DETERMINANT INEQUALITIES FOR POSITIVE DEFINITE MATRICES VIA HÖLDER'S WEIGHTED INEQUALITY

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Abstract. In this paper we prove among others that, if $\lambda_j > 0$, A_j, $B_j > 0$ for $j = 1, ..., m$ and p, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, then

$$\sum_{j=1}^{m} \lambda_j \left[\det\left(A_j + B_j\right)\right]^{-1} \leq \frac{1}{p^{n/p} q^{n/q}} \left(\sum_{j=1}^{m} \lambda_j \left[\det\left(A_j\right)\right]^{-1}\right)^{1/p} \left(\sum_{j=1}^{m} \lambda_j \left[\det\left(B_j\right)\right]^{-1}\right)^{1/q}.$$

In particular, for $p = q = 2$ we get

$$\left(\sum_{j=1}^{m} \lambda_j \left[\det\left(A_j + B_j\right)\right]^{-1}\right)^2 \leq \frac{1}{2^n} \sum_{j=1}^{m} \lambda_j \left[\det\left(A_j\right)\right]^{-1} \sum_{j=1}^{m} \lambda_j \left[\det\left(B_j\right)\right]^{-1}.$$

1. Introduction

A real square matrix $A = (a_{ij})$, $i, j = 1, ..., n$ is *symmetric* provided $a_{ij} = a_{ji}$ for all $i, j = 1, ..., n$. A real symmetric matrix is said to be *positive definite* provided the quadratic form $Q(x) = \sum_{i,j=1}^{n} a_{ij} x_i x_j$ is positive for all $x = (x_1, ..., x_n) \in \mathbb{R}^n \setminus \{0\}$. It is well known that a necessary and sufficient condition for the symmetric matrix A to be positive definite, and we write $A > 0$, is that all determinants

$$\det(A_k) = \det(a_{ij}), \; i, \; j = 1, ..., k; \; k = 1, ..., n$$

are positive.

It is know that the following integral representation is valid, see [1, pp. 61-62] or [8, pp. 211-212]

$$J_n(A) := \int_{\mathbb{R}^n} \exp(-\langle Ax, x\rangle) dx := \int_{-\infty}^{\infty} ... \int_{-\infty}^{\infty} \exp(-\langle Ax, x\rangle) dx = \frac{\pi^{n/2}}{\left[\det(A)\right]^{1/2}}, \tag{1.1}$$

where A is a positive definite matrix of order n and $\langle \cdot, \cdot \rangle$ is the usual inner product on $\mathbb{R}^n$.

By utilizing the representation (1.1) and Hölder's integral inequality for multiple integrals one can prove the *logarithmic concavity* of the determinant that is due to Ky Fan ([1, p. 63] or [8, p. 212]), namely

$$\det((1-\lambda) A + \lambda B) \geq \left[\det(A)\right]^{1-\lambda} \left[\det(B)\right]^{\lambda} \tag{1.2}$$

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for any positive definite matrices A, B and $\lambda \in [0,1]$.

By mathematical induction we can get a generalization of (1.2) which was obtained by L. Mirsky in [7], see also [8, p. 212]

$$\det\left(\sum_{j=1}^{m}\lambda_j A_j\right) \geq \prod_{j=1}^{m}\left[\det\left(A_j\right)\right]^{\lambda_j},\ m\geq 2, \tag{1.3}$$

where $\lambda_j > 0$, $j = 1,...,m$ with $\sum_{j=1}^{m}\lambda_j = 1$ and $A_j > 0$, $j = 1,...,m$.

