{j=1}^{m} p_j \exp\left(-\langle A_j x, x\rangle - \langle A_j y, y\rangle\right)$$
$$- \exp\left(-\left\langle \sum_{j=1}^{m} p_j A_j x, x\right\rangle - \left\langle \sum_{j=1}^{m} p_j A_j y, y\right\rangle\right)$$
$$\leq m \max_{j\in\{1,\dots,m\}} \{p_j\} \left[\frac{1}{m}\sum_{j=1}^{m} \exp\left(-\langle A_j x, x\rangle - \langle A_j y, y\rangle\right)\right.$$
$$\left. - \exp\left(-\left\langle \frac{1}{m}\sum_{j=1}^{m} A_j x, x\right\rangle - \left\langle \frac{1}{m}\sum_{j=1}^{m} A_j x, x\right\rangle\right)\right].$$

By taking the double integral $\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}$ in this inequality, we get

$$0 \leq m \min_{j\in\{1,\dots,m\}} \{p_j\} \left[\frac{1}{m}\sum_{j=1}^{m} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \exp\left(-\langle A_j x, x\rangle - \langle A_j y, y\rangle\right) dxdy\right.$$
$$\left. - \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \exp\left(-\left\langle \frac{1}{m}\sum_{j=1}^{m} A_j x, x\right\rangle - \left\langle \frac{1}{m}\sum_{j=1}^{m} A_j x, x\right\rangle\right) dxdy\right]$$
$$\leq \sum_{j=1}^{m} p_j \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \exp\left(-\langle A_j x, x\rangle - \langle A_j y, y\rangle\right) dxdy$$
$$- \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \exp\left(-\left\langle \sum_{j=1}^{m} p_j A_j x, x\right\rangle - \left\langle \sum_{j=1}^{m} p_j A_j y, y\right\rangle\right) dxdy$$
$$\leq m \max_{j\in\{1,\dots,m\}} \{p_j\} \left[\frac{1}{m}\sum_{j=1}^{m} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \exp\left(-\langle A_j x, x\rangle - \langle A_j y, y\rangle\right) dxdy\right.$$
$$\left. - \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \exp\left(-\left\langle \frac{1}{m}\sum_{j=1}^{m} A_j x, x\right\rangle - \left\langle \frac{1}{m}\sum_{j=1}^{m} A_j x, x\right\rangle\right) dxdy\right].$$

If we use the representation (3.1), then we get

$$0 \leq m \min_{j\in\{1,\dots,m\}} \{p_j\} \left[\frac{1}{m}\sum_{j=1}^{m} K_n(A_j) - K_n\left(\frac{1}{m}\sum_{j=1}^{m} A_j\right)\right]$$
$$\leq \sum_{j=1}^{m} p_j K_n(A_j) - K_n\left(\sum_{j=1}^{m} p_j A_j\right)$$
$$\leq m \max_{j\in\{1,\dots,m\}} \{p_j\} \left[\frac{1}{m}\sum_{j=1}^{m} K_n(A_j) - K_n\left(\frac{1}{m}\sum_{j=1}^{m} A_j\right)\right],$$

namely

$$0 \le m \min_{j\in\{1,...,m\}} \{p_j\} \left[\frac{1}{m} \sum_{j=1}^{m} \frac{1}{\det(A_j)} - \frac{1}{\det\left(\frac{1}{m}\sum_{j=1}^{m} A_j\right)} \right]$$
$$\le \sum_{j=1}^{m} \frac{p_j}{\det(A_j)} - \frac{1}{\det\left(\sum_{j=1}^{m} p_j A_j\right)}$$
$$\le m \max_{j\in\{1,...,m\}} \{p_j\} \left[\frac{1}{m} \sum_{j=1}^{m} \frac{1}{\det(A_j)} - \frac{1}{\det\left(\frac{1}{m}\sum_{j=1}^{m} A_j\right)} \right],$$

