SOME NEW NORM INEQUALITIES FOR THE NONCOMMUTATIVE ČEBYŠEV FUNCTIONAL IN BANACH ALGEBRAS

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Abstract. Let $\mathcal{B}$ be a complex Banach algebra. For two continuous functions $x,\ y:[a,b]\to\mathcal{B}$ we define the *noncommutative Čebyšev functional*

$$D(x,y):=(b-a)\int_a^b x(t)\,y(t)\,dt-\int_a^b x(t)\,dt\int_a^b y(t)\,dt.$$

In this paper we show among others that if x, y are strongly differentiable, then

$$\begin{aligned}&\|D(x,y)\|\\&\le\frac{1}{2}(b-a)\left\|x'\right\|_{[a,b],\infty}\int_a^b(b-t)(t-a)\left\|y'(t)\right\|dt\\&\le\begin{cases}\frac{1}{8}(b-a)^3\|x'\|_{[a,b],\infty}\|y'\|_{[a,b],1}\,,\\[2mm]\frac{1}{2}(b-a)^{3+1/q}\left[B(q+1,q+1)\right]^{1/q}\|x'\|_{[a,b],\infty}\|y'\|_{[a,b],p}\,,\\[2mm]\frac{1}{12}(b-a)^4\|x'\|_{[a,b],\infty}\|y'\|_{[a,b],\infty}\,,\end{cases}\end{aligned}$$

where

$$\left\|z'\right\|_{[a,b],r}:=\left(\int_a^b\left\|z'(t)\right\|^r dt\right)^{1/r}\ \text{ and }\ \left\|z'\right\|_{[a,b],\infty}:=\sup_{t\in(a,b)}\left\|z'(t)\right\|$$

for a strongly differentiable function z on (a,b), $B(\cdot,\cdot)$ is Beta function and p, $q>1$ with $\frac{1}{p}+\frac{1}{q}=1$. Some applications for analytic functions of elements in Banach algebras with examples for exponential function are also given.

1. Introduction

For two Lebesgue integrable functions $f,\ g:[a,b]\to\mathbb{C}$, in order to compare the integral mean of the product with the product of the integral means, we consider the *Čebyšev functional* defined by

$$D(f,g):=(b-a)\int_a^b f(t)\,g(t)\,dt-\int_a^b f(t)\,dt\int_a^b g(t)\,dt.$$

In 1934, G. Grüss [25] showed that

$$|D(f,g)|\le\frac{1}{4}(b-a)^2(M-m)(N-n)\,, \tag{1.1}$$

provided m, M, n, N are real numbers with the property that

$$-\infty<m\le f\le M<\infty,\quad -\infty<n\le g\le N<\infty\quad \text{a.e. on}\quad [a,b]\,. \tag{1.2}$$

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The constant $\frac{1}{4}$ is best possible in (1.1) in the sense that it cannot be replaced by a smaller one.

Another lesser known inequality for $D(f,g)$ was derived in 1882 by Čebyšev [7] under the assumption that f', g' exist and are continuous on $[a,b]$, and is given by

$$|D(f,g)| \le \frac{1}{12} \|f'\|_\infty \|g'\|_\infty (b-a)^4, \tag{1.3}$$

where $\|f'\|_\infty := \sup_{t\in[a,b]} |f'(t)| < \infty$.

The constant $\frac{1}{12}$ cannot be improved in general in (1.3).

Čebyšev's inequality (1.3) also holds if $f,\ g : [a,b] \to \mathbb{R}$ are assumed to be absolutely continuous and $f',\ g' \in L_\infty[a,b]$.

In 1970, A. M. Ostrowski [29] proved, amongst others, the following result that is in a sense a combination of the Čebyšev and Grüss results:

$$|D(f,g)| \le \frac{1}{8} (b-a)^3 (M-m) \|g'\|_\infty, \tag{1.4}$$

provided f is Lebesgue integrable on $[a,b]$ and satisfying (1.2) while $g : [a,b] \to \mathbb{R}$ is absolutely continuous and $g' \in L_\infty[a,b]$. Here the constant $\frac{1}{8}$ is also sharp.

In 1973, A. Lupaş [26] (see also [28, p. 210]) obtained the following result as well:

$$|D(f,g)| \le \frac{1}{\pi^2} \|f'\|_2 \|g'\|_2 (b-a)^3, \tag{1.5}$$

provided $f,\ g$ are absolutely continuous and $f',\ g' \in L_2[a,b]$.

Here the constant $\frac{1}{\pi^2}$ is the best possible as well.

In [3], P. Cerone and S. S. Dragomir proved the following inequalities:

$$|D(f,g)| \le (b-a) \begin{cases} \inf\limits_{\gamma\in\mathbb{R}} \|g-\gamma\|_\infty \int_a^b \left| f(t) - \frac{1}{b-a}\int_a^b f(s)\,ds \right| dt, \\ \\ \inf\limits_{\gamma\in\mathbb{R}} \|g-\gamma\|_q \left(\int_a^b \left| f(t) - \frac{1}{b-a}\int_a^b f(s)\,ds \right|^p dt \right)^{\frac{1}{p}}, \\ \text{where } p>1,\ 1/p+1/q=1. \end{cases} \tag{1.6}$$

For $\gamma = 0$, we get from the first inequality in (1.6)

$$|D(f,g)| \le (b-a) \|g\|_\infty \int_a^b \left| f(t) - \frac{1}{b-a}\int_a^b f(s)\,ds \right| dt \tag{1.7}$$

for which the constant 1 cannot be replaced by a smaller constant.

