

**DETERMINANT INEQUALITIES FOR POSITIVE DEFINITE  
MATRICES VIA KOBER'S RESULT FOR ARITHMETIC AND  
GEOMETRIC WEIGHTED MEANS**

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ABSTRACT. In this paper we prove among others that, if  $(A_j)_{j=1,\dots,m}$  are positive definite matrices of order  $n \geq 2$  and  $q_j \geq 0, j = 1, \dots, m$  with  $\sum_{j=1}^m q_j = 1$ , then

$$\begin{aligned} & \frac{1}{m-1} \min_{i \in \{1, \dots, m\}} \{q_i\} \\ & \times \left[ (m-1) \sum_{i=1}^m [\det(A_i)]^{-1} - 2^{n+1} \sum_{1 \leq i < j \leq m} [\det(A_i + A_j)]^{-1} \right] \\ & \leq \sum_{i=1}^m q_i [\det(A_i)]^{-1} - \left[ \det \sum_{i=1}^m q_i A_i \right]^{-1} \\ & \leq \max_{i \in \{1, \dots, m\}} \{q_i\} \\ & \times \left[ (m-1) \sum_{i=1}^m [\det(A_i)]^{-1} - 2^{n+1} \sum_{1 \leq i < j \leq m} [\det(A_i + A_j)]^{-1} \right]. \end{aligned}$$

1. INTRODUCTION

A real square matrix  $A = (a_{ij}), i, j = 1, \dots, n$  is *symmetric* provided  $a_{ij} = a_{ji}$  for all  $i, j = 1, \dots, n$ . A real symmetric matrix is said to be *positive definite* provided the quadratic form  $Q(x) = \sum_{i,j=1}^n a_{ij}x_i x_j$  is positive for all  $x = (x_1, \dots, x_n) \in \mathbb{R}^n \setminus \{0\}$ . It is well known that a necessary and sufficient condition for the symmetric matrix  $A$  to be positive definite, and we write  $A > 0$ , is that all determinants

$$\det(A_k) = \det(a_{ij}), \quad i, j = 1, \dots, k; \quad k = 1, \dots, n$$

are positive.

It is known that the following integral representation is valid, see [1, pp. 61-62] or [9, pp. 211-212]

$$\begin{aligned} (1.1) \quad J_n(A) & := \int_{\mathbb{R}^n} \exp(-\langle Ax, x \rangle) dx := \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} \exp(-\langle Ax, x \rangle) dx \\ & = \frac{\pi^{n/2}}{[\det(A)]^{1/2}}, \end{aligned}$$

where  $A$  is a positive definite matrix of order  $n$  and  $\langle \cdot, \cdot \rangle$  is the usual inner product on  $\mathbb{R}^n$ .

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By utilizing the representation (1.1) and Hölder's integral inequality for multiple integrals one can prove the *logarithmic concavity* of the determinant that is due to Ky Fan ([1, p. 63] or [9, p. 212]), namely

$$(1.2) \quad \det((1 - \lambda)A + \lambda B) \geq [\det(A)]^{1-\lambda} [\det(B)]^\lambda$$

for any positive definite matrices  $A, B$  and  $\lambda \in [0, 1]$ .

By mathematical induction we can get a generalization of (1.2) which was obtained by L. Mirsky in [8], see also [9, p. 212]

$$(1.3) \quad \det\left(\sum_{j=1}^m \lambda_j A_j\right) \geq \prod_{j=1}^m [\det(A_j)]^{\lambda_j}, \quad m \geq 2,$$

where  $\lambda_j > 0, j = 1, \dots, m$  with  $\sum_{j=1}^m \lambda_j = 1$  and  $A_j > 0, j = 1, \dots, m$ .

If we write (1.3) for  $A_j = B_j^{-1}$  we get

$$\det\left(\sum_{j=1}^m \lambda_j B_j^{-1}\right) \geq \prod_{j=1}^m [\det(B_j^{-1})]^{\lambda_j} = \left(\prod_{j=1}^m [\det(B_j)]^{\lambda_j}\right)^{-1},$$

which also gives

$$(1.4) \quad \prod_{j=1}^m [\det(A_j)]^{\lambda_j} \geq \det\left[\left(\sum_{j=1}^m \lambda_j A_j^{-1}\right)^{-1}\right],$$

where  $\lambda_j > 0, j = 1, \dots, m$  with  $\sum_{j=1}^m \lambda_j = 1$  and  $A_j > 0, j = 1, \dots, m$ .

Using the representation (1.1) one can also prove the result, see [9, p. 212],

$$(1.5) \quad \det(A) = \det(A_{1n}) \leq \det(A_{1k}) \det(A_{(k+1)n}), \quad k = 1, \dots, n;$$

where the determinant  $\det(A_{rs})$  is defined by

$$\det(A_{rs}) = \det(a_{ij}), \quad i, j = r, \dots, s.$$

In particular,

$$(1.6) \quad \det(A) \leq a_{11}a_{22}\dots a_{nn}.$$

We recall also the Minkowski's type inequality,

$$(1.7) \quad [\det(A + B)]^{1/n} \geq [\det(A)]^{1/n} + [\det(B)]^{1/n}$$

for  $A, B$  positive definite matrices of order  $n$ . For other determinant inequalities see Chapter VIII of the classic book [9]. For some recent results see [2]-[6].

