DETERMINANT INEQUALITIES FOR POSITIVE DEFINITE MATRICES VIA ČEBYŠEV'S AND GRÜSS' RESULTS

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Abstract. In this paper we prove among others that, if $(A_j)_{j=1,\ldots,m}$ and $(B_j)_{j=1,\ldots,m}$ are positive definite and have the same monotonicity in the operator order and $p_j \geq 0$, $j = 1, ..., m$ with $\sum_{j=1}^m p_j = 1$, then

$$\sum_{j=1}^{m} (1 - p_j) p_j [\det (A_j + B_j)]^{-1} \geq \sum_{1 \leq j \neq k \leq m} p_j p_k [\det (A_j + B_k)]^{-1}.$$

In particular, for $p_j = 1/m$, $j = 1, ..., m$ we get

$$(m - 1) \sum_{j=1}^{m} [\det (A_j + B_j)]^{-1} \geq \sum_{1 \leq j \neq k \leq m} [\det (A_j + B_k)]^{-1}.$$

1. Introduction

A real square matrix $A = (a_{ij})$, $i, j = 1, ..., n$ is *symmetric* provided $a_{ij} = a_{ji}$ for all $i, j = 1, ..., n$. A real symmetric matrix is said to be *positive definite* provided the quadratic form $Q(x) = \sum_{i,j=1}^n a_{ij} x_i x_j$ is positive for all $x = (x_1, ..., x_n) \in \mathbb{R}^n \setminus \{0\}$. It is well known that a necessary and sufficient condition for the symmetric matrix A to be positive definite, and we write $A > 0$, is that all determinants

$$\det(A_k) = \det (a_{ij}),\ i,\ j = 1, ..., k;\ k = 1, ..., n$$

are positive.

It is know that the following integral representation is valid, see [1, pp. 61-62] or [8, pp. 211-212]

$$J_n(A) := \int_{\mathbb{R}^n} \exp(- \langle Ax, x\rangle) dx := \int_{-\infty}^{\infty} ... \int_{-\infty}^{\infty} \exp(- \langle Ax, x\rangle) dx = \frac{\pi^{n/2}}{[\det (A)]^{1/2}}, \tag{1.1}$$

where A is a positive definite matrix of order n and $\langle \cdot, \cdot\rangle$ is the usual inner product on $\mathbb{R}^n$.

By utilizing the representation (1.1) and Hölder's integral inequality for multiple integrals one can prove the *logarithmic concavity* of the determinant that is due to Ky Fan ([1, p. 63] or [8, p. 212]), namely

$$\det ((1 - \lambda) A + \lambda B) \geq [\det (A)]^{1-\lambda} [\det (B)]^{\lambda} \tag{1.2}$$

for any positive definite matrices A, B and $\lambda \in [0, 1]$.

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By mathematical induction we can get a generalization of (1.2) which was obtained by L. Mirsky in [7], see also [8, p. 212]

$$\det\left(\sum_{j=1}^{m}\lambda_j A_j\right)\geq\prod_{j=1}^{m}\left[\det\left(A_j\right)\right]^{\lambda_j},\ m\geq 2, \tag{1.3}$$

where $\lambda_j>0$, $j=1,...,m$ with $\sum_{j=1}^{m}\lambda_j=1$ and $A_j>0$, $j=1,...,m$.

If we write (1.3) for $A_j=B_j^{-1}$ we get

$$\det\left(\sum_{j=1}^{m}\lambda_j B_j^{-1}\right)\geq\prod_{j=1}^{m}\left[\det\left(B_j^{-1}\right)\right]^{\lambda_j}=\left(\prod_{j=1}^{m}\left[\det\left(B_j\right)\right]^{\lambda_j}\right)^{-1},$$

which also gives

$$\prod_{j=1}^{m}\left[\det\left(A_j\right)\right]^{\lambda_j}\geq\det\left[\left(\sum_{j=1}^{m}\lambda_j A_j^{-1}\right)^{-1}\right], \tag{1.4}$$

where $\lambda_j>0$, $j=1,...,m$ with $\sum_{j=1}^{m}\lambda_j=1$ and $A_j>0$, $j=1,...,m$.

