DETERMINANT INEQUALITIES FOR POSITIVE DEFINITE MATRICES VIA DIANANDA'S RESULT FOR ARITHMETIC AND GEOMETRIC WEIGHTED MEANS

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Abstract. In this paper we prove among others that, if $(A_j)_{j=1,...,m}$ are positive definite matrices of order $n \geq 2$ and $q_j \geq 0$, $j = 1, ..., m$ with $\sum_{j=1}^{m} q_j = 1$, then

$$
\begin{aligned}
0 &\leq \frac{1}{1-\min_{i\in\{1,...,m\}}\{q_i\}} \\
&\times \left[\sum_{i=1}^{m} q_i(1-q_i)[\det(A_i)]^{-1} - 2^{n+1}\sum_{1\leq i<j\leq m} q_i q_j [\det(A_i+A_j)]^{-1}\right] \\
&\leq \sum_{i=1}^{m} q_i [\det(A_i)]^{-1} - \left[\det\left(\sum_{i=1}^{m} q_i A_i\right)\right]^{-1} \\
&\leq \frac{1}{\min_{i\in\{1,...,m\}}\{q_i\}} \\
&\times \left[\sum_{i=1}^{m} q_i(1-q_i)[\det(A_i)]^{-1} - 2^{n+1}\sum_{1\leq i<j\leq m} q_i q_j [\det(A_i+A_j)]^{-1}\right].
\end{aligned}
$$

1. Introduction

A real square matrix $A = (a_{ij})$, $i, j = 1, ..., n$ is *symmetric* provided $a_{ij} = a_{ji}$ for all $i, j = 1, ..., n$. A real symmetric matrix is said to be *positive definite* provided the quadratic form $Q(x) = \sum_{i,j=1}^{n} a_{ij}x_i x_j$ is positive for all $x = (x_1, ..., x_n) \in \mathbb{R}^n \setminus \{0\}$. It is well known that a necessary and sufficient condition for the symmetric matrix A to be positive definite, and we write $A > 0$, is that all determinants

$$\det(A_k) = \det(a_{ij}),\ i,\ j = 1, ..., k;\ k = 1, ..., n$$

are positive.

It is know that the following integral representation is valid, see [1, pp. 61-62] or [9, pp. 211-212]

$$
\begin{aligned}
J_n(A) &:= \int_{\mathbb{R}^n} \exp(-\langle Ax, x\rangle)dx := \int_{-\infty}^{\infty} ... \int_{-\infty}^{\infty} \exp(-\langle Ax, x\rangle)dx \\
&= \frac{\pi^{n/2}}{[\det(A)]^{1/2}},
\end{aligned}
\tag{1.1}
$$

where A is a positive definite matrix of order n and $\langle \cdot, \cdot \rangle$ is the usual inner product on $\mathbb{R}^n$.

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By utilizing the representation (1.1) and Hölder's integral inequality for multiple integrals one can prove the *logarithmic concavity* of the determinant that is due to Ky Fan ([1, p. 63] or [9, p. 212]), namely

$$\det\left(\left(1-\lambda\right)A+\lambda B\right)\geq\left[\det\left(A\right)\right]^{1-\lambda}\left[\det\left(B\right)\right]^{\lambda} \tag{1.2}$$

for any positive definite matrices A, B and $\lambda\in[0,1]$.

By mathematical induction we can get a generalization of (1.2) which was obtained by L. Mirsky in [8], see also [9, p. 212]

$$\det\left(\sum_{j=1}^{m}\lambda_j A_j\right)\geq\prod_{j=1}^{m}\left[\det\left(A_j\right)\right]^{\lambda_j},\ m\geq 2, \tag{1.3}$$

where $\lambda_j>0$, $j=1,...,m$ with $\sum_{j=1}^{m}\lambda_j=1$ and $A_j>0$, $j=1,...,m$.

If we write (1.3) for $A_j=B_j^{-1}$ we get

$$\det\left(\sum_{j=1}^{m}\lambda_j B_j^{-1}\right)\geq\prod_{j=1}^{m}\left[\det\left(B_j^{-1}\right)\right]^{\lambda_j}=\left(\prod_{j=1}^{m}\left[\det\left(B_j\right)\right]^{\lambda_j}\right)^{-1},$$

which also gives

$$\prod_{j=1}^{m}\left[\det\left(A_j\right)\right]^{\lambda_j}\geq\det\left[\left(\sum_{j=1}^{m}\lambda_j A_j^{-1}\right)^{-1}\right], \tag{1.4}$$

where $\lambda_j>0$, $j=1,...,m$ with $\sum_{j=1}^{m}\lambda_j=1$ and $A_j>0$, $j=1,...,m$.