In this paper we prove among others that, if  $(A_j)_{j=1,\dots,m}$  are positive definite matrices of order  $n \geq 2$  and  $q_j \geq 0$ ,  $j = 1, \dots, m$  with  $\sum_{j=1}^m q_j = 1$ , then

$$\begin{aligned}
 & 0 \leq \frac{1}{m-1} \min_{i \in \{1, \dots, m\}} \{q_i\} \\
 & \times \left[ (m-1) \sum_{i=1}^m [\det(A_i)]^{-1} - 2^{n+1} \sum_{1 \leq i < j \leq m} [\det(A_i + A_j)]^{-1} \right] \\
 & \leq \sum_{i=1}^m q_i [\det(A_i)]^{-1} - \left[ \det \left( \sum_{i=1}^m q_i A_i \right) \right]^{-1} \\
 & \leq \max_{i \in \{1, \dots, m\}} \{q_i\} \\
 & \times \left[ (m-1) \sum_{i=1}^m [\det(A_i)]^{-1} - 2^{n+1} \sum_{1 \leq i < j \leq m} [\det(A_i + A_j)]^{-1} \right].
 \end{aligned}$$

## 2. INEQUALITIES VIA SINGLE INTEGRAL REPRESENTATION

We recall the following classical result due to H. Kober, see [7]:

$$\begin{aligned}
 (2.1) \quad & 0 \leq \frac{1}{2(m-1)} \min_{i \in \{1, \dots, m\}} \{q_i\} \sum_{i,j=1}^m (\sqrt{x_i} - \sqrt{x_j})^2 \\
 & \leq \sum_{i=1}^m q_i x_i - \prod_{i=1}^m x_i^{q_i} \leq \frac{1}{2} \max_{i \in \{1, \dots, m\}} \{q_i\} \sum_{i,j=1}^m (\sqrt{x_i} - \sqrt{x_j})^2,
 \end{aligned}$$

where  $x_i, q_i \geq 0$ ,  $i \in \{1, \dots, m\}$  with  $\sum_{i=1}^m q_i = 1$ .

**Theorem 1.** *Assume that  $(A_j)_{j=1,\dots,m}$  are positive definite and  $q_j \geq 0$ ,  $j = 1, \dots, m$  with  $\sum_{j=1}^m q_j = 1$ . Then*

$$\begin{aligned}
 (2.2) \quad & 0 \leq \frac{1}{m-1} \min_{i \in \{1, \dots, m\}} \{q_i\} \\
 & \times \left[ (m-1) \sum_{i=1}^m [\det(A_i)]^{-1/2} - 2^{n/2+1} \sum_{1 \leq i < j \leq m} [\det(A_i + A_j)]^{-1/2} \right] \\
 & \leq \sum_{i=1}^m q_i [\det(A_i)]^{-1/2} - \left[ \det \left( \sum_{i=1}^m q_i A_i \right) \right]^{-1/2} \\
 & \leq \max_{i \in \{1, \dots, m\}} \{q_i\} \\
 & \times \left[ (m-1) \sum_{i=1}^m [\det(A_i)]^{-1/2} - 2^{n/2+1} \sum_{1 \leq i < j \leq m} [\det(A_i + A_j)]^{-1/2} \right].
 \end{aligned}$$

In particular, for  $m = 2$  and  $t \in (0, 1)$ ,

$$\begin{aligned}
(2.3) \quad & 0 \leq \min \{t, 1 - t\} \\
& \times \left[ [\det(A_1)]^{-1/2} + [\det(A_1)]^{-1/2} - 2^{n/2+1} [\det(A_1 + A_2)]^{-1/2} \right] \\
& \leq t [\det(A_1)]^{-1/2} + (1 - t) [\det(A_2)]^{-1/2} - [\det(tA_1 + (1 - t)A_2)]^{-1/2} \\
& \leq \min \{t, 1 - t\} \\
& \times \left[ [\det(A_1)]^{-1/2} + [\det(A_1)]^{-1/2} - 2^{n/2+1} [\det(A_1 + A_2)]^{-1/2} \right].
\end{aligned}$$

*Proof.* If we take  $x_i = \exp(-\langle A_i x, x \rangle)$ ,  $i \in \{1, \dots, m\}$ ,  $x \in \mathbb{R}^n$  in (2.1), then we get

$$\begin{aligned}
(2.4) \quad & \frac{1}{2(m-1)} \min_{i \in \{1, \dots, m\}} \{q_i\} \\
& \times \sum_{i,j=1}^m \left( \exp\left(-\left\langle \frac{1}{2} A_i x, x \right\rangle\right) - \exp\left(-\left\langle \frac{1}{2} A_j x, x \right\rangle\right) \right)^2 \\
& \leq \sum_{i=1}^m q_i \exp(-\langle A_i x, x \rangle) - \exp\left(-\left\langle \sum_{i=1}^m q_i A_i x, x \right\rangle\right) \\
& \leq \frac{1}{2} \max_{i \in \{1, \dots, m\}} \{q_i\} \\
& \times \sum_{i,j=1}^m \left( \exp\left(-\left\langle \frac{1}{2} A_i x, x \right\rangle\right) - \exp\left(-\left\langle \frac{1}{2} A_j x, x \right\rangle\right) \right)^2.
\end{aligned}$$