Using the representation (1.1) one can also prove the result, see [8, p. 212],

$$\det\left(A\right)=\det\left(A_{1n}\right)\leq\det\left(A_{1k}\right)\det\left(A_{(k+1)n}\right),\ k=1,...,n; \tag{1.5}$$

where the determinant $\det\left(A_{rs}\right)$ is defined by

$$\det\left(A_{rs}\right)=\det\left(a_{ij}\right),i,\ j=r,...,s.$$

In particular,

$$\det\left(A\right)\leq a_{11}a_{22}...a_{nn}. \tag{1.6}$$

We recall also the Minkowski's type inequality,

$$\left[\det\left(A+B\right)\right]^{1/n}\geq\left[\det\left(A\right)\right]^{1/n}+\left[\det\left(B\right)\right]^{1/n} \tag{1.7}$$

for A, B positive definite matrices of order n. For other determinant inequalities see Chapter VIII of the classic book [8]. For some recent results see [2]-[6].

Motivated by the above results, in this paper we prove among others that, if $\left(A_j\right)_{j=1,...,m}$ and $\left(B_j\right)_{j=1,...,m}$ are positive definite and have the same monotonicity in the operator order and $p_j\geq 0$, $j=1,...,m$ with $\sum_{j=1}^{m}p_j=1$, then

$$\sum_{j=1}^{m}\left(1-p_j\right)p_j\left[\det\left(A_j+B_j\right)\right]^{-1}\geq\sum_{1\leq j\neq k\leq m}p_jp_k\left[\det\left(A_j+B_k\right)\right]^{-1}.$$

In particular, for $p_j=1/m$, $j=1,...,m$ we get

$$\left(m-1\right)\sum_{j=1}^{m}\left[\det\left(A_j+B_j\right)\right]^{-1}\geq\sum_{1\leq j\neq k\leq m}\left[\det\left(A_j+B_k\right)\right]^{-1}.$$

2. Čebyšev Type Inequalities

We say that $(A_j)_{j=1,...,m}$ is *monotonic nondecreasing (nonincreasing)* in the operator order if $A_1 \leq (\geq) A_2 \leq (\geq) ... \leq (\geq) A_n$, namely

$$\langle A_1 x, x\rangle \leq (\geq) \langle A_2 x, x\rangle \leq (\geq) ... \leq (\geq) \langle A_n x, x\rangle$$

for all $x \in \mathbb{R}^n$.

The first main result is as follows:

Theorem 1. *Assume that $(A_j)_{j=1,...,m}$ and $(B_j)_{j=1,...,m}$ are positive definite and have the same monotonicity in the operator order and $p_j \geq 0$, $j = 1, ..., m$ with $\sum_{j=1}^m p_j = 1$. Then*

$$\sum_{j=1}^m (1 - p_j) p_j [\det(A_j + B_j)]^{-1/2} \geq \sum_{1\leq j\neq k\leq m} p_j p_k [\det(A_j + B_k)]^{-1/2}. \tag{2.1}$$

In particular, for $p_j = 1/m$, $j = 1, ..., m$ we get

$$(m-1)\sum_{j=1}^m [\det(A_j + B_j)]^{-1/2} \geq \sum_{1\leq j\neq k\leq m} [\det(A_j + B_k)]^{-1/2}. \tag{2.2}$$

If $(A_j)_{j=1,...,m}$ and $(B_j)_{j=1,...,m}$ have oposite monotonicity in the operator order, then the inequalities in (2.1) and (2.2) reverse.