Using the representation (1.1) one can also prove the result, see [9, p. 212],

$$\det\left(A\right)=\det\left(A_{1n}\right)\leq\det\left(A_{1k}\right)\det\left(A_{(k+1)n}\right),\ k=1,...,n; \tag{1.5}$$

where the determinant $\det\left(A_{rs}\right)$ is defined by

$$\det\left(A_{rs}\right)=\det\left(a_{ij}\right),i,\ j=r,...,s.$$

In particular,

$$\det\left(A\right)\leq a_{11}a_{22}...a_{nn}. \tag{1.6}$$

We recall also the Minkowski's type inequality,

$$\left[\det\left(A+B\right)\right]^{1/n}\geq\left[\det\left(A\right)\right]^{1/n}+\left[\det\left(B\right)\right]^{1/n} \tag{1.7}$$

for A, B positive definite matrices of order n. For other determinant inequalities see Chapter VIII of the classic book [9]. For some recent results see [2]-[7].

In this paper we prove among others that, if $(A_j)_{j=1,...,m}$ are positive definite matrices of order $n\geq 2$ and $q_j\geq 0$, $j=1,...,m$ with $\sum_{j=1}^m q_j=1$, then

$$\begin{aligned}0 &\leq \frac{1}{1-\min_{i\in\{1,...,m\}}\{q_i\}}\\ &\times\left[\sum_{i=1}^m q_i(1-q_i)[\det(A_i)]^{-1}-2^{n+1}\sum_{1\leq i<j\leq m}q_iq_j[\det(A_i+A_j)]^{-1}\right]\\ &\leq \sum_{i=1}^m q_i[\det(A_i)]^{-1}-\left[\det\left(\sum_{i=1}^m q_iA_i\right)\right]^{-1}\\ &\leq \frac{1}{\min_{i\in\{1,...,m\}}\{q_i\}}\\ &\times\left[\sum_{i=1}^m q_i(1-q_i)[\det(A_i)]^{-1}-2^{n+1}\sum_{1\leq i<j\leq m}q_iq_j[\det(A_i+A_j)]^{-1}\right].\end{aligned}$$

2. Inequalities Via Single Integral Representation

We recall the following classical result due to P. H. Diananda, see [3]:

$$\begin{aligned}0 &\leq \frac{1}{2\left(1-\min_{i\in\{1,...,m\}}\{q_i\}\right)}\sum_{i,j=1}^m q_iq_j(\sqrt{x_i}-\sqrt{x_j})^2\\ &\leq \sum_{i=1}^m q_ix_i-\prod_{i=1}^m x_i^{q_i}\\ &\leq \frac{1}{2\min_{i\in\{1,...,m\}}\{q_i\}}\sum_{i,j=1}^m q_iq_j(\sqrt{x_i}-\sqrt{x_j})^2,\end{aligned}\tag{2.1}$$

where $x_i,\ q_i\geq 0$, $i\in\{1,...,m\}$ with $\sum_{i=1}^m q_i=1$.

Theorem 1. *Assume that $(A_j)_{j=1,...,m}$ are positive definite and $q_j\geq 0$, $j=1,...,m$ with $\sum_{j=1}^m q_j=1$. Then*

$$\begin{aligned}0 &\leq \frac{1}{1-\min_{i\in\{1,...,m\}}\{q_i\}}\left[\sum_{i=1}^m q_i(1-q_i)[\det(A_i)]^{-1/2}\right.\\ &\left.-2^{n/2+1}\sum_{1\leq i<j\leq m}q_iq_j[\det(A_i+A_j)]^{-1/2}\right]\\ &\leq \sum_{i=1}^m q_i[\det(A_i)]^{-1/2}-\left[\det\left(\sum_{i=1}^m q_iA_i\right)\right]^{-1/2}\\ &\leq \frac{1}{\min_{i\in\{1,...,m\}}\{q_i\}}\left[\sum_{i=1}^m q_i(1-q_i)[\det(A_i)]^{-1/2}\right.\\ &\left.-2^{n/2+1}\sum_{1\leq i<j\leq m}q_iq_j[\det(A_i+A_j)]^{-1/2}\right].\end{aligned}\tag{2.2}$$

In particular, for $m=2$ *and* $t\in(0,1)$ *we get*

$$
\begin{aligned}
(2.3)\quad 0 &\le \min\{1-t,t\} \\
&\times\left[[\det(A_1)]^{-1/2}+[\det(A_2)]^{-1/2}-2^{n/2+1}[\det(A_1+A_2)]^{-1/2}\right] \\
&\le (1-t)[\det(A_1)]^{-1/2}+t[\det(A_2)]^{-1/2}-[\det((1-t)A_1+tA_2)]^{-1/2} \\
&\le \max\{1-t,t\} \\
&\times\left[[\det(A_1)]^{-1/2}+[\det(A_2)]^{-1/2}-2^{n/2+1}[\det(A_1+A_2)]^{-1/2}\right].
\end{aligned}
$$