If we write (1.3) for $A_j = B_j^{-1}$ we get

$$\det\left(\sum_{j=1}^{m}\lambda_j B_j^{-1}\right) \geq \prod_{j=1}^{m}\left[\det\left(B_j^{-1}\right)\right]^{\lambda_j} = \left(\prod_{j=1}^{m}\left[\det\left(B_j\right)\right]^{\lambda_j}\right)^{-1},$$

which also gives

$$\prod_{j=1}^{m}\left[\det\left(A_j\right)\right]^{\lambda_j} \geq \det\left[\left(\sum_{j=1}^{m}\lambda_j A_j^{-1}\right)^{-1}\right], \tag{1.4}$$

where $\lambda_j > 0$, $j = 1,...,m$ with $\sum_{j=1}^{m}\lambda_j = 1$ and $A_j > 0$, $j = 1,...,m$.

Using the representation (1.1) one can also prove the result, see [8, p. 212],

$$\det(A) = \det(A_{1n}) \leq \det(A_{1k})\det\left(A_{(k+1)n}\right),\ k = 1,...,n; \tag{1.5}$$

where the determinant $\det(A_{rs})$ is defined by

$$\det(A_{rs}) = \det(a_{ij}), i,\ j = r,...,s.$$

In particular,

$$\det(A) \leq a_{11}a_{22}...a_{nn}. \tag{1.6}$$

We recall also the Minkowski's type inequality,

$$\left[\det(A+B)\right]^{1/n} \geq \left[\det(A)\right]^{1/n} + \left[\det(B)\right]^{1/n} \tag{1.7}$$

for A, B positive definite matrices of order n. For other determinant inequalities see Chapter VIII of the classic book [8]. For some recent results see [2]-[6].

Motivated by the above results, in this paper we prove among others that, if $\lambda_j > 0$, A_j, $B_j > 0$ for $j = 1,...,m$ and p, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, then

$$\sum_{j=1}^{m}\lambda_j\left[\det\left(A_j+B_j\right)\right]^{-1}$$
$$\leq \frac{1}{p^{n/p}q^{n/q}}\left(\sum_{j=1}^{m}\lambda_j\left[\det\left(A_j\right)\right]^{-1}\right)^{1/p}\left(\sum_{j=1}^{m}\lambda_j\left[\det\left(B_j\right)\right]^{-1}\right)^{1/q}.$$

In particular, for $p = q = 2$ we get

$$\left(\sum_{j=1}^{m}\lambda_j\left[\det\left(A_j+B_j\right)\right]^{-1}\right)^2 \leq \frac{1}{2^n}\sum_{j=1}^{m}\lambda_j\left[\det\left(A_j\right)\right]^{-1}\sum_{j=1}^{m}\lambda_j\left[\det\left(B_j\right)\right]^{-1}.$$

2. Main Results

We have the first result as follows:

Theorem 1. *Assume that* $\lambda_j > 0$, A_j, $B_j > 0$ *for* $j = 1, ..., m$ *and* p, $q > 1$ *with* $\frac{1}{p} + \frac{1}{q} = 1$. *Then*

$$\sum_{j=1}^{m} \lambda_j \left[\det\left(A_j + B_j\right)\right]^{-1/2} \leq \frac{1}{p^{n/(2p)} q^{n/(2q)}} \left(\sum_{j=1}^{m} \lambda_j \left[\det\left(A_j\right)\right]^{-1/2}\right)^{1/p} \left(\sum_{j=1}^{m} \lambda_j \left[\det\left(B_j\right)\right]^{-1/2}\right)^{1/q}. \tag{2.1}$$

In particular, for $p = q = 2$ *we get*

$$\left(\sum_{j=1}^{m} \lambda_j \left[\det\left(A_j + B_j\right)\right]^{-1/2}\right)^2 \leq \frac{1}{2^n} \left(\sum_{j=1}^{m} \lambda_j \left[\det\left(A_j\right)\right]^{-1/2}\right) \left(\sum_{j=1}^{m} \lambda_j \left[\det\left(B_j\right)\right]^{-1/2}\right). \tag{2.2}$$

Proof. We use Hölder weighted discrete inequality

$$\sum_{j=1}^{m} \lambda_j a_j b_j \leq \left(\sum_{j=1}^{m} \lambda_j a_j^p\right)^{1/p} \left(\sum_{j=1}^{m} \lambda_j b_j^q\right)^{1/q} \tag{2.3}$$

that holds for λ_j, a_j, $b_j \geq 0$, $j = 1, ..., m$ and p, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$.