Since

$$\begin{aligned}
& \left( \exp\left(-\left\langle \frac{1}{2} A_i x, x \right\rangle\right) - \exp\left(-\left\langle \frac{1}{2} A_j x, x \right\rangle\right) \right)^2 \\
& = \exp(-\langle A_i x, x \rangle) - 2 \exp\left(-\left\langle \frac{1}{2} (A_i + A_j) x, x \right\rangle\right) + \exp(-\langle A_j x, x \rangle),
\end{aligned}$$

hence by (2.4) we get

$$\begin{aligned}
(2.5) \quad & \frac{1}{2(m-1)} \min_{i \in \{1, \dots, m\}} \{q_i\} \left[ \sum_{i,j=1}^m \exp(-\langle A_i x, x \rangle) \right. \\
& \left. - 2 \exp\left(-\left\langle \frac{1}{2} (A_i + A_j) x, x \right\rangle\right) + \exp(-\langle A_j x, x \rangle) \right] \\
& \leq \sum_{i=1}^m q_i \exp(-\langle A_i x, x \rangle) - \exp\left(-\left\langle \sum_{i=1}^m q_i A_i x, x \right\rangle\right) \\
& \leq \frac{1}{2} \max_{i \in \{1, \dots, m\}} \{q_i\} \left[ \sum_{i,j=1}^m \exp(-\langle A_i x, x \rangle) \right. \\
& \left. - 2 \exp\left(-\left\langle \frac{1}{2} (A_i + A_j) x, x \right\rangle\right) + \exp(-\langle A_j x, x \rangle) \right].
\end{aligned}$$

Observe that

$$\begin{aligned}
& \sum_{i,j=1}^m \left[ \exp(-\langle A_i x, x \rangle) - 2 \exp\left(-\left\langle \frac{1}{2}(A_i + A_j)x, x \right\rangle\right) + \exp(-\langle A_j x, x \rangle) \right] \\
&= m \sum_{i=1}^m \exp(-\langle A_i x, x \rangle) + n \sum_{j=1}^m \exp(-\langle A_j x, x \rangle) \\
&\quad - 2 \sum_{i,j=1}^m \exp\left(-\left\langle \frac{1}{2}(A_i + A_j)x, x \right\rangle\right) \\
&= 2m \sum_{i=1}^m \exp(-\langle A_i x, x \rangle) \\
&\quad - 2 \left[ \sum_{i=1}^m \exp(-\langle A_i x, x \rangle) + 2 \sum_{1 \leq i < j \leq m} \exp\left(-\left\langle \frac{1}{2}(A_i + A_j)x, x \right\rangle\right) \right] \\
&= 2(m-1) \sum_{i=1}^m \exp(-\langle A_i x, x \rangle) - 4 \sum_{1 \leq i < j \leq m} \exp\left(-\left\langle \frac{1}{2}(A_i + A_j)x, x \right\rangle\right),
\end{aligned}$$

therefore by (2.5) we derive

$$\begin{aligned}
(2.6) \quad & \frac{1}{m-1} \min_{i \in \{1, \dots, m\}} \{q_i\} \left[ (m-1) \sum_{i=1}^m \exp(-\langle A_i x, x \rangle) \right. \\
& \quad \left. - 2 \sum_{1 \leq i < j \leq m} \exp\left(-\left\langle \frac{1}{2}(A_i + A_j)x, x \right\rangle\right) \right] \\
& \leq \sum_{i=1}^m q_i \exp(-\langle A_i x, x \rangle) - \exp\left(-\left\langle \sum_{i=1}^m q_i A_i x, x \right\rangle\right) \\
& \leq \max_{i \in \{1, \dots, m\}} \{q_i\} \left[ (m-1) \sum_{i=1}^m \exp(-\langle A_i x, x \rangle) \right. \\
& \quad \left. - 2 \sum_{1 \leq i < j \leq m} \exp\left(-\left\langle \frac{1}{2}(A_i + A_j)x, x \right\rangle\right) \right].
\end{aligned}$$