Proof. We use the weighted Čebyšev inequality

$$\sum_{j=1}^m p_j a_j b_j \geq \sum_{j=1}^m p_j a_j \sum_{j=1}^m p_j b_j \tag{2.3}$$

provided that $p_j \geq 0$, $j = 1, ..., m$ with $\sum_{j=1}^m p_j = 1$ and $(a_j)_{j=1,...,n}$, $(b_j)_{j=1,...,n}$ have the same monotonicity, namely they are either both increasing or both decreasing. If they have opposite monotonicity, the sign of inequality reverses.

The inequality (2.3) can also be written as

$$\sum_{j=1}^m p_j a_j b_j \geq \sum_{j=1}^m \sum_{k=1}^m p_j p_k a_j b_k. \tag{2.4}$$

Assume that $A_1 \leq A_2 \leq ... \leq A_n$ and $B_1 \leq B_2 \leq ... \leq B_n$. Then for $x \in \mathbb{R}^n$ we have that $\langle A_j x, x\rangle$, and $\langle B_j x, x\rangle$ are both increasing. Therefore $\exp(-\langle A_j x, x\rangle)$ and $\exp(-\langle B_j x, x\rangle)$, $j = 1, ..., m$ are both decreasing and by (2.4) we get

$$\begin{aligned}&\sum_{j=1}^m p_j \exp(-\langle A_j x, x\rangle)\exp(-\langle B_j x, x\rangle)\\ &\geq \sum_{j=1}^m \sum_{k=1}^m p_j p_k \exp(-\langle A_j x, x\rangle)\exp(-\langle B_k x, x\rangle),\end{aligned}$$

namely

$$\sum_{j=1}^m p_j \exp(-\langle (A_j + B_j) x, x\rangle) \geq \sum_{j=1}^m \sum_{k=1}^m p_j p_k \exp(-\langle (A_j + B_k) x, x\rangle), \tag{2.5}$$

for $x \in \mathbb{R}^n$.

Now if we take the integral over $x \in \mathbb{R}^n$ in (2.5), then we get

$$\sum_{j=1}^{m} p_j \int_{\mathbb{R}^n} \exp\left(-\left\langle \left(A_j + B_j\right) x, x\right\rangle\right) dx \geq \sum_{j=1}^{m}\sum_{k=1}^{m} p_j p_k \int_{\mathbb{R}^n} \exp\left(-\left\langle \left(A_j + B_k\right) x, x\right\rangle\right) dx, \tag{2.6}$$

which by (1.1) we get

$$\sum_{j=1}^{m} p_j J_n\left(A_j + B_j\right) \geq \sum_{j=1}^{m}\sum_{k=1}^{m} p_j p_k J_n\left(A_j + B_k\right),$$

namely

$$\begin{aligned}
&\sum_{j=1}^{m} \frac{p_j}{\left[\det\left(A_j + B_j\right)\right]^{1/2}} \\
&\geq \sum_{j=1}^{m}\sum_{k=1}^{m} \frac{p_j p_k}{\left[\det\left(A_j + B_k\right)\right]^{1/2}} \\
&= \sum_{j=1}^{m} \frac{p_j^2}{\left[\det\left(A_j + B_j\right)\right]^{1/2}} + \sum_{1\leq j\neq k\leq m} \frac{p_j p_k}{\left[\det\left(A_j + B_k\right)\right]^{1/2}}.
\end{aligned}$$

This implies that

$$\sum_{j=1}^{m} \frac{\left(1 - p_j\right) p_j}{\left[\det\left(A_j + B_j\right)\right]^{1/2}} \geq \sum_{1\leq j\neq k\leq m} \frac{p_j p_k}{\left[\det\left(A_j + B_k\right)\right]^{1/2}}$$

and the inequality (2.1) is proved. □

If we take the square in the representation (1.1), then we get

$$\left(\int_{\mathbb{R}^n} \exp(-\langle x, Ax\rangle) dx\right)^2 = \frac{\pi^n}{\det(A)}.$$