or

$$0 \le m \min_{j\in\{1,...,m\}} \{p_j\} \left[\frac{1}{m} \sum_{j=1}^{m} [\det(A_j)]^{-1} - m^n \left[\det\left(\sum_{j=1}^{m} A_j \right) \right]^{-1} \right]$$
$$\le \sum_{j=1}^{m} p_j [\det(A_j)]^{-1} - \left[\det\left(\sum_{j=1}^{m} p_j A_j \right) \right]^{-1}$$
$$\le m \max_{j\in\{1,...,m\}} \{p_j\} \left[\frac{1}{m} \sum_{j=1}^{m} [\det(A_j)]^{-1} - m^n \left[\det\left(\sum_{j=1}^{m} A_j \right) \right]^{-1} \right],$$

which is equivalent to (3.2). □

Corollary 2. *If A_1, A_2 are positive definite, then*

$$0 \le \frac{1}{4} \left[[\det(A_1)]^{-1} + [\det(A_2)]^{-1} - 2^{n+1} [\det(A_1 + A_2)]^{-1} \right] \tag{3.4}$$
$$\le \frac{[\det(A_1)]^{-1} + [\det(A_2)]^{-1}}{2} - \int_0^1 [\det(tA_1 + (1-t) A_2)]^{-1} dt$$
$$\le \frac{3}{4} \left[[\det(A_1)]^{-1} + [\det(A_2)]^{-1} - 2^{n+1} [\det(A_1 + A_2)]^{-1} \right].$$

Remark 2. *If we take $A_j = B_j^{-1}$, where $B_j > 0$, $j = 1, ..., m$, then by (2.1) we get*

$$0 \le \min_{j\in\{1,...,m\}} \{p_j\} \left[\sum_{j=1}^{m} \det(B_j) - m^{n+1} \left[\det\left(\sum_{j=1}^{m} B_j^{-1} \right) \right]^{-1} \right] \tag{3.5}$$
$$\le \sum_{j=1}^{m} p_j \det(B_j) - \left[\det\left(\sum_{j=1}^{m} p_j B_j^{-1} \right) \right]^{-1}$$
$$\le \max_{j\in\{1,...,m\}} \{p_j\} \left[\sum_{j=1}^{m} \det(B_j) - m^{n+1} \left[\det\left(\sum_{j=1}^{m} B_j^{-1} \right) \right]^{-1} \right].$$

In particular, for $m=2$ and $t\in(0,1)$, we have

$$
\begin{aligned}
(3.6)\qquad & 0\le \min\{t,1-t\}\\
&\times\left[\det(B_1)+\det(B_2)-2^{n+1}\left[\det\left(B_1^{-1}+B_2^{-1}\right)\right]^{-1}\right]\\
&\le t\det(B_1)+(1-t)\det(B_2)-\left[\det\left(tB_1^{-1}+(1-t)B_2^{-1}\right)\right]^{-1}\\
&\le \max\{t,1-t\}\\
&\times\left[\det(B_1)+\det(B_2)-2^{n+1}\left[\det\left(B_1^{-1}+B_2^{-1}\right)\right]^{-1}\right].
\end{aligned}
$$

From (2.7) we also have

$$
\begin{aligned}
(3.7)\qquad & 0\le \frac{1}{4}\left[\det(B_1)+\det(B_2)-2^{n+1}\left[\det\left(B_1^{-1}+B_2^{-1}\right)\right]^{-1}\right]\\
&\le \frac{\det(B_1)+\det(B_2)}{2}-\int_0^1\left[\det\left(tB_1^{-1}+(1-t)B_2^{-1}\right)\right]^{-1}dt\\
&\le \frac{3}{4}\left[\det(B_1)+\det(B_2)-2^{n+1}\left[\det\left(B_1^{-1}+B_2^{-1}\right)\right]^{-1}\right].
\end{aligned}
$$

4. The Case of Hermitian Matrices

A complex square matrix $H=(h_{ij})$, $i,j=1,...,n$ is said to be Hermitian provided $h_{ij}=\overline{h_{ji}}$ for all i, $j=1,...,n$. A Hermitian matrix is said to be positive definite if the Hermitian form $P(z)=\sum_{i,j=1}^{n}a_{ij}z_i\overline{z_j}$ is positive for all $z=(z_1,...,z_n)\in\mathbb{C}^n\setminus\{0\}$.