If we integrate on  $\mathbb{R}^n$ , then we get

$$\begin{aligned}
& \frac{1}{m-1} \min_{i \in \{1, \dots, m\}} \{q_i\} \left[ (m-1) \sum_{i=1}^m \int_{\mathbb{R}^n} \exp(-\langle A_i x, x \rangle) dx \right. \\
& \quad \left. - 2 \sum_{1 \leq i < j \leq m} \int_{\mathbb{R}^n} \exp\left(-\left\langle \frac{1}{2}(A_i + A_j)x, x \right\rangle\right) dx \right] \\
& \leq \sum_{i=1}^m q_i \int_{\mathbb{R}^n} \exp(-\langle A_i x, x \rangle) dx - \int_{\mathbb{R}^n} \exp\left(-\left\langle \sum_{i=1}^m q_i A_i x, x \right\rangle\right) dx
\end{aligned}$$

$$\begin{aligned} &\leq \max_{i \in \{1, \dots, m\}} \{q_i\} \left[ (m-1) \sum_{i=1}^m \int_{\mathbb{R}^n} \exp(-\langle A_i x, x \rangle) dx \right. \\ &\quad \left. - 2 \sum_{1 \leq i < j \leq m} \exp\left(-\left\langle \frac{1}{2}(A_i + A_j)x, x \right\rangle\right) \right], \end{aligned}$$

and by (1.1)

$$\begin{aligned} &\frac{1}{m-1} \min_{i \in \{1, \dots, m\}} \{q_i\} \left[ (m-1) \sum_{i=1}^m J_n(A_i) - 2 \sum_{1 \leq i < j \leq m} J_n\left(\frac{1}{2}(A_i + A_j)\right) \right] \\ &\leq \sum_{i=1}^m q_i J_n(A_i) - J_n\left(\sum_{i=1}^m q_i A_i\right) \\ &\leq \max_{i \in \{1, \dots, m\}} \{q_i\} \left[ (m-1) \sum_{i=1}^m J_n(A_i) - 2 \sum_{1 \leq i < j \leq m} J_n\left(\frac{1}{2}(A_i + A_j)\right) \right]. \end{aligned}$$

By using (1.1) we have further

$$\begin{aligned} &\frac{1}{m-1} \min_{i \in \{1, \dots, m\}} \{q_i\} \\ &\times \left[ (m-1) \sum_{i=1}^m \frac{1}{[\det(A_i)]^{1/2}} - 2 \sum_{1 \leq i < j \leq m} \frac{1}{[\det(\frac{1}{2}(A_i + A_j))]^{1/2}} \right] \\ &\leq \sum_{i=1}^m \frac{q_i}{[\det(A_i)]^{1/2}} - \frac{1}{[\det(\sum_{i=1}^m q_i A_i)]^{1/2}} \\ &\leq \max_{i \in \{1, \dots, m\}} \{q_i\} \\ &\times \left[ (m-1) \sum_{i=1}^m \frac{1}{[\det(A_i)]^{1/2}} - 2 \sum_{1 \leq i < j \leq m} \frac{1}{[\det(\frac{1}{2}(A_i + A_j))]^{1/2}} \right], \end{aligned}$$

which is equivalent to (2.2).  $\square$

**Remark 1.** If we take  $q_i = \frac{1}{m}$ ,  $i = 1, \dots, m$  in (2.2), then we get

$$\begin{aligned} (2.7) \quad 0 &\leq \frac{1}{(m-1)m} \\ &\times \left[ (m-1) \sum_{i=1}^m [\det(A_i)]^{-1/2} - 2^{n/2+1} \sum_{1 \leq i < j \leq m} [\det(A_i + A_j)]^{-1/2} \right] \\ &\leq \frac{1}{m} \sum_{i=1}^m [\det(A_i)]^{-1/2} - \left[ \det\left(\frac{\sum_{i=1}^m A_i}{m}\right) \right]^{-1/2} \\ &\leq \frac{1}{m} \left[ (m-1) \sum_{i=1}^m [\det(A_i)]^{-1/2} - 2^{n/2+1} \sum_{1 \leq i < j \leq m} [\det(A_i + A_j)]^{-1/2} \right]. \end{aligned}$$

**Corollary 1.** *If  $A_1, A_2$  are positive definite, then*

$$\begin{aligned}
(2.8) \quad 0 &\leq \frac{1}{4} \left[ [\det(A_1)]^{-1/2} + [\det(A_2)]^{-1/2} - 2^{n/2+1} [\det(A_1 + A_2)]^{-1/2} \right] \\
&\leq \frac{[\det(A_1)]^{-1/2} + [\det(A_2)]^{-1/2}}{2} - \int_0^1 [\det(tA_1 + (1-t)A_2)]^{-1/2} dt \\
&\leq \frac{3}{4} \left[ [\det(A_1)]^{-1/2} + [\det(A_2)]^{-1/2} - 2^{n/2+1} [\det(A_1 + A_2)]^{-1/2} \right].
\end{aligned}$$