Since

$$\begin{aligned}
\left(\int_{\mathbb{R}^n} \exp(-\langle Ax, x\rangle) dx\right)^2 &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \exp(-\langle Ax, x\rangle)\exp(-\langle Ay, y\rangle) dx dy \\
&= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \exp(-\langle Ax, x\rangle - \langle Ay, y\rangle) dx dy,
\end{aligned}$$

hence

$$K_n(A) := \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \exp(-\langle Ax, x\rangle - \langle Ay, y\rangle) dx dy = \frac{\pi^n}{\det(A)} \tag{2.7}$$

for A a positive definite matrix of order n and $\langle\cdot,\cdot\rangle$ is the usual inner product on $\mathbb{R}^n$.

We also have:

Theorem 2. *Assume that* $\left(A_j\right)_{j=1,...,m}$ *and* $\left(B_j\right)_{j=1,...,m}$ *are positive definite and have the same monotonicity in the operator order and* $p_j \geq 0$, $j = 1, ..., m$ *with*

$\sum_{j=1}^{m} p_j = 1$. *Then*

$$\sum_{j=1}^{m} (1-p_j)\, p_j \left[\det\left(A_j+B_j\right)\right]^{-1} \geq \sum_{1\leq j\neq k\leq m} p_j p_k \left[\det\left(A_j+B_k\right)\right]^{-1}. \tag{2.8}$$

In particular, for $p_j = 1/m$, $j=1,...,m$ *we get*

$$(m-1)\sum_{j=1}^{m} \left[\det\left(A_j+B_j\right)\right]^{-1} \geq \sum_{1\leq j\neq k\leq m} \left[\det\left(A_j+B_k\right)\right]^{-1}. \tag{2.9}$$

If $(A_j)_{j=1,...,m}$ *and* $(B_j)_{j=1,...,m}$ *have oposite monotonicity in the operator order, then the inequalities in (2.8) and (2.9) reverse.*

Proof. Assume that $A_1 \leq A_2 \leq ... \leq A_n$ and $B_1 \leq B_2 \leq ... \leq B_n$. Then for x, $y \in \mathbb{R}^n$ we have that $\langle A_j x, x\rangle + \langle A_j y, y\rangle$, and $\langle B_j x, x\rangle + \langle B_j y, y\rangle$ are both increasing. Therefore $\exp\left(-\langle A_j x, x\rangle - \langle A_j y, y\rangle\right)$ and $\exp\left(-\langle B_j x, x\rangle - \langle B_j y, y\rangle\right)$, $j=1,...,m$ are both decreasing and by (2.4) we get

$$\begin{aligned}
&\sum_{j=1}^{m} p_j \exp\left(-\langle A_j x, x\rangle - \langle A_j y, y\rangle\right) \exp\left(-\langle B_j x, x\rangle - \langle B_j y, y\rangle\right) \\
&\geq \sum_{j=1}^{m}\sum_{k=1}^{m} p_j p_k \exp\left(-\langle A_j x, x\rangle - \langle A_j y, y\rangle\right) \exp\left(-\langle B_k x, x\rangle - \langle B_k y, y\rangle\right),
\end{aligned}$$

namely

$$\begin{aligned}
&\sum_{j=1}^{m} p_j \exp\left(-\langle (A_j+B_j)\, x, x\rangle - \langle (A_j+B_j)\, y, y\rangle\right) \\
&\geq \sum_{j=1}^{m}\sum_{k=1}^{m} p_j p_k \exp\left(-\langle (A_j+B_k)\, x, x\rangle - \langle (A_j+B_k)\, y, y\rangle\right),
\end{aligned}$$

for x, $y \in \mathbb{R}^n$.