It is known that, see for instance [9, p. 215], for a positive definite Hermitian matrix H, we have

$$
(4.1)\qquad K_n(H):=\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\exp\left(-\langle\overline{z},Hz\rangle\right)dxdy=\frac{\pi^n}{\det(H)},
$$

where $z=x+iy$ and dx and dy denote integration over real n-dimensional space $\mathbb{R}^n$. Here the inner product $\langle x,y\rangle$ is understood in the real sense, i.e. $\langle x,y\rangle=\sum_{k=1}^{n}x_ky_k$.

On making use of a similar argument to the one in Theorem 2 for the representation $K_n(\cdot)$ we can state the following result as well:

Theorem 3. *Assume that* $(H_j)_{j=1,...,m}$ *are Hermitian positive definite and* $p_j \geq 0$, $j = 1, ..., m$ *with* $\sum_{j=1}^{m} p_j = 1$. *Then*

$$0 \leq \min_{j\in\{1,...,m\}} \{p_j\} \left[\sum_{j=1}^{m} [\det(H_j)]^{-1} - m^{n+1} \left[\det \left(\sum_{j=1}^{m} H_j \right) \right]^{-1} \right] \tag{4.2}$$
$$\leq \sum_{j=1}^{m} p_j [\det(H_j)]^{-1} - \left[\det \left(\sum_{j=1}^{m} p_j H_j \right) \right]^{-1}$$
$$\leq \max_{j\in\{1,...,m\}} \{p_j\} \left[\sum_{j=1}^{m} [\det(H_j)]^{-1} - m^{n+1} \left[\det \left(\sum_{j=1}^{m} H_j \right) \right]^{-1} \right].$$

In particular, for $m = 2$ *and* $t \in (0, 1)$, *we have*

$$0 \leq \min\{t, 1-t\} \times \left[[\det(H_1)]^{-1} + [\det(H_2)]^{-1} - 2^{n+1} [\det(H_1 + H_2)]^{-1} \right] \tag{4.3}$$
$$\leq t [\det(H_1)]^{-1} + (1-t) [\det(H_2)]^{-1} - [\det(tH_1 + (1-t) H_2)]^{-1}$$
$$\leq \max\{t, 1-t\} \times \left[[\det(H_1)]^{-1} + [\det(H_2)]^{-1} - 2^{n+1} [\det(H_1 + H_2)]^{-1} \right].$$

We also have

$$0 \leq \frac{1}{4} \left[[\det(H_1)]^{-1} + [\det(H_2)]^{-1} - 2^{n+1} [\det(H_1 + H_2)]^{-1} \right] \tag{4.4}$$
$$\leq \frac{[\det(H_1)]^{-1} + [\det(H_2)]^{-1}}{2} - \int_0^1 [\det(tH_1 + (1-t) H_2)]^{-1} dt$$
$$\leq \frac{3}{4} \left[[\det(H_1)]^{-1} + [\det(H_2)]^{-1} - 2^{n+1} [\det(H_1 + H_2)]^{-1} \right].$$