*Proof.* If we take the integral in (2.3) we get

$$\begin{aligned}
(2.9) \quad 0 &\leq \int_0^1 \min\{t, 1-t\} dt \\
&\quad \times \left[ [\det(A_1)]^{-1/2} + [\det(A_2)]^{-1/2} - 2^{n/2+1} [\det(A_1 + A_2)]^{-1/2} \right] \\
&\leq \frac{[\det(A_1)]^{-1/2} + [\det(A_2)]^{-1/2}}{2} - \int_0^1 [\det(tA_1 + (1-t)A_2)]^{-1/2} dt \\
&\leq \int_0^1 \max\{t, 1-t\} dt \\
&\quad \times \left[ [\det(A_1)]^{-1/2} + [\det(A_2)]^{-1/2} - 2^{n/2+1} [\det(A_1 + A_2)]^{-1/2} \right].
\end{aligned}$$

Since

$$\int_0^1 \min\{t, 1-t\} dt = \frac{1}{4} \quad \text{and} \quad \int_0^1 \max\{t, 1-t\} dt = \frac{3}{4},$$

hence by (2.9) we get (2.8).  $\square$

**Remark 2.** *If we take  $A_j = B_j^{-2}$ , where  $B_j > 0$ ,  $j = 1, \dots, m$ , then by (2.2) we get*

$$\begin{aligned}
(2.10) \quad 0 &\leq \frac{1}{m-1} \min_{i \in \{1, \dots, m\}} \{q_i\} \\
&\quad \times \left[ (m-1) \sum_{i=1}^m \det(B_i) - 2^{n/2+1} \sum_{1 \leq i < j \leq m} [\det(B_i^{-2} + B_j^{-2})]^{-1/2} \right] \\
&\leq \sum_{i=1}^m q_i \det(B_i) - \left[ \det \left( \sum_{i=1}^m q_i B_i^{-2} \right) \right]^{-1/2} \\
&\leq \max_{i \in \{1, \dots, m\}} \{q_i\} \\
&\quad \times \left[ (m-1) \sum_{i=1}^m \det(B_i) - 2^{n/2+1} \sum_{1 \leq i < j \leq m} [\det(B_i^{-2} + B_j^{-2})]^{-1/2} \right].
\end{aligned}$$

In particular, for  $m = 2$  and  $t \in (0, 1)$ ,

$$\begin{aligned}
(2.11) \quad 0 &\leq \min \{t, 1-t\} \\
&\times \left[ \det(B_1) + \det(B_2) - 2^{n/2+1} [\det(B_1^{-2} + B_2^{-2})]^{-1/2} \right] \\
&\leq t \det(B_1) + (1-t) \det(B_2) - [\det(tB_1^{-2} + (1-t)B_2^{-2})]^{-1/2} \\
&\leq \min \{t, 1-t\} \\
&\times \left[ \det(B_1) + \det(B_2) - 2^{n/2+1} [\det(B_1^{-2} + B_2^{-2})]^{-1/2} \right].
\end{aligned}$$

From (2.8) we get

$$\begin{aligned}
(2.12) \quad 0 &\leq \frac{1}{4} \left[ \det(B_1) + \det(B_2) - 2^{n/2+1} [\det(B_1^{-2} + B_2^{-2})]^{-1/2} \right] \\
&\leq \frac{\det(B_1) + \det(B_2)}{2} - \int_0^1 [\det(tB_1^{-2} + (1-t)B_2^{-2})]^{-1/2} dt \\
&\leq \frac{3}{4} \left[ \det(B_1) + \det(B_2) - 2^{n/2+1} [\det(B_1^{-2} + B_2^{-2})]^{-1/2} \right].
\end{aligned}$$

### 3. INEQUALITIES VIA DOUBLE INTEGRAL REPRESENTATION

If we take the square in the representation (1.1), then we get

$$\left( \int_{\mathbb{R}^n} \exp(-\langle x, Ax \rangle) dx \right)^2 = \frac{\pi^n}{\det(A)}.$$

Since

$$\begin{aligned}
\left( \int_{\mathbb{R}^n} \exp(-\langle Ax, x \rangle) dx \right)^2 &= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \exp(-\langle Ax, x \rangle) \exp(-\langle Ay, y \rangle) dx dy \\
&= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \exp(-\langle Ax, x \rangle - \langle Ay, y \rangle) dx dy,
\end{aligned}$$

hence

$$(3.1) \quad K_n(A) := \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \exp(-\langle Ax, x \rangle - \langle Ay, y \rangle) dx dy = \frac{\pi^n}{\det(A)}$$

for  $A$  a positive definite matrix of order  $n$  and  $\langle \cdot, \cdot \rangle$  is the usual inner product on  $\mathbb{R}^n$ .