If $m \le g \le M$ for a.e. $x \in [a,b]$, then $\left\| g - \frac{m+M}{2} \right\|_\infty \le \frac{1}{2}(M-m)$ and by the first inequality in (1.6) we can deduce the following result obtained by Cheng and Sun [5]

$$|D(f,g)| \le \frac{1}{2}(b-a)(M-m) \int_a^b \left| f(t) - \frac{1}{b-a}\int_a^b f(s)\,ds \right| dt. \tag{1.8}$$

The constant $\frac{1}{2}$ is best in (1.8) as shown by Cerone and Dragomir in [4].

The following result holds [15].

Theorem 1. *Let $f:[a,b]\to\mathbb{C}$ be of bounded variation on $[a,b]$ and $g:[a,b]\to\mathbb{C}$ a Lebesgue integrable function on $[a,b]$. Then*

$$|D(f,g)|\le\frac{1}{2}(b-a)\bigvee_a^b(f)\int_a^b\left|g(t)-\frac{1}{b-a}\int_a^b g(s)\,ds\right|dt \tag{1.9}$$

where $\bigvee_a^b(f)$ denotes the total variation of f on the interval $[a,b]$. The constant $\frac{1}{2}$ is best possible in (1.9).

For more recent upper bounds related to the Čebyšev functional see [3], [4] and [10]-[15].

In order to obtain similar results for two functions with values in Banach algebras, we need the following preparations.

Let $\mathcal{B}$ be an algebra. An *algebra norm* on $\mathcal{B}$ is a map $\|\cdot\|:\mathcal{B}\to[0,\infty)$ such that $(\mathcal{B},\|\cdot\|)$ is a normed space, and, further: $\|ab\|\le\|a\|\,\|b\|$ for any $a,b\in\mathcal{B}$. The normed algebra $(\mathcal{B},\|\cdot\|)$ is a *Banach algebra* if $\|\cdot\|$ is a *complete norm.* We assume that the Banach algebra is *unital,* this means that $\mathcal{B}$ has an identity 1 and that $\|1\|=1$.

Let $\mathcal{B}$ be a unital algebra. An element $a\in\mathcal{B}$ is *invertible* if there exists an element $b\in\mathcal{B}$ with $ab=ba=1$. The element b is unique; it is called the *inverse* of a and written a^{-1} or $\frac{1}{a}$. The set of invertible elements of $\mathcal{B}$ is denoted by $\operatorname{Inv}(\mathcal{B})$. If $a,b\in\operatorname{Inv}(\mathcal{B})$ then $ab\in\operatorname{Inv}(\mathcal{B})$ and $(ab)^{-1}=b^{-1}a^{-1}$.

For a unital Banach algebra we also have:

(i) If $a\in\mathcal{B}$ and $\lim_{n\to\infty}\|a^n\|^{1/n}<1$, then $1-a\in\operatorname{Inv}(\mathcal{B})$;
(ii) $\{b\in\mathcal{B}:\ \|1-b\|<1\}\subset\operatorname{Inv}(\mathcal{B})$;
(iii) $\operatorname{Inv}(\mathcal{B})$ is an *open subset* of $\mathcal{B}$;
(iv) The map $\operatorname{Inv}(\mathcal{B})\ni a\longmapsto a^{-1}\in\operatorname{Inv}(\mathcal{B})$ is continuous.

For simplicity, we denote $z1$, where $z\in\mathbb{C}$ and 1 is the identity of $\mathcal{B}$, by z. The *resolvent set* of $a\in\mathcal{B}$ is defined by

$$\rho(a):=\{z\in\mathbb{C}:\ z-a\in\operatorname{Inv}(\mathcal{B})\};$$

the *spectrum* of a is $\sigma(a)$, the complement of $\rho(a)$ in $\mathbb{C}$, and the *resolvent function* of a is $R_a:\rho(a)\to\operatorname{Inv}(\mathcal{B})$, $R_a(z):=(z-a)^{-1}$. For each $z,w\in\rho(a)$ we have the identity

$$R_a(w)-R_a(z)=(z-w)R_a(z)R_a(w).$$

We also have that

$$\sigma(a)\subset\{z\in\mathbb{C}:\ |z|\le\|a\|\}.$$

The *spectral radius* of a is defined as

$$\nu(a)=\sup\{|z|:z\in\sigma(a)\}.$$

Let $\mathcal{B}$ a unital Banach algebra and $a\in\mathcal{B}$. Then

(i) The resolvent set $\rho(a)$ is open in $\mathbb{C}$;
(ii) For any *bounded linear functionals* $\lambda:\mathcal{B}\to\mathbb{C}$, the function $\lambda\circ R_a$ is analytic on $\rho(a)$;
(iii) The spectrum $\sigma(a)$ is compact and nonempty in $\mathbb{C}$;
(iv) For each $n\in\mathbb{N}$ and $r>\nu(a)$, we have $a^n=\frac{1}{2\pi i}\int_{|\xi|=r}\xi^n(\xi-a)^{-1}d\xi$;
(v) We have $\nu(a)=\lim_{n\to\infty}\|a^n\|^{1/n}$.