Let $x \in \mathbb{R}^n$ and take

$$a_j = \exp\left(-\langle A_j x, x\rangle\right) \text{ and } b_j = \exp\left(-\langle B_j x, x\rangle\right),$$

for $j = 1, ..., m$ to get

$$\sum_{j=1}^{m} \lambda_j \exp\left(-\langle \left(A_j + B_j\right) x, x\rangle\right) \leq \left(\sum_{j=1}^{m} \lambda_j \exp\left(-\langle p A_j x, x\rangle\right)\right)^{1/p} \left(\sum_{j=1}^{m} \lambda_j \exp\left(-\langle q B_j x, x\rangle\right)\right)^{1/q}$$

for all $x \in \mathbb{R}^n$.

If we take the integral on $\mathbb{R}^n$, then we get

$$\sum_{j=1}^{m} \lambda_j \int_{\mathbb{R}^n} \exp\left(-\langle \left(A_j + B_j\right) x, x\rangle\right) dx \leq \int_{\mathbb{R}^n} \left(\sum_{j=1}^{m} \lambda_j \exp\left(-\langle p A_j x, x\rangle\right)\right)^{1/p} \times \left(\sum_{j=1}^{m} \lambda_j \exp\left(-\langle q B_j x, x\rangle\right)\right)^{1/q} dx. \tag{2.4}$$

Using the Hölder's integral inequality we have

$$\int_{\mathbb{R}^n}\left(\sum_{j=1}^{m}\lambda_j \exp\left(-\langle pA_jx,x\rangle\right)\right)^{1/p}\left(\sum_{j=1}^{m}\lambda_j \exp\left(-\langle qB_jx,x\rangle\right)\right)^{1/q}dx \tag{2.5}$$
$$\leq\left(\int_{\mathbb{R}^n}\left[\left(\sum_{j=1}^{m}\lambda_j \exp\left(-\langle pA_jx,x\rangle\right)\right)^{1/p}\right]^{p}dx\right)^{1/p}$$
$$\times\left(\int_{\mathbb{R}^n}\left[\left(\sum_{j=1}^{m}\lambda_j \exp\left(-\langle qB_jx,x\rangle\right)\right)^{1/q}\right]^{q}dx\right)^{1/q}$$
$$=\left(\sum_{j=1}^{m}\lambda_j\int_{\mathbb{R}^n}\exp\left(-\langle pA_jx,x\rangle\right)dx\right)^{1/p}$$
$$\times\left(\sum_{j=1}^{m}\lambda_j\int_{\mathbb{R}^n}\exp\left(-\langle qB_jx,x\rangle\right)dx\right)^{1/q},$$

which gives that

$$\sum_{j=1}^{m}\lambda_j\int_{\mathbb{R}^n}\exp\left(-\langle\left(A_j+B_j\right)x,x\rangle\right)dx \tag{2.6}$$
$$\leq\left(\sum_{j=1}^{m}\lambda_j\int_{\mathbb{R}^n}\exp\left(-\langle pA_jx,x\rangle\right)dx\right)^{1/p}$$
$$\times\left(\sum_{j=1}^{m}\lambda_j\int_{\mathbb{R}^n}\exp\left(-\langle qB_jx,x\rangle\right)dx\right)^{1/q}.$$