We have by integration that

$$\begin{aligned}
&\sum_{j=1}^{m} p_j \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \exp\left(-\langle (A_j+B_j)\, x, x\rangle - \langle (A_j+B_j)\, y, y\rangle\right) dxdy \\
&\geq \sum_{j=1}^{m}\sum_{k=1}^{m} p_j p_k \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \exp\left(-\langle (A_j+B_k)\, x, x\rangle - \langle (A_j+B_k)\, y, y\rangle\right) dxdy.
\end{aligned}$$

By making use of (2.7), we get

$$\sum_{j=1}^{m} p_j \frac{\pi^n}{\det\left(A_j+B_j\right)} \geq \sum_{j=1}^{m}\sum_{k=1}^{m} p_j p_k \frac{\pi^n}{\det\left(A_j+B_k\right)},$$

namely

$$\begin{aligned}
\sum_{j=1}^{m} \frac{p_j}{\det\left(A_j+B_j\right)} &\geq \sum_{j=1}^{m}\sum_{k=1}^{m} \frac{p_j p_k}{\det\left(A_j+B_k\right)} \\
&= \sum_{j=1}^{m} \frac{p_j^2}{\det\left(A_j+B_j\right)} + \sum_{1\leq j\neq k\leq m} \frac{p_j p_k}{\det\left(A_j+B_k\right)},
\end{aligned}$$

which is equivalent to (2.8). □

Remark 1. *If $0 < A_1 \leq A_2$ and $0 < B_1 \leq B_2$ in the operator order, then by taking $m = 2$ and $p_1 = p$, $p_2 = 1 - p$, with $p \in (0,1)$ in (2.1) we have*

$$\begin{aligned} &[\det(A_1 + B_1)]^{-1/2} + [\det(A_2 + B_2)]^{-1/2} \\ &\geq [\det(A_1 + B_2)]^{-1/2} + [\det(A_2 + B_1)]^{-1/2}. \end{aligned} \tag{2.10}$$

Also, by (2.8), we get

$$\begin{aligned} &[\det(A_1 + B_1)]^{-1} + [\det(A_2 + B_2)]^{-1} \\ &\geq [\det(A_1 + B_2)]^{-1} + [\det(A_2 + B_1)]^{-1}. \end{aligned} \tag{2.11}$$

3. Grüss' Type Inequalities

We have the following Grüss' type inequalities as well:

Theorem 3. *Assume that $0 < m_1 I_n \leq A_j \leq M_1 I_n$ and $0 < m_2 I_n \leq B_j \leq M_2 I_n$ for $j = 1, ..., m$. If $p_j \geq 0$, $j = 1, ..., m$ with $\sum_{j=1}^m p_j = 1$, then*

$$\begin{aligned} &\left| \sum_{j=1}^m (1 - p_j) p_j [\det(A_j + B_j)]^{-1/2} - \sum_{1 \leq j \neq k \leq m} p_j p_k [\det(A_j + B_k)]^{-1/2} \right| \\ &\leq \frac{1}{4} \left[(m_1 + m_2)^{-n} + (M_1 + M_2)^{-n} - (M_1 + m_2)^{-n} - (M_2 + m_1)^{-m} \right]. \end{aligned} \tag{3.1}$$

In particular, for $p_j = 1/m$, $j = 1, ..., m$ we get

$$\begin{aligned} &\left| (m - 1) \sum_{j=1}^m [\det(A_j + B_j)]^{-1/2} - \sum_{1 \leq j \neq k \leq m} [\det(A_j + B_k)]^{-1/2} \right| \\ &\leq \frac{1}{4} m^2 \left[(m_1 + m_2)^{-n} + (M_1 + M_2)^{-n} - (M_1 + m_2)^{-n} - (M_2 + m_1)^{-m} \right]. \end{aligned} \tag{3.2}$$

Proof. We use the Grüss' type inequality

$$\left| \sum_{j=1}^m p_j a_j b_j - \sum_{j=1}^m p_j a_j \sum_{j=1}^m p_j b_j \right| \leq \frac{1}{4} (Q_1 - q_1)(Q_2 - q_2) \tag{3.3}$$

provided $q_1 \leq a_j \leq Q_1$, $q_2 \leq b_j \leq Q_2$ and $p_j \geq 0$, $j = 1, ..., m$ with $\sum_{j=1}^m p_j = 1$.