Proof. Now we observe that

$$
\begin{aligned}
\frac{1}{2}\sum_{i,j=1}^{m} q_iq_j(\sqrt{x_i}-\sqrt{x_j})^2 &= \frac{1}{2}\sum_{i,j=1}^{m} q_iq_j\left(x_i-2\sqrt{x_i}\sqrt{x_j}+x_j\right) \\
&= \frac{1}{2}\sum_{i=1}^{m} q_ix_i-\sum_{i,j=1}^{m} q_iq_j\sqrt{x_i}\sqrt{x_j}+\frac{1}{2}\sum_{j=1}^{m} q_jx_j \\
&= \sum_{i=1}^{m} q_ix_i-\sum_{i,j=1}^{m} q_iq_j\sqrt{x_i}\sqrt{x_j} \\
&= \sum_{i=1}^{m} q_ix_i-\sum_{i=1}^{m} q_i^2x_i-2\sum_{1\le i<j\le m} q_iq_j\sqrt{x_i}\sqrt{x_j} \\
&= \sum_{i=1}^{m} q_i(1-q_i)x_i-2\sum_{1\le i<j\le m} q_iq_j\sqrt{x_i}\sqrt{x_j},
\end{aligned}
$$

therefore (2.1) becomes

$$
\begin{aligned}
(2.4)\quad 0 &\le \frac{1}{1-\min_{i\in\{1,\dots,m\}}\{q_i\}}\left[\sum_{i=1}^{m} q_i(1-q_i)x_i-2\sum_{1\le i<j\le m} q_iq_j\sqrt{x_i}\sqrt{x_j}\right] \\
&\le \sum_{i=1}^{m} q_ix_i-\prod_{i=1}^{m} x_i^{q_i} \\
&\le \frac{1}{\min_{i\in\{1,\dots,m\}}\{q_i\}}\left[\sum_{i=1}^{m} q_i(1-q_i)x_i-2\sum_{1\le i<j\le m} q_iq_j\sqrt{x_i}\sqrt{x_j}\right],
\end{aligned}
$$

which is a more convenient inequality for our considerations below.

If we take $x_i = \exp(-\langle A_i x, x\rangle)$, $i \in \{1, ..., m\}$, $x \in \mathbb{R}^n$ in (2.4), then we get

$$0 \le \frac{1}{1-\min_{i\in\{1,\dots,m\}}\{q_i\}}\left[\sum_{i=1}^m q_i(1-q_i)\exp(-\langle A_i x, x\rangle) - 2\sum_{1\le i<j\le m} q_i q_j \exp\left(-\left\langle \frac{1}{2}(A_i+A_j)x, x\right\rangle\right)\right] \le \sum_{i=1}^m q_i \exp(-\langle A_i x, x\rangle) - \exp\left(-\left\langle \sum_{i=1}^m q_i A_i x, x\right\rangle\right) \le \frac{1}{\min_{i\in\{1,\dots,m\}}\{q_i\}}\left[\sum_{i=1}^m q_i(1-q_i)\exp(-\langle A_i x, x\rangle), -2\sum_{1\le i<j\le m} q_i q_j \exp\left(-\left\langle \frac{1}{2}(A_i+A_j)x, x\right\rangle\right)\right]. \tag{2.5}$$

Integrating (2.5) on $\mathbb{R}^n$, we get

$$0 \le \frac{1}{1-\min_{i\in\{1,\dots,m\}}\{q_i\}}\left[\sum_{i=1}^m q_i(1-q_i)\int_{\mathbb{R}^n}\exp(-\langle A_i x, x\rangle)\,dx - 2\sum_{1\le i<j\le m} q_i q_j \int_{\mathbb{R}^n}\exp\left(-\left\langle \frac{1}{2}(A_i+A_j)x, x\right\rangle\right)dx\right] \le \sum_{i=1}^m q_i \int_{\mathbb{R}^n}\exp(-\langle A_i x, x\rangle)\,dx - \int_{\mathbb{R}^n}\exp\left(-\left\langle \sum_{i=1}^m q_i A_i x, x\right\rangle\right)dx \le \frac{1}{\min_{i\in\{1,\dots,m\}}\{q_i\}}\left[\sum_{i=1}^m q_i(1-q_i)\int_{\mathbb{R}^n}\exp(-\langle A_i x, x\rangle)\,dx, -2\sum_{1\le i<j\le m} q_i q_j \int_{\mathbb{R}^n}\exp\left(-\left\langle \frac{1}{2}(A_i+A_j)x, x\right\rangle\right)dx\right]. \tag{2.6}$$

By using the representation (1.1) we obtain from (2.6) that

$$0 \le \frac{1}{1-\min_{i\in\{1,\dots,m\}}\{q_i\}} \times \left[\sum_{i=1}^m q_i(1-q_i)J_n(A_i) - 2\sum_{1\le i<j\le m} q_i q_j J_n\left(\frac{1}{2}(A_i+A_j)\right)\right] \le \sum_{i=1}^m q_i J_n(A_i) - J_n\left(\sum_{i=1}^m q_i A_i\right) \le \frac{1}{\min_{i\in\{1,\dots,m\}}\{q_i\}} \times \left[\sum_{i=1}^m q_i(1-q_i)J_n(A_i) - 2\sum_{1\le i<j\le m} q_i q_j J_n\left(\frac{1}{2}(A_i+A_j)\right)\right],$$