Remark 3. *Now, if we take $H_j = K_j^{-1}$, $j = 1, ..., m$ with K_j are Hermitian positive definite matrices, then we have the inequalities*

$$0 \leq \min_{j\in\{1,\ldots,m\}} \{p_j\} \left[\sum_{j=1}^{m} \det(K_j) - m^{n+1} \left[\det\left(\sum_{j=1}^{m} K_j^{-1} \right) \right]^{-1} \right] \tag{4.5}$$
$$\leq \sum_{j=1}^{m} p_j \det(K_j) - \left[\det\left(\sum_{j=1}^{m} p_j K_j^{-1} \right) \right]^{-1}$$
$$\leq \max_{j\in\{1,\ldots,m\}} \{p_j\} \left[\sum_{j=1}^{m} \det(K_j) - m^{n+1} \left[\det\left(\sum_{j=1}^{m} K_j^{-1} \right) \right]^{-1} \right].$$

In particular, for $m = 2$ and $t \in (0, 1)$, we have

$$0 \leq \min\{t, 1-t\} \tag{4.6}$$
$$\times \left[\det(K_1) + \det(K_2) - 2^{n+1} \left[\det\left(K_1^{-1} + K_2^{-1}\right) \right]^{-1} \right]$$
$$\leq t \det(K_1) + (1-t)\det(K_2) - \left[\det\left(tK_1^{-1} + (1-t) K_2^{-1}\right) \right]^{-1}$$
$$\leq \max\{t, 1-t\}$$
$$\times \left[\det(K_1) + \det(K_2) - 2^{n+1} \left[\det\left(K_1^{-1} + K_2^{-1}\right) \right]^{-1} \right].$$

We also have

$$0 \leq \frac{1}{4} \left[\det(K_1) + \det(K_2) - 2^{n+1} \left[\det\left(K_1^{-1} + K_2^{-1}\right) \right]^{-1} \right] \tag{4.7}$$
$$\leq \frac{\det(K_1) + \det(K_2)}{2} - \int_0^1 \left[\det\left(tK_1^{-1} + (1-t) K_2^{-1}\right) \right]^{-1} dt$$
$$\leq \frac{3}{4} \left[\det(K_1) + \det(K_2) - 2^{n+1} \left[\det\left(K_1^{-1} + K_2^{-1}\right) \right]^{-1} \right].$$

References

[1] E. F. Beckenbach and R. Bellman, Inequalities, Berlin-Heidelberg-New York, 1971.
[2] S. S. Dragomir, Bounds for the normalized Jensen functional, *Bull. Austral. Math. Soc.* **74** (3) (2006), 471-476.
[3] Y. Li, L. Yongtao, Z. Huang Feng and W. Liu, Inequalities regarding partial trace and partial determinant. *Math. Inequal. Appl.* **23** (2020), no. 2, 477–485.
[4] M. Lin and G. Sinnamon, Revisiting a sharpened version of Hadamard's determinant inequality. *Linear Algebra Appl.* **606** (2020), 192–200
[5] J.-T. Liu, Q.-W. Wang and F.-F. Sun, Determinant inequalities for Hadamard product of positive definite matrices. *Math. Inequal. Appl.* **20** (2017), no. 2, 537–542.
[6] W. Luo, Further extensions of Hartfiel's determinant inequality to multiple matrices. Spec. Matrices 9 (2021), 78–82.
[7] M. Ito, Estimations of the weighted power mean by the Heron mean and related inequalities for determinants and traces. *Math. Inequal. Appl.* **22** (2019), no. 3, 949–966.

[8] L. Mirsky, An inequality for positive definite matricies, *Amer. Math. Monthly*, **62** (1955), 428-430.

[9] D. S. Mitrinović, J. E. Pečarić and A.M. Fink, Classical and New Inequalities in Analysis, Kluwer Acedemi Publishers, 1993

[1]Mathematics, College of Engineering & Science, Victoria University, PO Box 14428, Melbourne City, MC 8001, Australia.

E-mail address: sever.dragomir@vu.edu.au

URL: http://rgmia.org/dragomir

[2]DST-NRF Centre of Excellence in the Mathematical, and Statistical Sciences, School of Computer Science, & Applied Mathematics, University of the Witwatersrand,, Private Bag 3, Johannesburg 2050, South Africa