Let $\mathcal{B}$ be a unital Banach algebra, $a \in \mathcal{B}$ and G be a domain of $\mathbb{C}$ with $\sigma(a) \subset G$. If $f : G \to \mathbb{C}$ is analytic on G, we define an element $f(a)$ in $\mathcal{B}$ by

$$f(a) := \frac{1}{2\pi i} \int_{\delta} f(\xi)(\xi - a)^{-1} d\xi, \tag{1.10}$$

where $\delta \subset G$ is taken to be close rectifiable curve in G and such that $\sigma(a) \subset \operatorname{ins}(\delta)$, the inside of δ.

It is well known (see for instance [6, pp. 201-204]) that $f(a)$ does not depend on the choice of δ and the *Spectral Mapping Theorem* (SMT)

$$\sigma(f(a)) = f(\sigma(a)) \tag{1.11}$$

holds.

Concerning other basic definitions and facts in the theory of Banach algebras, the reader can consult the classical books [8] and [30].

For some recent norm inequalities for functions on Banach algebras, see [19], [2] and [16]-[24].

2. Main Results

For two continuous functions $x, y : [a,b] \to \mathcal{B}$ we define the *noncommutative Čebyšev functional*

$$D(x,y) := (b-a) \int_a^b x(t) y(t) dt - \int_a^b x(t) dt \int_a^b y(t) dt.$$

We have the following result of interest:

Theorem 2. *Let $x, y : [a,b] \to \mathcal{B}$ be a strongly differentiable functions on the interval (a,b). If $\|x'\|_{[a,b],\infty} := \sup_{t \in (a,b)} \|x'(u)\| < \infty$, then*

$$\begin{aligned} \|D(x,y)\| &\le \|x'\|_{[a,b],\infty} D\left(\ell, \int_a^{\cdot} \|y'(u)\| du\right) \\ &\le \frac{1}{8}(b-a)^3 \|x'\|_{[a,b],\infty} \|y'\|_{[a,b],1}, \end{aligned} \tag{2.1}$$

where $\|z'\|_{[a,b],1} := \int_a^b \|z'(u)\| du$.

Proof. Observe that

$$\begin{aligned} &\int_a^b \int_a^b [x(t) - x(s)][y(t) - y(s)] dtds \\ &= \int_a^b \int_a^b (x(t) y(t) - x(s) y(t) - x(t) y(s) + x(s) y(s)) dtds \\ &= (b-a) \int_a^b x(t) y(t) dt - \int_a^b x(s) ds \int_a^b y(t) dt \\ &- \int_a^b x(t) dt \int_a^b y(s) ds + (b-a) \int_a^b x(s) y(s) ds \\ &= 2(b-a) \int_a^b x(t) y(t) dt - 2 \int_a^b x(t) dt \int_a^b y(t) dt = 2D(x,y), \end{aligned}$$

which give the Korkine's noncommutative identity for functions with values in Banach algebras

$$D(x,y)=\frac{1}{2}\int_a^b\int_a^b [x(t)-x(s)][y(t)-y(s)]\,dtds.$$

For Korkine's classical identity for real-valued functions, see [28, p. 242].

If we take the norm and use the integral's properties, we get

$$\begin{aligned}\|D(x,y)\| &\le \frac{1}{2}\int_a^b\int_a^b \|[x(t)-x(s)][y(t)-y(s)]\|\,dtds \\ &\le \frac{1}{2}\int_a^b\int_a^b \|x(t)-x(s)\|\,\|y(t)-y(s)\|\,dtds.\end{aligned}\tag{2.2}$$

Observe that for $s,\,t\in[a,b]$

$$x(t)-x(s)=\int_s^t x'(u)\,du,\ \ y(t)-y(s)=\int_s^t y'(u)\,du,$$

which implies that

$$\begin{aligned}\|x(t)-x(s)\|\,\|y(t)-y(s)\| &= \left\|\int_s^t x'(u)\,du\right\|\left\|\int_s^t y'(u)\,du\right\| \\ &\le \left|\int_s^t \|x'(u)\|\,du\right|\left|\int_s^t \|y'(u)\|\,du\right| \\ &\le \sup_{t\in(a,b)}\|x'(u)\|\,|t-s|\left|\int_s^t \|y'(u)\|\,du\right| \\ &= \sup_{t\in(a,b)}\|x'(u)\|\,(t-s)\int_s^t \|y'(u)\|\,du,\end{aligned}$$

for all $s,\,t\in[a,b]$.

By (2.2) we get

$$\|D(x,y)\|\le \sup_{t\in(a,b)}\|x'(u)\|\,\frac{1}{2}\int_a^b\int_a^b (t-s)\left(\int_s^t \|y'(u)\|\,du\right)dtds.\tag{2.3}$$

Since

$$(t-s)\left(\int_s^t \|y'(u)\|\,du\right)=(t-s)\left(\int_a^t \|y'(u)\|\,du-\int_a^s \|y'(u)\|\,du\right),$$

hence by Korkine's identity for real valued functions $f(t)=\ell(t)$ and $g(t)=\int_a^t \|y'(u)\|\,du$, we have

$$\begin{aligned}&\frac{1}{2}\int_a^b\int_a^b (t-s)\left(\int_s^t \|y'(u)\|\,du\right) \\ &=(b-a)\int_a^b \ell(t)\left(\int_a^t \|y'(u)\|\,du\right)dt-\int_a^b \ell(t)\,dt\int_a^b\left(\int_a^t \|y'(u)\|\,du\right)dt \\ &=D\left(\ell,\int_a^{\cdot} \|y'(u)\|\,du\right).\end{aligned}\tag{2.4}$$

By utilising (2.3) and (2.4), we deduce the first inequality in (2.1).