which is equivalent to

$$\begin{aligned}
0 &\leq \frac{1}{1-\min_{i\in\{1,\dots,m\}}\{q_i\}} \\
&\times \left[\sum_{i=1}^{m} \frac{q_i(1-q_i)}{[\det(A_i)]^{1/2}} - 2\sum_{1\leq i<j\leq m} \frac{q_i q_j}{\left[\det\left(\frac{1}{2}(A_i+A_j)\right)\right]^{1/2}}\right] \\
&\leq \sum_{i=1}^{m} \frac{q_i}{[\det(A_i)]^{1/2}} - \frac{1}{\left[\det\left(\sum_{i=1}^{m} q_i A_i\right)\right]^{1/2}} \\
&\leq \frac{1}{\min_{i\in\{1,\dots,m\}}\{q_i\}} \\
&\times \left[\sum_{i=1}^{m} \frac{q_i(1-q_i)}{[\det(A_i)]^{1/2}} - 2\sum_{1\leq i<j\leq m} \frac{q_i q_j}{\left[\det\left(\frac{1}{2}(A_i+A_j)\right)\right]^{1/2}}\right],
\end{aligned}$$

and the inequality (2.2) is proved.

Now, by taking $m=2$, $q_1=1-t$ and $q_2=t$, then by (2.2) we get

$$\begin{aligned}
0 &\leq \frac{(1-t)t}{1-\min\{1-t,t\}} \\
&\times \left[[\det(A_1)]^{-1/2} + [\det(A_2)]^{-1/2} - 2^{n/2+1}[\det(A_1+A_2)]^{-1/2}\right] \\
&\leq (1-t)[\det(A_1)]^{-1/2} + t[\det(A_2)]^{-1/2} - [\det((1-t)A_1+tA_2)]^{-1/2} \\
&\leq \frac{(1-t)t}{\min\{1-t,t\}} \\
&\times \left[[\det(A_1)]^{-1/2} + [\det(A_2)]^{-1/2} - 2^{n/2+1}[\det(A_1+A_2)]^{-1/2}\right].
\end{aligned} \tag{2.7}$$

Observe that

$$\frac{(1-t)t}{\min\{1-t,t\}} = \frac{\min\{1-t,t\}\max\{1-t,t\}}{\min\{1-t,t\}} = \max\{1-t,t\}$$

and

$$\frac{(1-t)t}{1-\min\{1-t,t\}} = \frac{(1-t)t}{1-t} = t, \text{ for } t\in(0,1/2)$$

while

$$\frac{(1-t)t}{1-\min\{1-t,t\}} = \frac{(1-t)t}{t} = 1-t, \text{ for } t\in(1/2,1).$$

Therefore

$$\frac{(1-t)t}{1-\min\{1-t,t\}} = \min\{1-t,t\}, \ t\in(0,1)$$

and by (2.7) we get (2.3). □

Remark 1. *If we take in (2.2) $q_j = \frac{1}{m}$, $j = 1, ..., m$, then we get*

$$
\begin{aligned}
(2.8)\quad 0 &\leq \frac{1}{(m-1)\,m} \\
&\times \left[(m-1)\sum_{i=1}^{m}\left[\det(A_i)\right]^{-1/2} - 2^{n/2+1}\sum_{1\leq i<j\leq m}\left[\det(A_i + A_j)\right]^{-1/2}\right] \\
&\leq \frac{1}{m}\sum_{i=1}^{m}\left[\det(A_i)\right]^{-1/2} - \left[\det\left(\frac{\sum_{i=1}^{m} A_i}{m}\right)\right]^{-1/2} \\
&\leq \frac{1}{m}\left[(m-1)\sum_{i=1}^{m}\left[\det(A_i)\right]^{-1/2} - 2^{n/2+1}\sum_{1\leq i<j\leq m}\left[\det(A_i + A_j)\right]^{-1/2}\right].
\end{aligned}
$$

Corollary 1. *If A_1, A_2 are positive definite, then*

$$
\begin{aligned}
(2.9)\quad 0 &\leq \frac{1}{4}\left[\left[\det(A_1)\right]^{-1/2} + \left[\det(A_2)\right]^{-1/2} - 2^{n/2+1}\left[\det(A_1 + A_2)\right]^{-1/2}\right] \\
&\leq \frac{\left[\det(A_1)\right]^{-1/2} + \left[\det(A_2)\right]^{-1/2}}{2} - \int_0^1 \left[\det(tA_1 + (1-t)A_2)\right]^{-1/2} dt \\
&\leq \frac{3}{4}\left[\left[\det(A_1)\right]^{-1/2} + \left[\det(A_2)\right]^{-1/2} - 2^{n/2+1}\left[\det(A_1 + A_2)\right]^{-1/2}\right].
\end{aligned}
$$