Now, by (1.1) we get

$$\int_{\mathbb{R}^n}\exp\left(-\langle\left(A_j+B_j\right)x,x\rangle\right)dx=J_n\left(A_j+B_j\right)=\frac{\pi^{n/2}}{\left[\det\left(A_j+B_j\right)\right]^{1/2}},$$

$$\begin{aligned}\int_{\mathbb{R}^n}\exp\left(-\langle pA_jx,x\rangle\right)dx &= J_n\left(pA_j\right)=\frac{\pi^{n/2}}{\left[\det\left(pA_j\right)\right]^{1/2}}\\ &= \frac{\pi^{n/2}}{p^{n/2}\left[\det\left(A_j\right)\right]^{1/2}}\end{aligned}$$

and

$$\begin{aligned}\int_{\mathbb{R}^n}\exp\left(-\langle qB_jx,x\rangle\right)dx &= J_n\left(qB_j\right)=\frac{\pi^{n/2}}{\left[\det\left(qB_j\right)\right]^{1/2}}\\ &= \frac{\pi^{n/2}}{q^{n/2}\left[\det\left(B_j\right)\right]^{1/2}}\end{aligned}$$

and by (2.6), that

$$\begin{aligned}
&\pi^{n/2}\sum_{j=1}^{m}\lambda_j\left[\det\left(A_j+B_j\right)\right]^{-1/2}\\
&\le\left(\sum_{j=1}^{m}\lambda_j\frac{\pi^{n/2}}{p^{n/2}\left[\det\left(A_j\right)\right]^{1/2}}\right)^{1/p}\left(\sum_{j=1}^{m}\lambda_j\frac{\pi^{n/2}}{q^{n/2}\left[\det\left(B_j\right)\right]^{1/2}}\right)^{1/q}\\
&=\frac{\pi^{n/2}}{p^{n/(2p)}q^{n/(2q)}}\left(\sum_{j=1}^{m}\lambda_j\left[\det\left(A_j\right)\right]^{-1/2}\right)^{1/p}\left(\sum_{j=1}^{m}\lambda_j\left[\det\left(B_j\right)\right]^{-1/2}\right)^{1/q},
\end{aligned}$$

which is equivalent to (2.1). □

Remark 1. *For $m=1$ we derive that*

$$\left[\det\left(A+B\right)\right]^{-1/2}\le\frac{1}{p^{n/(2p)}q^{n/(2q)}}\left[\det\left(A\right)\right]^{-1/(2p)}\left[\det\left(B\right)\right]^{-1/(2q)},$$

which is equivalent to

$$\left[\det\left(A+B\right)\right]^{-1}\le\frac{1}{p^{n/p}q^{n/q)}}\left[\det\left(A\right)\right]^{-1/p}\left[\det\left(B\right)\right]^{-1/q},$$

and to

$$p^{n/p}q^{n/q}\left[\det\left(A\right)\right]^{1/p}\left[\det\left(B\right)\right]^{1/q}\le\det\left(A+B\right)\tag{2.7}$$

for $A,\ B>0$ and $p,\ q>1$ with $\frac{1}{p}+\frac{1}{q}=1$.

If we take $(1-t)\,A$ instead of A and tB instead of B we get

$$p^{n/p}q^{n/q}\left[\det\left(\left(1-t\right)A\right)\right]^{1/p}\left[\det\left(tB\right)\right]^{1/q}\le\det\left(\left(1-t\right)A+tB\right),$$

namely

$$p^{n/p}q^{n/q}\left(1-t\right)^{n/p}t^{n/q}\left[\det\left(A\right)\right]^{1/p}\left[\det\left(B\right)\right]^{1/q}\le\det\left(\left(1-t\right)A+tB\right),\tag{2.8}$$

for $A,\ B>0$, $t\in(0,1)$ and $p,\ q>1$ with $\frac{1}{p}+\frac{1}{q}=1$.

If we take in (2.8) $p=1/\left(1-t\right)$ and $q=1/t$, then we get Ky Fan's result (1.2).

Remark 2. *If in (2.1) we take A_j^{-2} instead of A_j and B_j^{-2} instead of B_j, $j=1,...,m$ we get*

$$\begin{aligned}
&\sum_{j=1}^{m}\lambda_j\left[\det\left(A_j^{-2}+B_j^{-2}\right)\right]^{-1/2}\\
&\le\frac{1}{p^{n/(2p)}q^{n/(2q)}}\left(\sum_{j=1}^{m}\lambda_j\det\left(A_j\right)\right)^{1/p}\left(\sum_{j=1}^{m}\lambda_j\det\left(B_j\right)\right)^{1/q}
\end{aligned}\tag{2.9}$$

for $\lambda_j>0$, $A_j,\ B_j>0$ for $j=1,...,m$ and $p,\ q>1$ with $\frac{1}{p}+\frac{1}{q}=1$.