Observe that

$$0 \leq \int_a^t \|y'(u)\| \, du \leq \int_a^b \|y'(u)\| \, du$$

for all $t \in [a,b]$, then by (1.8) for the functions $f(t) = \ell(t)$ and $g(t) = \int_a^t \|y'(u)\| \, du$, $t \in [a,b]$, we get

$$\begin{aligned}
&\left| D\left(\ell, \int_a^{\cdot} \|y'(u)\| \, du\right) \right| \\
&\leq \frac{1}{2}(b-a) \int_a^b \|y'(u)\| \, du \int_a^b \left| t - \frac{1}{b-a} \int_a^b s ds \right| dt \\
&= \frac{1}{2}(b-a) \int_a^b \|y'(u)\| \, du \int_a^b \left| t - \frac{a+b}{2} \right| dt = \frac{1}{8}(b-a)^3 \int_a^b \|y'(u)\| \, du,
\end{aligned}$$

which proves the first part of (2.1). □

Remark 1. *If we apply the same inequality (1.8) for the functions $f(t) = \int_a^t \|y'(u)\| \, du$ and $g(t) = \ell(t)$, $t \in [a,b]$, then we get*

$$\begin{aligned}
(2.5) \qquad &\left| D\left(\ell, \int_a^{\cdot} \|y'(u)\| \, du\right) \right| \\
&\leq \frac{1}{2}(b-a)^2 \int_a^b \left| \int_a^t \|y'(u)\| \, du - \frac{1}{b-a} \int_a^b \left(\int_a^s \|y'(u)\| \, du \right) ds \right| dt.
\end{aligned}$$

Observe that

$$\begin{aligned}
&\int_a^b \left| \int_a^t \|y'(u)\| \, du - \frac{1}{b-a} \int_a^b \left(\int_a^s \|y'(u)\| \, du \right) ds \right| dt \\
&= \int_a^b \left| \int_a^t \|y'(u)\| \, du - \frac{1}{b-a} \left(\left(\int_a^b \|y'(u)\| \, du \right) b - \int_a^b \|y'(s)\| \, s ds \right) \right| dt \\
&= \int_a^b \left| \int_a^t \|y'(u)\| \, du - \frac{1}{b-a} \left(\int_a^b (b-u) \|y'(u)\| \, du \right) \right| dt \\
&= \frac{1}{b-a} \int_a^b \left| (b-a) \int_a^t \|y'(u)\| \, du - \int_a^b (b-u) \|y'(u)\| \, du \right| dt \\
&= \frac{1}{b-a} \int_a^b \left| \int_a^t (u-a) \|y'(u)\| \, du - \int_t^b (b-u) \|y'(u)\| \, du \right| dt.
\end{aligned}$$

Then by (2.1) and (2.5) we obtain

$$\begin{aligned}
(2.6) \qquad &\|D(x,y)\| \\
&\leq \|x'\|_{[a,b],\infty} D\left(\ell, \int_a^{\cdot} \|y'(u)\| \, du\right) \\
&\leq \frac{1}{2}(b-a) \int_a^b \left| \int_a^t (u-a) \|y'(u)\| \, du - \int_t^b (b-u) \|y'(u)\| \, du \right| dt.
\end{aligned}$$

Remark 2. *Using (1.3) we have*

$$0 \le D\left(\ell, \int_a^{\cdot} \|y'(u)\| \, du\right) \le \frac{1}{12} \sup_{t\in(a,b)} \|y'(u)\| (b-a)^4 ,$$

and by (2.1) we derive

$$\begin{aligned} \|D(x,y)\| &\le \|x'\|_{[a,b],\infty} D\left(\ell, \int_a^{\cdot} \|y'(u)\| \, du\right) \\ &\le \frac{1}{12} \|x'\|_{[a,b],\infty} \|y'\|_{[a,b],\infty} (b-a)^4 \end{aligned} \tag{2.7}$$

provided that $\|x'\|_{[a,b],\infty}, \|y'\|_{[a,b],\infty} < \infty$.

Using (1.4) we have

$$0 \le D\left(\ell, \int_a^{\cdot} \|y'(u)\| \, du\right) \le \frac{1}{8} (b-a)^3 \int_a^b \|y'(u)\| \, du,$$

and by (2.1) we obtain

$$\begin{aligned} \|D(x,y)\| &\le \|x'\|_{[a,b],\infty} D\left(\ell, \int_a^{\cdot} \|y'(u)\| \, du\right) \\ &\le \frac{1}{8} (b-a)^3 \|x'\|_{[a,b],\infty} \|y'\|_{[a,b],1} , \end{aligned} \tag{2.8}$$

provided that $\|x'\|_{[a,b],\infty} < \infty$.