Proof. If we take the integral in (2.3) we get

$$
\begin{aligned}
(2.10)\quad 0 &\leq \int_0^1 \min\{t, 1-t\}\, dt \\
&\times \left[\left[\det(A_1)\right]^{-1/2} + \left[\det(A_2)\right]^{-1/2} - 2^{n/2+1}\left[\det(A_1 + A_2)\right]^{-1/2}\right] \\
&\leq \frac{\left[\det(A_1)\right]^{-1/2} + \left[\det(A_2)\right]^{-1/2}}{2} - \int_0^1 \left[\det(tA_1 + (1-t)A_2)\right]^{-1/2} dt \\
&\leq \int_0^1 \max\{t, 1-t\}\, dt \\
&\times \left[\left[\det(A_1)\right]^{-1/2} + \left[\det(A_2)\right]^{-1/2} - 2^{n/2+1}\left[\det(A_1 + A_2)\right]^{-1/2}\right].
\end{aligned}
$$

Since

$$\int_0^1 \min\{t, 1-t\}\, dt = \frac{1}{4} \text{ and } \int_0^1 \max\{t, 1-t\}\, dt = \frac{3}{4},$$

hence by (2.10) we get (2.9). □

Remark 2. *If we take* $A_j = B_j^{-2}$*, where* $B_j > 0$*,* $j = 1, ..., m$*, then by (2.5) we obtain*

$$0 \le \frac{1}{1 - \min_{i \in \{1,...,m\}} \{q_i\}} \left[\sum_{i=1}^{m} q_i (1 - q_i) \det (B_i) - 2^{n/2+1} \sum_{1 \le i < j \le m} q_i q_j \left[\det \left(B_i^{-2} + B_j^{-2} \right) \right]^{-1/2} \right] \le \sum_{i=1}^{m} q_i \det (B_i) - \left[\det \left(\sum_{i=1}^{m} q_i B_i^{-2} \right) \right]^{-1/2} \le \frac{1}{\min_{i \in \{1,...,m\}} \{q_i\}} \left[\sum_{i=1}^{m} q_i (1 - q_i) \det (B_i) - 2^{n/2+1} \sum_{1 \le i < j \le m} q_i q_j \left[\det \left(B_i^{-2} + B_j^{-2} \right) \right]^{-1/2} \right]. \tag{2.11}$$

In particular, for $m = 2$ *and* $t \in (0, 1)$ *we derive*

$$0 \le \min \{1 - t, t\} \times \left[\det (B_1) + \det (B_2) - 2^{n/2+1} \left[\det \left(B_1^{-2} + B_2^{-2} \right) \right]^{-1/2} \right] \le (1 - t) \det (B_1) + t \det (B_2) - \left[\det \left((1 - t) B_1^{-2} + t B_2^{-2} \right) \right]^{-1/2} \le \max \{1 - t, t\} \times \left[\det (B_1) + \det (B_2) - 2^{n/2+1} \left[\det \left(B_1^{-2} + B_2^{-2} \right) \right]^{-1/2} \right]. \tag{2.12}$$

From (2.9) we also obtain

$$0 \le \frac{1}{4} \left[\det (B_1) + \det (B_2) - 2^{n/2+1} \left[\det \left(B_1^{-2} + B_2^{-2} \right) \right]^{-1/2} \right] \le \frac{\det (B_1) + \det (B_2)}{2} - \int_0^1 \left[\det \left((1 - t) B_1^{-2} + t B_2^{-2} \right) \right]^{-1/2} dt \le \frac{3}{4} \left[\det (B_1) + \det (B_2) - 2^{n/2+1} \left[\det \left(B_1^{-2} + B_2^{-2} \right) \right]^{-1/2} \right]. \tag{2.13}$$

3. Inequalities Via Double Integral Representation

If we take the square in the representation (1.1), then we get

$$\left(\int_{\mathbb{R}^n} \exp(- \langle x, Ax \rangle) dx \right)^2 = \frac{\pi^n}{\det (A)}.$$

Since

$$\left(\int_{\mathbb{R}^n} \exp(-\langle Ax,x\rangle)dx\right)^2 = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \exp(-\langle Ax,x\rangle)\exp(-\langle Ay,y\rangle)dxdy$$
$$= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \exp(-\langle Ax,x\rangle - \langle Ay,y\rangle))dxdy,$$

hence

$$K_n(A) := \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \exp(-\langle Ax,x\rangle - \langle Ay,y\rangle))dxdy = \frac{\pi^n}{\det(A)} \tag{3.1}$$

for A a positive definite matrix of order n and $\langle\cdot,\cdot\rangle$ is the usual inner product on $\mathbb{R}^n$.