**Theorem 2.** Assume that  $(A_j)_{j=1,\dots,m}$  are positive definite and  $q_j \geq 0$ ,  $j = 1, \dots, m$  with  $\sum_{j=1}^m q_j = 1$ . Then

$$\begin{aligned}
(3.2) \quad 0 &\leq \frac{1}{m-1} \min_{i \in \{1, \dots, m\}} \{q_i\} \\
&\times \left[ (m-1) \sum_{i=1}^m [\det(A_i)]^{-1} - 2^{n+1} \sum_{1 \leq i < j \leq m} [\det(A_i + A_j)]^{-1} \right] \\
&\leq \sum_{i=1}^m q_i [\det(A_i)]^{-1} - \left[ \det \left( \sum_{i=1}^m q_i A_i \right) \right]^{-1} \\
&\leq \max_{i \in \{1, \dots, m\}} \{q_i\} \\
&\times \left[ (m-1) \sum_{i=1}^m [\det(A_i)]^{-1} - 2^{n+1} \sum_{1 \leq i < j \leq m} [\det(A_i + A_j)]^{-1} \right].
\end{aligned}$$

In particular, for  $m = 2$  and  $t \in (0, 1)$ ,

$$\begin{aligned}
(3.3) \quad 0 &\leq \min \{t, 1-t\} \\
&\times \left[ [\det(A_1)]^{-1} + [\det(A_2)]^{-1} - 2^{n+1} [\det(A_1 + A_2)]^{-1} \right] \\
&\leq t [\det(A_1)]^{-1} + (1-t) [\det(A_2)]^{-1} - [\det(tA_1 + (1-t)A_2)]^{-1} \\
&\leq \min \{t, 1-t\} \\
&\times \left[ [\det(A_1)]^{-1} + [\det(A_2)]^{-1} - 2^{n+1} [\det(A_1 + A_2)]^{-1} \right].
\end{aligned}$$

*Proof.* If we take  $x_i = \exp(-\langle A_i x, x \rangle - \langle A_i y, y \rangle)$ ,  $i \in \{1, \dots, m\}$ ,  $x, y \in \mathbb{R}^n$  in (2.1), then we get

$$\begin{aligned}
(3.4) \quad &\frac{1}{2(m-1)} \min_{i \in \{1, \dots, m\}} \{q_i\} \sum_{i,j=1}^m \left[ \exp \left( - \left\langle \frac{1}{2} A_i x, x \right\rangle - \left\langle \frac{1}{2} A_i y, y \right\rangle \right) \right. \\
&\quad \left. - \exp \left( - \left\langle \frac{1}{2} A_j x, x \right\rangle - \left\langle \frac{1}{2} A_j y, y \right\rangle \right) \right]^2 \\
&\leq \sum_{i=1}^m q_i \exp(-\langle A_i x, x \rangle - \langle A_i y, y \rangle) \\
&\quad - \exp \left( - \left\langle \sum_{i=1}^m q_i A_i x, x \right\rangle - \left\langle \sum_{i=1}^m q_i A_i y, y \right\rangle \right) \\
&\leq \frac{1}{2} \max_{i \in \{1, \dots, m\}} \{q_i\} \sum_{i,j=1}^m \left[ \exp \left( - \left\langle \frac{1}{2} A_i x, x \right\rangle - \left\langle \frac{1}{2} A_i y, y \right\rangle \right) \right. \\
&\quad \left. - \exp \left( - \left\langle \frac{1}{2} A_j x, x \right\rangle - \left\langle \frac{1}{2} A_j y, y \right\rangle \right) \right]^2.
\end{aligned}$$

By performing similar calculation to the ones in the proof of Theorem 1, we get

$$\begin{aligned}
& \frac{1}{m-1} \min_{i \in \{1, \dots, m\}} \{q_i\} \left[ (m-1) \sum_{i=1}^m \exp(-\langle A_i x, x \rangle - \langle A_i y, y \rangle) \right. \\
& \quad \left. - 2 \sum_{1 \leq i < j \leq m} \exp\left(-\left\langle \frac{1}{2}(A_i + A_j)x, x \right\rangle - \left\langle \frac{1}{2}(A_i + A_j)y, y \right\rangle\right) \right] \\
& \leq \sum_{i=1}^m q_i \exp(-\langle A_i x, x \rangle - \langle A_i y, y \rangle) \\
& \quad - \exp\left(-\left\langle \sum_{i=1}^m q_i A_i, x \right\rangle - \left\langle \sum_{i=1}^m q_i A_i, y \right\rangle\right) \\
& \leq \max_{i \in \{1, \dots, m\}} \{q_i\} \left[ (m-1) \sum_{i=1}^m \exp(-\langle A_i x, x \rangle - \langle A_i y, y \rangle) \right. \\
& \quad \left. - 2 \sum_{1 \leq i < j \leq m} \exp\left(-\left\langle \frac{1}{2}(A_i + A_j)x, x \right\rangle - \left\langle \frac{1}{2}(A_i + A_j)y, y \right\rangle\right) \right]
\end{aligned}$$

for all  $x, y \in \mathbb{R}^n$ .