Corollary 1. *Let* $x, y : [a,b] \to \mathcal{B}$ *be a strongly differentiable functions on the interval* (a,b). *If*

$$\|y'\|_{[a,b],r} := \left(\int_a^b \|y'(u)\|^r \, du\right)^{1/r}, \; r \ge 1,$$

then

$$\begin{aligned} \|D(x,y)\| &\le \frac{1}{2} (b-a) \|x'\|_{[a,b],\infty} \int_a^b (b-t)(t-a) \|y'(t)\| \, dt \\ &\le \begin{cases} \frac{1}{8} (b-a)^3 \|x'\|_{[a,b],\infty} \|y'\|_{[a,b],1} , \\ \\ \frac{1}{2} (b-a)^{3+1/q} [B(q+1,q+1)]^{1/q} \|x'\|_{[a,b],\infty} \|y'\|_{[a,b],p} , \\ \\ \frac{1}{12} (b-a)^4 \|x'\|_{[a,b],\infty} \|y'\|_{[a,b],\infty} , \end{cases} \end{aligned} \tag{2.9}$$

where $B(\cdot,\cdot)$ *is Beta function and* $p, q > 1$ *with* $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. Observe that, integrating by parts, we have

$$\begin{aligned}
&\frac{1}{2}\int_a^b (b-t)(t-a)\|y'(t)\|\,dt \\
&= \frac{1}{2}\int_a^b (b-t)(t-a)\,d\left(\int_a^t \|y'(u)\|\,du\right) \\
&= \frac{1}{2}\left[(b-t)(t-a)\int_a^t \|y'(u)\|\,du\Big|_a^b + \int_a^b (2t-a-b)\left(\int_a^t \|y'(u)\|\,du\right)dt\right] \\
&= \int_a^b \left(t-\frac{a+b}{2}\right)\left(\int_a^t \|y'(u)\|\,du\right)dt \\
&= \int_a^b t\left(\int_a^t \|y'(u)\|\,du\right)dt - \frac{a+b}{2}\int_a^b \left(\int_a^t \|y'(u)\|\,du\right)dt \\
&= \frac{1}{b-a}D\left(\ell, \int_a^{\cdot} \|y'(u)\|\,du\right),
\end{aligned}$$

namely

$$D\left(\ell, \int_a^{\cdot} \|y'(u)\|\,du\right) = \frac{1}{2}(b-a)\int_a^b (b-t)(t-a)\|y'(t)\|\,dt. \tag{2.10}$$

By utilising the first inequality (2.1) we deduce the first inequality in (2.9).

By Hölder's integral inequality we have for $p,\ q>1$ with $\frac{1}{p}+\frac{1}{q}=1$,

$$\begin{aligned}
&\int_a^b (b-t)(t-a)\|y'(t)\|\,dt \\
&\le \begin{cases}
\sup_{t\in[a,b]}\left[(b-t)(t-a)\right]\int_a^b \|y'(t)\|\,dt, \\[2ex]
\left(\int_a^b \left[(b-t)(t-a)\right]^q dt\right)^{1/q}\left(\int_a^b \|y'(t)\|^p dt\right)^{1/p}, \\[2ex]
\int_a^b (b-t)(t-a)\,dt \sup_{t\in[a,b]}\|y'(t)\|,
\end{cases} \\
&= \begin{cases}
\frac{1}{4}(b-a)^2\int_a^b \|y'(t)\|\,dt, \\[2ex]
(b-a)^{2+1/q}\left[B(q+1,q+1)\right]^{1/q}\left(\int_a^b \|y'(t)\|^p dt\right)^{1/p}, \\[2ex]
\frac{1}{6}(b-a)^3 \sup_{t\in[a,b]}\|y'(t)\|,
\end{cases}
\end{aligned}$$

which proves the last part of (2.9). □

Theorem 3. *Let* $x,\ y : [a,b] \to \mathcal{B}$ *be a strongly differentiable functions on the interval* (a,b). *If* $p,\ q > 1$ *with* $\frac{1}{p}+\frac{1}{q}=1$, *then*

$$\begin{aligned}
\|D(x,y)\| &\le \left[D\left(\ell,\int_a^{\cdot}\|x'(u)\|^p\,du\right)\right]^{1/p}\left[D\left(\ell,\int_a^{\cdot}\|x'(u)\|^q\,du\right)\right]^{1/q} \\
&= \frac{1}{2}(b-a)\left[\int_a^b (b-t)(t-a)\|x'(t)\|^p\,dt\right]^{1/p} \\
&\quad\times\left[\int_a^b (b-t)(t-a)\|y'(t)\|^q\,dt\right]^{1/q} \\
&\le \frac{1}{8}(b-a)^3\|x'\|_{[a,b],p}\|y'\|_{[a,b],q}\,.
\end{aligned} \tag{2.11}$$

In particular, we have for $p=q=2$

$$\begin{aligned}
\|D(x,y)\| &\le \left[D\left(\ell,\int_a^{\cdot}\|x'(u)\|^2\,du\right)\right]^{1/2}\left[D\left(\ell,\int_a^{\cdot}\|x'(u)\|^2\,du\right)\right]^{1/2} \\
&= \frac{1}{2}(b-a)\left[\int_a^b (b-t)(t-a)\|x'(t)\|^2\,dt\right]^{1/2} \\
&\quad\times\left[\int_a^b (b-t)(t-a)\|y'(t)\|^2\,dt\right]^{1/2} \\
&\le \frac{1}{8}(b-a)^3\|x'\|_{[a,b],2}\|y'\|_{[a,b],2}\,.
\end{aligned} \tag{2.12}$$