From (2.2) we get

$$\left(\sum_{j=1}^{m}\lambda_j\left[\det\left(A_j^{-2}+B_j^{-2}\right)\right]^{-1/2}\right)^{2}\le\frac{1}{2^n}\sum_{j=1}^{m}\lambda_j\det\left(A_j\right)\sum_{j=1}^{m}\lambda_j\det\left(B_j\right),\tag{2.10}$$

for $\lambda_j>0$, $A_j,\ B_j>0$ for $j=1,...,m$ and $p,\ q>1$ with $\frac{1}{p}+\frac{1}{q}=1$.

If we take the square in the representation (1.1), then we get

$$\left(\int_{\mathbb{R}^n} \exp(-\langle x, Ax\rangle)dx\right)^2 = \frac{\pi^n}{\det(A)}.$$

Since

$$\begin{aligned}\left(\int_{\mathbb{R}^n} \exp(-\langle Ax, x\rangle)dx\right)^2 &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \exp(-\langle Ax, x\rangle)\exp(-\langle Ay, y\rangle)dxdy \\ &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \exp(-\langle Ax, x\rangle - \langle Ay, y\rangle))dxdy,\end{aligned}$$

hence

$$K_n(A) := \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \exp(-\langle Ax, x\rangle - \langle Ay, y\rangle))dxdy = \frac{\pi^n}{\det(A)} \tag{2.11}$$

for A a positive definite matrix of order n and $\langle \cdot, \cdot \rangle$ is the usual inner product on $\mathbb{R}^n$.

Theorem 2. *Assume that $\lambda_j > 0$, A_j, $B_j > 0$ for $j = 1, ..., m$ and p, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$. Then*

$$\begin{aligned}&\sum_{j=1}^m \lambda_j [\det(A_j + B_j)]^{-1} \\ &\le \frac{1}{p^{n/p}q^{n/q}}\left(\sum_{j=1}^m \lambda_j [\det(A_j)]^{-1}\right)^{1/p}\left(\sum_{j=1}^m \lambda_j [\det(B_j)]^{-1}\right)^{1/q}.\end{aligned} \tag{2.12}$$

In particular, for $p = q = 2$ we get

$$\left(\sum_{j=1}^m \lambda_j [\det(A_j + B_j)]^{-1}\right)^2 \le \frac{1}{2^n}\sum_{j=1}^m \lambda_j [\det(A_j)]^{-1}\sum_{j=1}^m \lambda_j [\det(B_j)]^{-1}. \tag{2.13}$$

Proof. Let x, $y \in \mathbb{R}^n$ and take

$$a_j = \exp(-\langle A_j x, x\rangle - \langle A_j y, y\rangle) \text{ and } b_j = \exp(-\langle B_j x, x\rangle - \langle B_j y, y\rangle),$$

for $j = 1, ..., m$, in (2.3) to get

$$\begin{aligned}&\sum_{j=1}^m \lambda_j \exp(-\langle (A_j + B_j)x, x\rangle - \langle (A_j + B_j)y, y\rangle) \\ &\le \left(\sum_{j=1}^m \lambda_j \exp(-\langle pA_j x, x\rangle - \langle pA_j y, y\rangle)\right)^{1/p} \\ &\times \left(\sum_{j=1}^m \lambda_j \exp(-\langle qB_j x, x\rangle - \langle qB_j y, y\rangle)\right)^{1/q}.\end{aligned} \tag{2.14}$$