If we integrate on  $\mathbb{R}^n \times \mathbb{R}^n$ , then we get

$$\begin{aligned}
& \frac{1}{m-1} \min_{i \in \{1, \dots, m\}} \{q_i\} \left[ (m-1) \sum_{i=1}^m \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \exp(-\langle A_i x, x \rangle - \langle A_i y, y \rangle) dx dy \right. \\
& \quad \left. - 2 \sum_{1 \leq i < j \leq m} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \exp\left(-\left\langle \frac{A_i + A_j}{2}x, x \right\rangle - \left\langle \frac{A_i + A_j}{2}y, y \right\rangle\right) dx dy \right] \\
& \leq \sum_{i=1}^m q_i \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \exp(-\langle A_i x, x \rangle - \langle A_i y, y \rangle) dx dy \\
& \quad - \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \exp\left(-\left\langle \sum_{i=1}^m q_i A_i, x \right\rangle - \left\langle \sum_{i=1}^m q_i A_i, y \right\rangle\right) dx dy \\
& \leq \max_{i \in \{1, \dots, m\}} \{q_i\} \left[ (m-1) \sum_{i=1}^m \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \exp(-\langle A_i x, x \rangle - \langle A_i y, y \rangle) dx dy \right. \\
& \quad \left. - 2 \sum_{1 \leq i < j \leq m} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \exp\left(-\left\langle \frac{A_i + A_j}{2}x, x \right\rangle - \left\langle \frac{A_i + A_j}{2}y, y \right\rangle\right) dx dy \right]
\end{aligned}$$

and by (3.1) we get

$$\begin{aligned}
& \frac{1}{m-1} \min_{i \in \{1, \dots, m\}} \{q_i\} \left[ (m-1) \sum_{i=1}^m K_n(A_i) - 2 \sum_{1 \leq i < j \leq m} K_n\left(\frac{1}{2}(A_i + A_j)\right) \right] \\
& \leq \sum_{i=1}^m q_i K_n(A_i) - K_n\left(\sum_{i=1}^m q_i A_i\right) \\
& \leq \max_{i \in \{1, \dots, m\}} \{q_i\} \left[ (m-1) \sum_{i=1}^m K_n(A_i) - 2 \sum_{1 \leq i < j \leq m} K_n\left(\frac{1}{2}(A_i + A_j)\right) \right].
\end{aligned}$$

By using (3.1) we have further

$$\begin{aligned}
& \frac{1}{m-1} \min_{i \in \{1, \dots, m\}} \{q_i\} \\
& \times \left[ (m-1) \sum_{i=1}^m \frac{1}{\det(A_i)} - 2 \sum_{1 \leq i < j \leq m} \frac{1}{\det\left(\frac{1}{2}(A_i + A_j)\right)} \right] \\
& \leq \sum_{i=1}^m \frac{q_i}{\det(A_i)} - \frac{1}{\det\left(\sum_{i=1}^m q_i A_i\right)} \\
& \leq \max_{i \in \{1, \dots, m\}} \{q_i\} \\
& \times \left[ (m-1) \sum_{i=1}^m \frac{1}{\det(A_i)} - 2 \sum_{1 \leq i < j \leq m} \frac{1}{\det\left(\frac{1}{2}(A_i + A_j)\right)} \right],
\end{aligned}$$

which is equivalent to (3.2).  $\square$

**Corollary 2.** *If  $A_1, A_2$  are positive definite, then*

$$\begin{aligned}
(3.5) \quad 0 & \leq \frac{1}{4} \left[ [\det(A_1)]^{-1} + [\det(A_2)]^{-1} - 2^{n+1} [\det(A_1 + A_2)]^{-1} \right] \\
& \leq \frac{[\det(A_1)]^{-1} + [\det(A_2)]^{-1}}{2} - \int_0^1 [\det(tA_1 + (1-t)A_2)]^{-1} dt \\
& \leq \frac{3}{4} \left[ [\det(A_1)]^{-1} + [\det(A_2)]^{-1} - 2^{n+1} [\det(A_1 + A_2)]^{-1/2} \right].
\end{aligned}$$

**Remark 3.** If we take  $A_j = B_j^{-1}$ , where  $B_j > 0$ ,  $j = 1, \dots, m$ , then by (2.2) we get

$$\begin{aligned}
(3.6) \quad 0 &\leq \frac{1}{m-1} \min_{i \in \{1, \dots, m\}} \{q_i\} \\
&\times \left[ (m-1) \sum_{i=1}^m \det(B_i) - 2^{n+1} \sum_{1 \leq i < j \leq m} [\det(B_i^{-1} + B_j^{-1})]^{-1} \right] \\
&\leq \sum_{i=1}^m q_i \det(B_i) - \left[ \det \left( \sum_{i=1}^m q_i B_i^{-1} \right) \right]^{-1} \\
&\leq \max_{i \in \{1, \dots, m\}} \{q_i\} \\
&\times \left[ (m-1) \sum_{i=1}^m \det(B_i) - 2^{n+1} \sum_{1 \leq i < j \leq m} [\det(B_i^{-1} + B_j^{-1})]^{-1} \right].
\end{aligned}$$

In particular, for  $m = 2$  and  $t \in (0, 1)$ ,

$$\begin{aligned}
(3.7) \quad 0 &\leq \min \{t, 1-t\} \\
&\times \left[ \det(B_1) + \det(B_2) - 2^{n+1} [\det(B_1^{-1} + B_2^{-1})]^{-1} \right] \\
&\leq t \det(B_1) + (1-t) \det(B_2) - [\det(tB_1^{-1} + (1-t)B_2^{-1})]^{-1} \\
&\leq \min \{t, 1-t\} \\
&\times \left[ \det(B_1) + \det(B_2) - 2^{n+1} [\det(B_1^{-2} + B_2^{-2})]^{-1} \right].
\end{aligned}$$