Proof. Using Hölder's inequality for $p,\ q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, then

$$\begin{aligned}
&\|x(t)-x(s)\|\,\|y(t)-y(s)\| \\
&= \left\|\int_s^t x'(u)\,du\right\|\left\|\int_s^t y'(u)\,du\right\| \\
&\le \left|\int_s^t \|x'(u)\|\,du\right|\left|\int_s^t \|y'(u)\|\,du\right| \\
&\le |t-s|^{1/q}\left|\int_s^t \|x'(u)\|^p\,du\right|^{1/p}|t-s|^{1/p}\left|\int_s^t \|y'(u)\|^q\,du\right|^{1/q} \\
&= |t-s|\left|\int_s^t \|x'(u)\|^p\,du\right|^{1/p}\left|\int_s^t \|y'(u)\|^q\,du\right|^{1/q}.
\end{aligned}$$

By the weighted Hölder's inequality for double integral, we also have

$$\begin{aligned}
&\int_a^b \int_a^b \|x(t)-x(s)\| \|y(t)-y(s)\| dtds \\
&\leq \int_a^b \int_a^b |t-s| \left| \int_s^t \|x'(u)\|^p du \right|^{1/p} \left| \int_s^t \|y'(u)\|^q du \right|^{1/q} dtds \\
&\leq \left(\int_a^b \int_a^b |t-s| \left(\left| \int_s^t \|x'(u)\|^p du \right|^{1/p} \right)^p dtds \right)^{1/p} \\
&\times \left(\int_a^b \int_a^b |t-s| \left(\left| \int_s^t \|y'(u)\|^q du \right|^{1/q} \right)^q dtds \right)^{1/q} \\
&= \left(\int_a^b \int_a^b |t-s| \left| \int_s^t \|x'(u)\|^p du \right| dtds \right)^{1/p} \\
&\times \left(\int_a^b \int_a^b |t-s| \left| \int_s^t \|y'(u)\|^q du \right| dtds \right)^{1/q}.
\end{aligned} \tag{2.13}$$

Observe that

$$\begin{aligned}
&\int_a^b \int_a^b |t-s| \left| \int_s^t \|x'(u)\|^p du \right| dtds \\
&= \int_a^b \int_a^b (t-s) \left(\int_s^t \|x'(u)\|^p du \right) dtds \\
&= \int_a^b \int_a^b (t-s) \left(\int_a^t \|x'(u)\|^p du - \int_a^s \|x'(u)\|^p du \right) dtds \\
&= 2D\left(\ell, \int_a^{\cdot} \|x'(u)\|^p du\right)
\end{aligned}$$

and

$$\int_a^b \int_a^b |t-s| \left| \int_s^t \|y'(u)\|^q du \right| dtds = 2D\left(\ell, \int_a^{\cdot} \|y'(u)\|^q du\right).$$

Therefore, by (2.2)

$$\begin{aligned}
\|D(x,y)\| &\leq \frac{1}{2} \int_a^b \int_a^b \|x(t)-x(s)\| \|y(t)-y(s)\| dtds \\
&\leq \frac{1}{2} \left[2D\left(\ell, \int_a^{\cdot} \|x'(u)\|^p du\right) \right]^{1/p} \left[2D\left(\ell, \int_a^{\cdot} \|y'(u)\|^q du\right) \right]^{1/q} \\
&= \left[D\left(\ell, \int_a^{\cdot} \|x'(u)\|^p du\right) \right]^{1/p} \left[D\left(\ell, \int_a^{\cdot} \|y'(u)\|^q du\right) \right]^{1/q}.
\end{aligned}$$

From (2.10) we have

$$D\left(\ell, \int_a^{\cdot} \|x'(u)\|^p du\right) = \frac{1}{2}(b-a) \int_a^b (b-t)(t-a) \|x'(t)\|^p dt$$

and

$$D\left(\ell, \int_a^{\cdot} \|y'(u)\|^q du\right) = \frac{1}{2}(b-a) \int_a^b (b-t)(t-a) \|y'(t)\|^q dt.$$

Therefore

$$\left[D\left(\ell,\int_a^{\cdot}\|x'(u)\|^p\,du\right)\right]^{1/p}\left[D\left(\ell,\int_a^{\cdot}\|y'(u)\|^q\,du\right)\right]^{1/q}$$
$$=\left[\frac{1}{2}(b-a)\int_a^b(b-t)(t-a)\|x'(t)\|^p\,dt\right]^{1/p}$$
$$\times\left[\frac{1}{2}(b-a)\int_a^b(b-t)(t-a)\|y'(t)\|^q\,dt\right]^{1/q}$$
$$=\frac{1}{2}(b-a)\left[\int_a^b(b-t)(t-a)\|x'(t)\|^p\,dt\right]^{1/p}\left[\int_a^b(b-t)(t-a)\|y'(t)\|^q\,dt\right]^{1/q}$$

and the first part of the theorem is proved.

Now, observe that

$$\int_a^b(b-t)(t-a)\|x'(t)\|^p\,dt\le\frac{1}{4}(b-a)^2\int_a^b\|x'(t)\|^p\,dt$$

and

$$\int_a^b(b-t)(t-a)\|y'(t)\|^q\,dt\le\frac{1}{4}(b-a)^2\int_a^b\|y'(t)\|^q\,dt,$$

which gives the last part of (2.11). □

Remark 3. *Assume that α, $\beta>1$ with $\frac{1}{\alpha}+\frac{1}{\beta}=1$ and γ, $\delta>1$ with $\frac{1}{\gamma}+\frac{1}{\delta}=1$. Then by Hölder's inequality we get*

$$\int_a^b(b-t)(t-a)\|x'(t)\|^p\,dt$$
$$\le(b-a)^{2+1/\beta}\left[B(\beta+1,\beta+1)\right]^{1/\beta}\left(\int_a^b\|x'(t)\|^{\alpha p}\,dt\right)^{1/\alpha}$$

and

$$\int_a^b(b-t)(t-a)\|y'(t)\|^q\,dt$$
$$\le(b-a)^{2+1/\delta}\left[B(\delta+1,\delta+1)\right]^{1/\delta}\left(\int_a^b\|y'(t)\|^{\gamma q}\,dt\right)^{1/\gamma}.$$