Taking the double integral $\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}$ and using Hölder's inequality we get successively

$$\sum_{j=1}^{m}\lambda_j\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\exp\left(-\left\langle\left(A_j+B_j\right)x,x\right\rangle-\left\langle\left(A_j+B_j\right)y,y\right\rangle\right)dxdy$$
$$\leq\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\left(\sum_{j=1}^{m}\lambda_j\exp\left(-\left\langle pA_jx,x\right\rangle-\left\langle pA_jy,y\right\rangle\right)\right)^{1/p}$$
$$\times\left(\sum_{j=1}^{m}\lambda_j\exp\left(-\left\langle qB_jx,x\right\rangle-\left\langle qB_jy,y\right\rangle\right)\right)^{1/q}dxdy$$

$$\leq\left(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\left[\left(\sum_{j=1}^{m}\lambda_j\exp\left(-\left\langle pA_jx,x\right\rangle-\left\langle pA_jy,y\right\rangle\right)\right)^{1/p}\right]^{p}dxdy\right)^{1/p}$$
$$\times\left(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\left[\left(\sum_{j=1}^{m}\lambda_j\exp\left(-\left\langle qB_jx,x\right\rangle-\left\langle qB_jy,y\right\rangle\right)\right)^{1/q}\right]^{q}dxdy\right)^{1/q}$$
$$=\left(\sum_{j=1}^{m}\lambda_j\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\exp\left(-\left\langle pA_jx,x\right\rangle-\left\langle pA_jy,y\right\rangle\right)dxdy\right)^{1/p}$$
$$\times\left(\sum_{j=1}^{m}\lambda_j\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\exp\left(-\left\langle qB_jx,x\right\rangle-\left\langle qB_jy,y\right\rangle\right)dxdy\right)^{1/q},$$

namely

$$\sum_{j=1}^{m}\lambda_jK_n\left(A_j+B_j\right)\leq\left(\sum_{j=1}^{m}\lambda_jK_n\left(pA_j\right)\right)^{1/p}\left(\sum_{j=1}^{m}\lambda_jK_n\left(qB_j\right)\right)^{1/q}.$$

By (2.11) we derive

$$\sum_{j=1}^{m}\frac{\lambda_j\pi^n}{\det\left(A_j+B_j\right)}\leq\left(\sum_{j=1}^{m}\frac{\lambda_j\pi^n}{\det\left(pA_j\right)}\right)^{1/p}\left(\sum_{j=1}^{m}\frac{\lambda_j\pi^n}{\det\left(qB_j\right)}\right)^{1/q},$$

which is equivalent to

$$\sum_{j=1}^{m}\frac{\lambda_j}{\det\left(A_j+B_j\right)}\leq\frac{1}{p^{n/p}q^{n/q}}\left(\sum_{j=1}^{m}\frac{\lambda_j}{\det\left(A_j\right)}\right)^{1/p}\left(\sum_{j=1}^{m}\frac{\lambda_j}{\det\left(B_j\right)}\right)^{1/q}$$

and the inequality (2.12) is proved. □

Remark 3. *If in (2.12) we take A_j^{-1} instead of A_j and B_j^{-1} instead of B_j, $j = 1, ..., m$ we get*

$$\sum_{j=1}^{m} \lambda_j \left[\det\left(A_j^{-1} + B_j^{-1}\right)\right]^{-1} \leq \frac{1}{p^{n/p} q^{n/q}} \left(\sum_{j=1}^{m} \lambda_j \det(A_j)\right)^{1/p} \left(\sum_{j=1}^{m} \lambda_j \det(B_j)\right)^{1/q} \tag{2.15}$$

for $\lambda_j > 0$, A_j, $B_j > 0$ for $j = 1, ..., m$ and p, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$.

From (2.13) we get

$$\left(\sum_{j=1}^{m} \lambda_j \left[\det\left(A_j^{-1} + B_j^{-1}\right)\right]^{-1}\right)^2 \leq \frac{1}{2^n} \sum_{j=1}^{m} \lambda_j \det(A_j) \sum_{j=1}^{m} \lambda_j \det(B_j), \tag{2.16}$$

where $\lambda_j > 0$, A_j, $B_j > 0$ for $j = 1, ..., m$ and p, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$.