From (2.8) we get

$$\begin{aligned}
(3.8) \quad 0 &\leq \frac{1}{4} \left[ \det(B_1) + \det(B_2) - 2^{n+1} [\det(B_1^{-1} + B_2^{-1})]^{-1} \right] \\
&\leq \frac{\det(B_1) + \det(B_2)}{2} - \int_0^1 [\det(tB_1^{-1} + (1-t)B_2^{-1})]^{-1} dt \\
&\leq \frac{3}{4} \left[ \det(B_1) + \det(B_2) - 2^{n+1} [\det(B_1^{-1} + B_2^{-1})]^{-1} \right].
\end{aligned}$$

#### 4. THE CASE OF HERMITIAN MATRICES

A complex square matrix  $H = (h_{ij})$ ,  $i, j = 1, \dots, n$  is said to be Hermitian provided  $h_{ij} = \overline{h_{ji}}$  for all  $i, j = 1, \dots, n$ . A Hermitian matrix is said to be positive definite if the Hermitian form  $P(z) = \sum_{i,j=1}^n a_{ij} z_i \overline{z_j}$  is positive for all  $z = (z_1, \dots, z_n) \in \mathbb{C}^n \setminus \{0\}$ .

It is known that, see for instance [9, p. 215], for a positive definite Hermitian matrix  $H$ , we have

$$(4.1) \quad K_n(H) := \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \exp(-\langle \bar{z}, Hz \rangle) dx dy = \frac{\pi^n}{\det(H)},$$

where  $z = x + iy$  and  $dx$  and  $dy$  denote integration over real  $n$ -dimensional space  $\mathbb{R}^n$ . Here the inner product  $\langle x, y \rangle$  is understood in the real sense, i.e.  $\langle x, y \rangle = \sum_{k=1}^n x_k y_k$ .

On making use of a similar argument to the one in Theorem 2 for the representation  $K_n(\cdot)$  we can state the following result as well:

**Theorem 3.** *Assume that  $(H_j)_{j=1, \dots, m}$  are Hermitian positive definite and  $q_j \geq 0$ ,  $j = 1, \dots, m$  with  $\sum_{j=1}^m q_j = 1$ . Then*

$$\begin{aligned}
 (4.2) \quad 0 &\leq \frac{1}{m-1} \min_{i \in \{1, \dots, m\}} \{q_i\} \\
 &\times \left[ (m-1) \sum_{i=1}^m [\det(H_i)]^{-1} - 2^{n+1} \sum_{1 \leq i < j \leq m} [\det(H_i + H_j)]^{-1} \right] \\
 &\leq \sum_{i=1}^m q_i [\det(H_i)]^{-1} - \left[ \det \left( \sum_{i=1}^m q_i H_i \right) \right]^{-1} \\
 &\leq \max_{i \in \{1, \dots, m\}} \{q_i\} \\
 &\times \left[ (m-1) \sum_{i=1}^m [\det(H_i)]^{-1} - 2^{n+1} \sum_{1 \leq i < j \leq m} [\det(H_i + H_j)]^{-1} \right].
 \end{aligned}$$

In particular, for  $m = 2$  and  $t \in (0, 1)$ ,

$$\begin{aligned}
 (4.3) \quad 0 &\leq \min \{t, 1-t\} \\
 &\times \left[ [\det(H_1)]^{-1} + [\det(H_2)]^{-1} - 2^{n+1} [\det(H_1 + H_2)]^{-1} \right] \\
 &\leq t [\det(H_1)]^{-1} + (1-t) [\det(H_2)]^{-1} - [\det(tH_1 + (1-t)H_2)]^{-1} \\
 &\leq \min \{t, 1-t\} \\
 &\times \left[ [\det(H_1)]^{-1} + [\det(H_2)]^{-1} - 2^{n+1} [\det(H_1 + H_2)]^{-1} \right].
 \end{aligned}$$

Also

$$\begin{aligned}
 (4.4) \quad 0 &\leq \frac{1}{4} \left[ [\det(H_1)]^{-1} + [\det(H_2)]^{-1} - 2^{n+1} [\det(H_1 + H_2)]^{-1} \right] \\
 &\leq \frac{[\det(H_1)]^{-1} + [\det(H_2)]^{-1}}{2} - \int_0^1 [\det(tH_1 + (1-t)H_2)]^{-1} dt \\
 &\leq \frac{3}{4} \left[ [\det(H_1)]^{-1} + [\det(H_2)]^{-1} - 2^{n+1} [\det(H_1 + H_2)]^{-1/2} \right].
 \end{aligned}$$

By taking  $H_i = K_i^{-1}$ ,  $i = 1, \dots, m$  with  $K_i$  Hermitian and positive definite we can obtain similar inequalities to the ones from Remark 3.

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