The inequality (3.3) can be written as

$$\left| \sum_{j=1}^m p_j a_j b_j - \sum_{j=1}^m \sum_{k=1}^m p_j p_k a_j b_k \right| \leq \frac{1}{4} (Q_1 - q_1)(Q_2 - q_2). \tag{3.4}$$

Since $0 < m_1 I_n \leq A_j \leq M_1 I_n$ and $0 < m_2 I_n \leq B_j \leq M_2 I_n$ for $j = 1, ..., m$, then we have for $x \in \mathbb{R}^n$ that

$$q_1 = \exp\left(-M_1 \|x\|^2\right) \leq a_j := \exp(-\langle A_j x, x \rangle) \leq \exp\left(-m_1 \|x\|^2\right) = Q_1$$

and

$$q_2 = \exp\left(-M_2 \|x\|^2\right) \leq b_j := \exp(-\langle B_j x, x \rangle) \leq \exp\left(-m_2 \|x\|^2\right) = Q_2$$

for $j = 1, ..., m$.

By utilising the inequality (3.4) we derive

$$
\begin{aligned}
&\left|\sum_{j=1}^{m} p_j \exp\left(-\left\langle \left(A_j+B_j\right)x,x\right\rangle\right)-\sum_{j=1}^{m}\sum_{k=1}^{m} p_j p_k \exp\left(-\left\langle \left(A_j+B_k\right)x,x\right\rangle\right)\right| \\
&\leq \frac{1}{4}\left(\exp\left(-m_1\left\|x\right\|^2\right)-\exp\left(-M_1\left\|x\right\|^2\right)\right) \\
&\times\left(\exp\left(-m_2\left\|x\right\|^2\right)-\exp\left(-M_2\left\|x\right\|^2\right)\right) \\
&=\frac{1}{4}\left[\exp\left(-\left(m_1+m_2\right)\left\|x\right\|^2\right)+\exp\left(-\left(M_1+M_2\right)\left\|x\right\|^2\right)\right. \\
&\left.-\exp\left(-\left(M_1+m_2\right)\left\|x\right\|^2\right)-\exp\left(-\left(M_2+m_1\right)\left\|x\right\|^2\right)\right]
\end{aligned}
\tag{3.5}
$$

for $x\in\mathbb{R}^n$.

If we take the integral on $\mathbb{R}^n$, then we get

$$
\begin{aligned}
&\left|\sum_{j=1}^{m} p_j \int_{\mathbb{R}^n}\exp\left(-\left\langle \left(A_j+B_j\right)x,x\right\rangle\right)dx\right. \\
&\left.-\sum_{j=1}^{m}\sum_{k=1}^{m} p_j p_k \int_{\mathbb{R}^n}\exp\left(-\left\langle \left(A_j+B_k\right)x,x\right\rangle\right)dx\right| \\
&\leq \frac{1}{4}\left[\int_{\mathbb{R}^n}\exp\left(-\left(m_1+m_2\right)\left\|x\right\|^2\right)dx+\int_{\mathbb{R}^n}\exp\left(-\left(M_1+M_2\right)\left\|x\right\|^2\right)dx\right. \\
&\left.-\int_{\mathbb{R}^n}\exp\left(-\left(M_1+m_2\right)\left\|x\right\|^2\right)dx-\int_{\mathbb{R}^n}\exp\left(-\left(M_2+m_1\right)\left\|x\right\|^2\right)dx\right],
\end{aligned}
$$

which is equivalent to

$$
\begin{aligned}
&\left|\sum_{j=1}^{m} p_j J_n\left(A_j+B_j\right)-\sum_{j=1}^{m}\sum_{k=1}^{m} p_j p_k J_n\left(A_j+B_k\right)\right| \\
&\leq \frac{1}{4}\left[J_n\left(\left(m_1+m_2\right)I_n\right)+J_n\left(\left(M_1+M_2\right)I_n\right)\right. \\
&\left.-J_n\left(\left(M_1+m_2\right)I_n\right)-J_n\left(\left(M_2+m_1\right)I_n\right)\right],
\end{aligned}
$$