We also have:

Theorem 3. *Let $p_j > 0$, $j = 1, ..., m$, with $\sum_{k=1}^{m} p_j = 1$ and $A_j > 0$, $j = 1, ..., m$, then*

$$\sum_{1 \leq j < k \leq m} p_j p_k \left[\det(A_j + A_k)\right]^{-1/2} \leq \frac{1}{2^{n/2+1}} \sum_{j=1}^{m} p_j (1 - p_j) \left[\det(A_j)\right]^{-1/2}. \tag{2.17}$$

In particular, for $p_j = 1/m$,

$$\sum_{1 \leq j < k \leq m} \left[\det(A_j + A_k)\right]^{-1/2} \leq \frac{m-1}{2^{n/2+1}} \sum_{j=1}^{m} \left[\det(A_j)\right]^{-1/2}. \tag{2.18}$$

Proof. We use the Cauchy-Bunyakovsky-Schwarz type inequality

$$\left(\sum_{j=1}^{m} p_j a_j\right)^2 \leq \sum_{j=1}^{m} p_j a_j^2,$$

where $p_j > 0$, $j = 1, ..., m$, $\sum_{k=1}^{m} p_j = 1$ and $a_j \geq 0$, $j = 1, ..., m$.

This inequality can be written as

$$\sum_{j=1}^{m} \sum_{k=1}^{m} p_j p_k a_j a_k \leq \sum_{k=1}^{m} p_j a_j^2. \tag{2.19}$$

Let $x \in \mathbb{R}^n$ and take $a_j = \exp(-\langle A_j x, x \rangle)$, $j = 1, ..., m$ in (2.19) to get

$$\sum_{j=1}^{m} \sum_{k=1}^{m} p_j p_k \exp(-\langle (A_j + A_k) x, x \rangle) \leq \sum_{j=1}^{m} p_j \exp(-\langle 2A_j x, x \rangle). \tag{2.20}$$

Now, if we take the integral $\int_{\mathbb{R}^n}$ in (2.20), then

$$\sum_{j=1}^{m} \sum_{k=1}^{m} p_j p_k \int_{\mathbb{R}^n} \exp(-\langle (A_j + A_k) x, x \rangle) \, dx \leq \sum_{j=1}^{m} p_j \int_{\mathbb{R}^n} \exp(-\langle 2A_j x, x \rangle) \, dx.$$

This inequality is equivalent, by (1.1), to

$$\sum_{j=1}^{m}\sum_{k=1}^{m} p_j p_k J_n\left(A_j + A_k\right) \le \sum_{j=1}^{m} p_j J_n\left(2A_j\right). \tag{2.21}$$

Observe that

$$\sum_{j=1}^{m}\sum_{k=1}^{m} p_j p_k J_n\left(A_j + A_k\right) = \sum_{j=1}^{m} p_j^2 J_n\left(2A_j\right) + 2\sum_{1\le j<k\le m} p_j p_k J_n\left(A_j + A_k\right),$$

then by (2.21) we get

$$2\sum_{1\le j<k\le m} p_j p_k J_n\left(A_j + A_k\right) \le \sum_{j=1}^{m} p_j\left(1-p_j\right) J_n\left(2A_j\right).$$

By (1.1) we get

$$2\sum_{1\le j<k\le m} \frac{p_j p_k}{\left[\det\left(A_j + A_k\right)\right]^{1/2}} \le \frac{1}{2^{n/2}}\sum_{j=1}^{m} \frac{p_j\left(1-p_j\right)}{\left[\det\left(A_j\right)\right]^{1/2}},$$

which is equivalent to (2.17) □

By utilising the representation (2.11) we also have

Theorem 4. *Let $p_j > 0$, $j = 1, ..., m$, with $\sum_{k=1}^{m} p_j = 1$ and $A_j > 0$, $j = 1, ..., m$, then*

$$\sum_{1\le j<k\le m} p_j p_k \left[\det\left(A_j + A_k\right)\right]^{-1} \le \frac{1}{2^{n+1}}\sum_{j=1}^{m} p_j\left(1-p_j\right)\left[\det\left(A_j\right)\right]^{-1}. \tag{2.22}$$