Theorem 2. *Assume that* $(A_j)_{j=1,...,m}$ *are positive definite and* $q_j \geq 0$, $j = 1,...,m$ *with* $\sum_{j=1}^m q_j = 1$. *Then*

$$\begin{aligned} 0 &\leq \frac{1}{1-\min_{i\in\{1,...,m\}}\{q_i\}} \\ &\times \left[\sum_{i=1}^m q_i(1-q_i)[\det(A_i)]^{-1} - 2^{n+1}\sum_{1\leq i<j\leq m} q_iq_j[\det(A_i+A_j)]^{-1}\right] \\ &\leq \sum_{i=1}^m q_i[\det(A_i)]^{-1} - \left[\det\left(\sum_{i=1}^m q_iA_i\right)\right]^{-1} \\ &\leq \frac{1}{\min_{i\in\{1,...,m\}}\{q_i\}} \\ &\times \left[\sum_{i=1}^m q_i(1-q_i)[\det(A_i)]^{-1} - 2^{n+1}\sum_{1\leq i<j\leq m} q_iq_j[\det(A_i+A_j)]^{-1}\right]. \end{aligned} \tag{3.2}$$

In particular, for $m = 2$ *and* $t \in (0,1)$ *we get*

$$\begin{aligned} 0 &\leq \min\{1-t,t\} \\ &\times \left[[\det(A_1)]^{-1} + [\det(A_2)]^{-1} - 2^{n+1}[\det(A_1+A_2)]^{-1}\right] \\ &\leq (1-t)[\det(A_1)]^{-1} + t[\det(A_2)]^{-1} - [\det((1-t)A_1+tA_2)]^{-1} \\ &\leq \max\{1-t,t\} \\ &\times \left[[\det(A_1)]^{-1} + [\det(A_2)]^{-1} - 2^{n+1}[\det(A_1+A_2)]^{-1}\right]. \end{aligned} \tag{3.3}$$

Proof. If we use the inequality (2.4) for $x_i = \exp\left(-\langle A_i x, x\rangle - \langle A_j y, y\rangle\right),\ i \in \{1, ..., m\},\ x,\ y \in \mathbb{R}^n$, then we get

$$
\begin{aligned}
(3.4)\qquad 0 &\le \frac{1}{1-\min_{i\in\{1,\ldots,m\}}\{q_i\}}\left[\sum_{i=1}^{m} q_i\left(1-q_i\right)\exp\left(-\langle A_i x, x\rangle - \langle A_j y, y\rangle\right)\right.\\
&\left. -2\sum_{1\le i<j\le m} q_i q_j \exp\left(-\left\langle \frac{1}{2}\left(A_i + A_j\right)x, x\right\rangle - \left\langle \frac{1}{2}\left(A_i + A_j\right)y, y\right\rangle\right)\right]\\
&\le \sum_{i=1}^{m} q_i \exp\left(-\langle A_i x, x\rangle - \langle A_j y, y\rangle\right)\\
&- \exp\left(-\left\langle \sum_{i=1}^{m} q_i A_i x, x\right\rangle - \left\langle \sum_{i=1}^{m} q_i A_i x, x\right\rangle\right)\\
&\le \frac{1}{\min_{i\in\{1,\ldots,m\}}\{q_i\}}\left[\sum_{i=1}^{m} q_i\left(1-q_i\right)\exp\left(-\langle A_i x, x\rangle - \langle A_j y, y\rangle\right),\right.\\
&\left. -2\sum_{1\le i<j\le m} q_i q_j \exp\left(-\left\langle \frac{1}{2}\left(A_i + A_j\right)x, x\right\rangle - \left\langle \frac{1}{2}\left(A_i + A_j\right)y, y\right\rangle\right)\right].
\end{aligned}
$$

If we integrate (3.4) on $\mathbb{R}^n \times \mathbb{R}^n$, then we obtain

$$
\begin{aligned}
(3.5)\qquad 0 &\le \frac{1}{1-\min_{i\in\{1,\ldots,m\}}\{q_i\}}\\
&\times\left[\sum_{i=1}^{m} q_i\left(1-q_i\right)\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\exp\left(-\langle A_i x, x\rangle - \langle A_j y, y\rangle\right)dxdy\right.\\
&- 2\sum_{1\le i<j\le m} q_i q_j\\
&\left.\times\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\exp\left(-\left\langle \frac{1}{2}\left(A_i + A_j\right)x, x\right\rangle - \left\langle \frac{1}{2}\left(A_i + A_j\right)y, y\right\rangle\right)dxdy\right]\\
&\le \sum_{i=1}^{m} q_i \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\exp\left(-\langle A_i x, x\rangle - \langle A_j y, y\rangle\right)dxdy\\
&- \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\exp\left(-\left\langle \sum_{i=1}^{m} q_i A_i x, x\right\rangle - \left\langle \sum_{i=1}^{m} q_i A_i x, x\right\rangle\right)dxdy\\
&\le \frac{1}{\min_{i\in\{1,\ldots,m\}}\{q_i\}}\\
&\left[\sum_{i=1}^{m} q_i\left(1-q_i\right)\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\exp\left(-\langle A_i x, x\rangle - \langle A_j y, y\rangle\right)dxdy\ ,\right.\\
&- 2\sum_{1\le i<j\le m} q_i q_j\\
&\left.\times\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\exp\left(-\left\langle \frac{1}{2}\left(A_i + A_j\right)x, x\right\rangle - \left\langle \frac{1}{2}\left(A_i + A_j\right)y, y\right\rangle\right)dxdy\right].
\end{aligned}
$$