namely

$$
\begin{aligned}
&\left|\sum_{j=1}^{m}\frac{\left(1-p_j\right)p_j}{\left[\det\left(A_j+B_j\right)\right]^{1/2}}-\sum_{1\leq j\neq k\leq m}\frac{p_j p_k}{\left[\det\left(A_j+B_k\right)\right]^{1/2}}\right| \\
&\leq \frac{1}{4}\left[\frac{1}{\det\left(\left(m_1+m_2\right)I_n\right)}+\frac{1}{\det\left(\left(M_1+M_2\right)I_n\right)}\right. \\
&\left.-\frac{1}{\det\left(\left(M_1+m_2\right)I_n\right)}-\frac{1}{\det\left(\left(M_2+m_1\right)I_n\right)}\right] \\
&=\frac{1}{4}\left[\left(m_1+m_2\right)^{-n}+\left(M_1+M_2\right)^{-n}-\left(M_1+m_2\right)^{-n}-\left(M_2+m_1\right)^{-m}\right],
\end{aligned}
$$

which proves the desired result (3.1). □

By utilising a similar argument to the one in the proof of Theorem 3 and making use of the representation (2.7) we can also state that

Theorem 4. *Assume that* $0 < m_1 I_n \leq A_j \leq M_1 I_n$ *and* $0 < m_2 I_n \leq B_j \leq M_2 I_n$ *for* $j = 1, ..., m$. *If* $p_j \geq 0$, $j = 1, ..., m$ *with* $\sum_{j=1}^{m} p_j = 1$, *then*

$$\left| \sum_{j=1}^{m} (1 - p_j) p_j \left[\det (A_j + B_j)\right]^{-1} - \sum_{1 \leq j \neq k \leq m} p_j p_k \left[\det (A_j + B_k)\right]^{-1} \right| \leq \frac{1}{4} \left[(m_1 + m_2)^{-n} + (M_1 + M_2)^{-n} - (M_1 + m_2)^{-n} - (M_2 + m_1)^{-m} \right]. \tag{3.6}$$

In particular,

$$\left| (m - 1) \sum_{j=1}^{m} \left[\det (A_j + B_j)\right]^{-1} - \sum_{1 \leq j \neq k \leq m} \left[\det (A_j + B_k)\right]^{-1} \right| \leq \frac{1}{4} m^2 \left[(m_1 + m_2)^{-n} + (M_1 + M_2)^{-n} - (M_1 + m_2)^{-n} - (M_2 + m_1)^{-m} \right]. \tag{3.7}$$

Remark 2. *Assume that* $0 < m_1 I_n \leq A_1$, $A_2 \leq M_1 I_n$ *and* $0 < m_2 I_n \leq B_1$, $B_2 \leq M_2 I_n$. *Then by (3.1) we have*

$$\Big| \left[\det (A_1 + B_1)\right]^{-1/2} + \left[\det (A_2 + B_2)\right]^{-1/2} - \left[\det (A_1 + B_2)\right]^{-1/2} - \left[\det (A_2 + B_1)\right]^{-1/2} \Big| \leq \frac{1}{4} \left[(m_1 + m_2)^{-n} + (M_1 + M_2)^{-n} - (M_1 + m_2)^{-n} - (M_2 + m_1)^{-m} \right]. \tag{3.8}$$

By utilising (3.6) we get

$$\Big| \left[\det (A_1 + B_1)\right]^{-1} + \left[\det (A_2 + B_2)\right]^{-1} - \left[\det (A_1 + B_2)\right]^{-1} - \left[\det (A_2 + B_1)\right]^{-1} \Big| \leq \frac{1}{4} \left[(m_1 + m_2)^{-n} + (M_1 + M_2)^{-n} - (M_1 + m_2)^{-n} - (M_2 + m_1)^{-m} \right]. \tag{3.9}$$