In particular, for $p_j = 1/m$,

$$\sum_{1\le j<k\le m} \left[\det\left(A_j + A_k\right)\right]^{-1} \le \frac{m-1}{2^{n+1}}\sum_{j=1}^{m} \left[\det\left(A_j\right)\right]^{-1}. \tag{2.23}$$

3. The Case of Hermitian Matrices

A complex square matrix $H = \left(h_{ij}\right)$, $i, j = 1, ..., n$ is said to be Hermitian provided $h_{ij} = \overline{h_{ji}}$ for all i, $j = 1, ..., n$. A Hermitian matrix is said to be positive definite if the Hermitian form $P\left(z\right) = \sum_{i,j=1}^{n} a_{ij} z_i \overline{z_j}$ is positive for all $z = \left(z_1, ..., z_n\right) \in \mathbb{C}^n \setminus \{0\}$.

It is known that, see for instance [8, p. 215], for a positive definite Hermitian matrix H, we have

$$K_n\left(H\right) := \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \exp\left(-\left\langle \overline{z}, Hz\right\rangle\right) dxdy = \frac{\pi^n}{\det\left(H\right)}, \tag{3.1}$$

where $z = x + iy$ and dx and dy denote integration over real n-dimensional space $\mathbb{R}^n$. Here the inner product $\langle x, y\rangle$ is understood in the real sense, i.e. $\langle x, y\rangle = \sum_{k=1}^{n} x_k y_k$.

Theorem 5. *Assume that* $\lambda_j > 0$, H_j, K_j *are Hermitian and positive definite for* $j = 1, ..., m$ *and* p, $q > 1$ *with* $\frac{1}{p} + \frac{1}{q} = 1$. *Then*

$$\sum_{j=1}^{m} \lambda_j \left[\det\left(H_j + K_j\right)\right]^{-1} \leq \frac{1}{p^{n/p} q^{n/q}} \left(\sum_{j=1}^{m} \lambda_j \left[\det\left(H_j\right)\right]^{-1}\right)^{1/p} \left(\sum_{j=1}^{m} \lambda_j \left[\det\left(K_j\right)\right]^{-1}\right)^{1/q}. \tag{3.2}$$

In particular, for $p = q = 2$ *we get*

$$\left(\sum_{j=1}^{m} \lambda_j \left[\det\left(H_j + K_j\right)\right]^{-1}\right)^{2} \leq \frac{1}{2^n} \sum_{j=1}^{m} \lambda_j \left[\det\left(H_j\right)\right]^{-1} \sum_{j=1}^{m} \lambda_j \left[\det\left(K_j\right)\right]^{-1}. \tag{3.3}$$

The proof is similar to the one in Theorem 2 by employing the representation (3.1).

Finally, we can state:

Theorem 6. *Let* $p_j > 0$, $j = 1, ..., m$, *with* $\sum_{k=1}^{m} p_j = 1$ *and* $H_j > 0$, $j = 1, ..., m$, *then*

$$\sum_{1 \leq j < k \leq m} p_j p_k \left[\det\left(H_j + H_k\right)\right]^{-1} \leq \frac{1}{2^{n+1}} \sum_{j=1}^{m} p_j \left(1 - p_j\right) \left[\det\left(H_j\right)\right]^{-1}. \tag{3.4}$$

In particular, for $p_j = 1/m$,

$$\sum_{1 \leq j < k \leq m} \left[\det\left(H_j + H_k\right)\right]^{-1} \leq \frac{m-1}{2^{n+1}} \sum_{j=1}^{m} \left[\det\left(H_j\right)\right]^{-1}. \tag{3.5}$$

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