Then

$$\left[\int_a^b(b-t)(t-a)\|x'(t)\|^p\,dt\right]^{1/p}$$
$$\le(b-a)^{(2\beta+1)/(\beta p)}\left[B(\beta+1,\beta+1)\right]^{1/(\beta p)}\left(\int_a^b\|x'(t)\|^{\alpha p}\,dt\right)^{1/(\alpha p)}$$

and

$$\left[\int_a^b (b-t)(t-a)\|y'(t)\|^q dt\right]^{1/q}$$
$$\leq (b-a)^{(2\delta+1)/(\delta q)}[B(\delta+1,\delta+1)]^{1/(\delta q)}\left(\int_a^b \|y'(t)\|^{\gamma q} dt\right)^{1/(\gamma q)}.$$

Therefore

$$\frac{1}{2}(b-a)\left[\int_a^b (b-t)(t-a)\|x'(t)\|^p dt\right]^{1/p}$$
$$\times\left[\int_a^b (b-t)(t-a)\|y'(t)\|^q dt\right]^{1/q}$$
$$\leq \frac{1}{2}(b-a)(b-a)^{(2\beta+1)/(\beta p)}[B(\beta+1,\beta+1)]^{1/(\beta p)}\left(\int_a^b \|x'(t)\|^{\alpha p} dt\right)^{1/(\alpha p)}$$
$$\times(b-a)^{(2\delta+1)/(\delta q)}[B(\delta+1,\delta+1)]^{1/(\delta q)}\left(\int_a^b \|y'(t)\|^{\gamma q} dt\right)^{1/(\gamma q)}$$
$$=\frac{1}{2}[B(\beta+1,\beta+1)]^{1/(\beta p)}[B(\delta+1,\delta+1)]^{1/(\delta q)}$$
$$\times(b-a)^{1+(2\beta+1)/(\beta p)+(2\delta+1)/(\delta q)}$$
$$\times\left(\int_a^b \|x'(t)\|^{\alpha p} dt\right)^{1/(\alpha p)}\left(\int_a^b \|y'(t)\|^{\gamma q} dt\right)^{1/(\gamma q)}$$

and by (2.11) we get

$$\|D(x,y)\| \leq \frac{1}{2}[B(\beta+1,\beta+1)]^{1/(\beta p)}[B(\delta+1,\delta+1)]^{1/(\delta q)}$$
$$\times(b-a)^{1+(2\beta+1)/(\beta p)+(2\delta+1)/(\delta q)}\|x'\|_{[a,b],\alpha p}\|y'\|_{[a,b],\gamma q}, \tag{2.14}$$

where $p, q > 1$ *with* $\frac{1}{p}+\frac{1}{q}=1$, $\alpha, \beta > 1$ *with* $\frac{1}{\alpha}+\frac{1}{\beta}=1$ *and* $\gamma, \delta > 1$ *with* $\frac{1}{\gamma}+\frac{1}{\delta}=1$.

3. Applications for Analytic Functions

Assume that $f: G \to \mathbb{C}$ is analytic on the convex domain G and $x, y \in \mathcal{B}$, $x \neq y$ with $\sigma(x), \sigma(y) \subset G$. By the convexity of G we have that $\sigma((1-t)x+ty) \subset G$ for all $t \in [0,1]$ and we can define the auxiliary function $f_{x,y}: [0,1] \to \mathcal{B}$ by

$$f_{x,y}(t) := f((1-t)x+ty), \ t \in [0,1]. \tag{3.1}$$

We list here some facts that were established in [20]:

Lemma 1. *Assume that* $f: G \to \mathbb{C}$ *is analytic on the convex domain* G *and* x, $y \in \mathcal{B}$, $x \neq y$ *with* $\sigma(x), \sigma(y) \subset G$. *The function* $f_{x,y}: [0,1] \to \mathcal{B}$ *is differentiable on* $(0,1)$ *as a function of* t *and we have*

$$f'_{x,y}(t) = D(f)((1-t)x+ty)(y-x) \tag{3.2}$$

for all $t \in (0,1)$, *where* $D(f)(\cdot)(\cdot)$ *is the Fréchet derivative of function* f *as a function defined on the Banach algebra* $\mathcal{B}$ *by equation (1.10).*

We also have the lateral derivatives

$$f_{x,y}^{'}(0+) = D(f)(x)(y-x) \text{ and } f_{x,y}^{'}(1-) = D(f)(y)(y-x). \tag{3.3}$$

Lemma 2. *Assume that $f : G \to \mathbb{C}$ is analytic on the domain G and $x \in \mathcal{B}$, with $\sigma(x) \subset G$, then for $v \in \mathcal{B}$ we have*

$$D(f)(x)(v) = \frac{1}{2\pi i}\int_{\gamma} f(\xi)(\xi - x)^{-1} v (\xi - x)^{-1} d\xi, \tag{3.4}$$

where $\gamma \subset G$ is taken to be close rectifiable curve in G and such that $\sigma(x) \subset \operatorname{ins}(\gamma)$, the inside of γ.