By utilizing the representation (3.1), we derive

$$\begin{aligned} 0 &\leq \frac{1}{1-\min_{i\in\{1,...,m\}}\{q_i\}} \\ &\quad \times \left[\sum_{i=1}^{m} q_i(1-q_i) K_n(A_i) - 2\sum_{1\leq i<j\leq m} q_i q_j K_n\left(\frac{1}{2}(A_i+A_j)\right)\right] \\ &\leq \sum_{i=1}^{m} q_i K_n(A_i) - K_n\left(\sum_{i=1}^{m} q_i A_i\right) \\ &\leq \frac{1}{\min_{i\in\{1,...,m\}}\{q_i\}} \\ &\quad \times \left[\sum_{i=1}^{m} q_i(1-q_i) K_n(A_i) - 2\sum_{1\leq i<j\leq m} q_i q_j K_n\left(\frac{1}{2}(A_i+A_j)\right)\right], \end{aligned}$$

namely

$$\begin{aligned} 0 &\leq \frac{1}{1-\min_{i\in\{1,...,m\}}\{q_i\}} \left[\sum_{i=1}^{m} \frac{q_i(1-q_i)}{\det(A_i)} - 2\sum_{1\leq i<j\leq m} \frac{q_i q_j}{\det\left(\frac{1}{2}(A_i+A_j)\right)}\right] \\ &\leq \sum_{i=1}^{m} \frac{q_i}{\det(A_i)} - \frac{1}{\det\left(\sum_{i=1}^{m} q_i A_i\right)} \\ &\leq \frac{1}{\min_{i\in\{1,...,m\}}\{q_i\}} \left[\sum_{i=1}^{m} \frac{q_i(1-q_i)}{\det(A_i)} - 2\sum_{1\leq i<j\leq m} \frac{q_i q_j}{\det\left(\frac{1}{2}(A_i+A_j)\right)}\right], \end{aligned}$$

which is equivalent to (3.2). □

Remark 3. *If we take in (3.2) $q_j = \frac{1}{m}$, $j = 1, ..., m$, then we get*

$$\begin{aligned} 0 &\leq \frac{1}{(m-1)m} \\ &\quad \times \left[(m-1)\sum_{i=1}^{m} [\det(A_i)]^{-1} - 2^{n+1}\sum_{1\leq i<j\leq m} [\det(A_i+A_j)]^{-1}\right] \\ &\leq \frac{1}{m}\sum_{i=1}^{m} [\det(A_i)]^{-1} - \left[\det\left(\frac{\sum_{i=1}^{m} A_i}{m}\right)\right]^{-1} \\ &\leq \frac{1}{m}\left[(m-1)\sum_{i=1}^{m} [\det(A_i)]^{-1} - 2^{n+1}\sum_{1\leq i<j\leq m} [\det(A_i+A_j)]^{-1}\right]. \end{aligned} \tag{3.6}$$

Corollary 2. *If A_1, A_2 are positive definite, then*

$$0 \leq \frac{1}{4}\left[[\det(A_1)]^{-1} + [\det(A_2)]^{-1} - 2^{n+1}[\det(A_1 + A_2)]^{-1}\right] \tag{3.7}$$

$$\leq \frac{[\det(A_1)]^{-1} + [\det(A_2)]^{-1}}{2} - \int_0^1 [\det(tA_1 + (1-t)A_2)]^{-1} dt$$

$$\leq \frac{3}{4}\left[[\det(A_1)]^{-1} + [\det(A_2)]^{-1} - 2^{n+1}[\det(A_1 + A_2)]^{-1}\right].$$

Remark 4. *If we take $A_j = B_j^{-1}$, where $B_j > 0$, $j = 1, ..., m$, then by (2.5) we get*

$$0 \leq \frac{1}{1 - \min_{i\in\{1,...,m\}}\{q_i\}}\left[\sum_{i=1}^m q_i(1-q_i)\det(B_i)\right. \tag{3.8}$$

$$\left. -2^{n+1}\sum_{1\leq i<j\leq m} q_i q_j \left[\det\left(B_i^{-1} + B_j^{-1}\right)\right]^{-1}\right]$$

$$\leq \sum_{i=1}^m q_i \det(B_i) - \left[\det\left(\sum_{i=1}^m q_i B_i^{-1}\right)\right]^{-1}$$

$$\leq \frac{1}{\min_{i\in\{1,...,m\}}\{q_i\}}\left[\sum_{i=1}^m q_i(1-q_i)\det(B_i)\right.$$

$$\left. -2^{n+1}\sum_{1\leq i<j\leq m} q_i q_j \left[\det\left(B_i^{-1} + B_j^{-1}\right)\right]^{-1}\right].$$