4. The Case of Hermitian Matrices

A complex square matrix $H = (h_{ij})$, $i, j = 1, ..., n$ is said to be Hermitian provided $h_{ij} = \overline{h_{ji}}$ for all i, $j = 1, ..., n$. A Hermitian matrix is said to be positive definite if the Hermitian form $P(z) = \sum_{i,j=1}^{n} a_{ij} z_i \overline{z_j}$ is positive for all $z = (z_1, ..., z_n) \in \mathbb{C}^n \setminus \{0\}$.

It is known that, see for instance [8, p. 215], for a positive definite Hermitian matrix H, we have

$$K_n(H) := \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \exp\left(-\langle \overline{z}, Hz \rangle\right) dx dy = \frac{\pi^n}{\det(H)}, \tag{4.1}$$

where $z = x + iy$ and dx and dy denote integration over real n-dimensional space $\mathbb{R}^n$. Here the inner product $\langle x, y \rangle$ is understood in the real sense, i.e. $\langle x, y \rangle = \sum_{k=1}^{n} x_k y_k$.

By a similar argument to the one in Theorem 2 we can prove that:

Theorem 5. *Assume that* $(H_j)_{j=1,...,m}$ *and* $(K_j)_{j=1,...,m}$ *are positive definite and have the same monotonicity in the operator order and* $p_j \geq 0$, $j = 1, ..., m$ *with*

$\sum_{j=1}^{m} p_j = 1$. *Then*

$$\sum_{j=1}^{m} (1-p_j)\, p_j \left[\det (H_j + K_j)\right]^{-1} \geq \sum_{1\leq j\neq k\leq m} p_j p_k \left[\det (H_j + K_k)\right]^{-1}. \tag{4.2}$$

In particular, for $p_j = 1/m$, $j = 1, ..., m$ we get

$$(m-1) \sum_{j=1}^{m} \left[\det (H_j + K_j)\right]^{-1} \geq \sum_{1\leq j\neq k\leq m} \left[\det (H_j + K_k)\right]^{-1}. \tag{4.3}$$

If $(H_j)_{j=1,...,m}$ and $(K_j)_{j=1,...,m}$ have oposite monotonicity in the operator order, then the inequalities in (4.2) and (4.3) reverse.

Finally, we also have:

Theorem 6. *Assume that $0 < m_1 I_n \leq H_j \leq M_1 I_n$ and $0 < m_2 I_n \leq K_j \leq M_2 I_n$ for $j = 1, ..., m$. If $p_j \geq 0$, $j = 1, ..., m$ with $\sum_{j=1}^{m} p_j = 1$, then*

$$\begin{aligned} &\left| \sum_{j=1}^{m} (1-p_j)\, p_j \left[\det (H_j + K_j)\right]^{-1} - \sum_{1\leq j\neq k\leq m} p_j p_k \left[\det (H_j + K_k)\right]^{-1} \right| \\ &\leq \frac{1}{4} \left[(m_1 + m_2)^{-n} + (M_1 + M_2)^{-n} - (M_1 + m_2)^{-n} - (M_2 + m_1)^{-m} \right]. \end{aligned} \tag{4.4}$$

In particular,

$$\begin{aligned} &\left| (m-1) \sum_{j=1}^{m} \left[\det (A_j + B_j)\right]^{-1} - \sum_{1\leq j\neq k\leq m} \left[\det (A_j + B_k)\right]^{-1} \right| \\ &\leq \frac{1}{4} m^2 \left[(m_1 + m_2)^{-n} + (M_1 + M_2)^{-n} - (M_1 + m_2)^{-n} - (M_2 + m_1)^{-m} \right]. \end{aligned} \tag{4.5}$$

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