In particular, for $m = 2$ and $t \in (0,1)$ we get

$$0 \leq \min\{1-t, t\} \tag{3.9}$$

$$\times \left[\det(B_1) + \det(B_2) - 2^{n+1}\left[\det\left(B_1^{-1} + B_2^{-1}\right)\right]^{-1}\right]$$

$$\leq (1-t)\det(B_1) + t\det(B_2) - \left[\det\left((1-t)B_1^{-1} + tB_2^{-1}\right)\right]^{-1}$$

$$\leq \max\{1-t, t\}$$

$$\times \left[\det(B_1) + \det(B_2) - 2^{n+1}\left[\det\left(B_1^{-1} + B_2^{-1}\right)\right]^{-1}\right].$$

From (2.9) we also obtain

$$0 \le \frac{1}{4}\left[\det(B_1) + \det(B_2) - 2^{n+1}\left[\det\left(B_1^{-1} + B_2^{-1}\right)\right]^{-1}\right] \tag{3.10}$$
$$\le \frac{\det(B_1) + \det(B_2)}{2} - \int_0^1 \left[\det\left((1-t)B_1^{-1} + tB_2^{-2}\right)\right]^{-1} dt$$
$$\le \frac{3}{4}\left[\det(B_1) + \det(B_2) - 2^{n+1}\left[\det\left(B_1^{-1} + B_2^{-1}\right)\right]^{-1}\right].$$

4. The Case of Hermitian Matrices

A complex square matrix $H = (h_{ij})$, $i, j = 1, ..., n$ is said to be Hermitian provided $h_{ij} = \overline{h_{ji}}$ for all i, $j = 1, ..., n$. A Hermitian matrix is said to be positive definite if the Hermitian form $P(z) = \sum_{i,j=1}^{n} a_{ij} z_i \overline{z_j}$ is positive for all $z = (z_1, ..., z_n) \in \mathbb{C}^n \setminus \{0\}$.

It is known that, see for instance [9, p. 215], for a positive definite Hermitian matrix H, we have

$$K_n(H) := \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \exp\left(-\langle \overline{z}, Hz\rangle\right) dxdy = \frac{\pi^n}{\det(H)}, \tag{4.1}$$

where $z = x + iy$ and dx and dy denote integration over real n-dimensional space $\mathbb{R}^n$. Here the inner product $\langle x, y\rangle$ is understood in the real sense, i.e. $\langle x, y\rangle = \sum_{k=1}^{n} x_k y_k$.

On making use of a similar argument to the one in Theorem 2 for the representation $K_n(\cdot)$ we can state the following result as well:

Theorem 3. *Assume that $(H_j)_{j=1,...,m}$ are positive definite and $q_j \ge 0$, $j = 1, ..., m$ with $\sum_{j=1}^{m} q_j = 1$. Then*

$$0 \le \frac{1}{1 - \min_{i\in\{1,...,m\}}\{q_i\}} \tag{4.2}$$
$$\times \left[\sum_{i=1}^{m} q_i(1-q_i)\left[\det(H_i)\right]^{-1} - 2^{n+1}\sum_{1\le i<j\le m} q_i q_j \left[\det(H_i + H_j)\right]^{-1}\right]$$
$$\le \sum_{i=1}^{m} q_i\left[\det(H_i)\right]^{-1} - \left[\det\left(\sum_{i=1}^{m} q_i H_i\right)\right]^{-1}$$
$$\le \frac{1}{\min_{i\in\{1,...,m\}}\{q_i\}}$$
$$\times \left[\sum_{i=1}^{m} q_i(1-q_i)\left[\det(H_i)\right]^{-1} - 2^{n+1}\sum_{1\le i<j\le m} q_i q_j \left[\det(H_i + H_j)\right]^{-1}\right].$$

In particular, for $m=2$ and $t\in(0,1)$ we get

$$
\begin{aligned}
(4.3)\qquad 0 &\leq \min\{1-t,t\} \\
&\times\left[[\det(H_1)]^{-1}+[\det(H_2)]^{-1}-2^{n+1}[\det(H_1+H_2)]^{-1}\right] \\
&\leq (1-t)[\det(H_1)]^{-1}+t[\det(H_2)]^{-1}-[\det((1-t)H_1+tH_2)]^{-1} \\
&\leq \max\{1-t,t\} \\
&\times\left[[\det(H_1)]^{-1}+[\det(H_2)]^{-1}-2^{n+1}[\det(H_1+H_2)]^{-1}\right].
\end{aligned}
$$

Also,

$$
\begin{aligned}
(4.4)\qquad 0 &\leq \frac{1}{4}\left[[\det(H_1)]^{-1}+[\det(H_2)]^{-1}-2^{n+1}[\det(H_1+H_2)]^{-1}\right] \\
&\leq \frac{[\det(H_1)]^{-1}+[\det(H_2)]^{-1}}{2}-\int_0^1[\det(tH_1+(1-t)H_2)]^{-1}dt \\
&\leq \frac{3}{4}\left[[\det(H_1)]^{-1}+[\det(H_2)]^{-1}-2^{n+1}[\det(H_1+H_2)]^{-1}\right].
\end{aligned}
$$

By taking $H_i=K_i^{-1}$, $i=1,...,m$ with K_i Hermitian and positive definite we can obtain similar inequalities to the ones from Remark 4.

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