Lemma 3. *Assume that $f : G \to \mathbb{C}$ is analytic on the convex domain G and x, $y \in \mathcal{B}$, $x \neq y$ with $\sigma(x)$, $\sigma(y) \subset G$ while $\gamma \subset G$ is taken to be close rectifiable curve in G and such that $\sigma(x)$, $\sigma(y) \subset \operatorname{ins}(\gamma)$. Then*

$$f_{x,y}^{'}(t) = \frac{1}{2\pi i}\int_{\gamma} f(\xi)(\xi - (1-t)x - ty)^{-1}(y-x)(\xi - (1-t)x - ty)^{-1} d\xi \tag{3.5}$$

for all $t \in (0,1)$.

We also have the lateral derivatives

$$f_{x,y}^{'}(0+) = \frac{1}{2\pi i}\int_{\gamma} f(\xi)(\xi - x)^{-1}(y-x)(\xi - x)^{-1} d\xi, \tag{3.6}$$

and

$$f_{x,y}^{'}(1-) = \frac{1}{2\pi i}\int_{\gamma} f(\xi)(\xi - y)^{-1}(y-x)(\xi - y)^{-1} d\xi. \tag{3.7}$$

The proof is obvious by Lemmas 1 and 2.

Lemma 4. *With the assumptions of Lemma 3 we have the bounds*

$$\begin{aligned} &\left\| f_{x,y}^{'}(t) \right\| \\ &\leq \frac{1}{2\pi}\|y-x\|\int_{\gamma}|f(\xi)|\left\|(\xi-(1-t)x-ty)^{-1}\right\|^{2}|d\xi| \\ &\leq \frac{1}{2\pi}\|y-x\|\int_{\gamma}|f(\xi)|(|\xi|-\|(1-t)x+ty\|)^{-2}|d\xi| \\ &\leq \frac{1}{2\pi}\|y-x\|\int_{\gamma}|f(\xi)|[(1-t)(|\xi|-\|x\|)+t(|\xi|-\|y\|)]^{-2}|d\xi| \\ &\leq \frac{1}{2\pi}\|y-x\|\int_{\gamma}|f(\xi)|\left[(1-t)(|\xi|-\|x\|)^{-2}+t(|\xi|-\|y\|)^{-2}\right]|d\xi| \\ &\leq \frac{1}{2\pi}\|y-x\|\int_{\gamma}\frac{|f(\xi)|}{\min\left\{(|\xi|-\|x\|)^{2},(|\xi|-\|y\|)^{2}\right\}}|d\xi| \end{aligned} \tag{3.8}$$

for all $t \in [0,1]$.

We have the following bounds for the p-norm of $f_{x,y}^{'}$, see also [20]:

Proposition 1. *Assume that $f : G \to \mathbb{C}$ is analytic on the convex domain G and x, $y \in \mathcal{B}$, $x \neq y$ with $\sigma(x)$, $\sigma(y) \subset G$ while $\gamma \subset G$ is taken to be close rectifiable curve in G and such that $\sigma(x)$, $\sigma(y) \subset \operatorname{ins}(\gamma)$. Then*

$$\sup_{t\in[0,1]}\left\| f_{x,y}^{'}(t) \right\| \leq \frac{1}{2\pi}\|y-x\|\int_{\gamma}\frac{|f(\xi)|}{\min\left\{(|\xi|-\|x\|)^{2},(|\xi|-\|y\|)^{2}\right\}}|d\xi|, \tag{3.9}$$

$$\int_0^1 \left\| f'_{x,y}(t) \right\| dt \leq \frac{1}{2\pi} \|y-x\| \int_\gamma \frac{|f(\xi)|}{(|\xi|-\|y\|)(|\xi|-\|x\|)} |d\xi| \tag{3.10}$$

and

$$\begin{aligned} &\left(\int_0^1 \left\| f'_{x,y}(t) \right\|^p dt \right)^{1/p} \\ &\leq \frac{1}{2\pi} \|y-x\| \left(\int_\gamma |f(\xi)|^q |d\xi| \right)^{1/q} \\ &\times \left(\frac{1}{(2p-1)(\|y\|-\|x\|)} \int_\gamma \frac{(|\xi|-\|x\|)^{2p-1} - (|\xi|-\|y\|)^{2p-1}}{(|\xi|-\|x\|)^{2p-1}(|\xi|-\|y\|)^{2p-1}} |d\xi| \right)^{1/p}, \end{aligned} \tag{3.11}$$

for $p, q > 1$ *with* $\frac{1}{p} + \frac{1}{q} = 1$.

We can state now the main result of this section:

Theorem 4. *Assume that* $f, g : G \to \mathbb{C}$ *are analytic on the convex domain* G *and* $x, y \in \mathcal{B}$, $x \neq y$ *with* $\sigma(x), \sigma(y) \subset G$. *Define*

$$\begin{aligned} D(f,g,x,y) &:= \int_0^1 f((1-t)x+ty)\, g((1-t)x+ty)\, dt \\ &\quad - \int_0^1 f((1-t)x+ty)\, dt \int_0^1 g((1-t)x+ty)\, dt. \end{aligned} \tag{3.12}$$

Then we have the norm inequalities

$$\begin{aligned} &\|D(f,g,x,y)\| \\ &\leq \frac{1}{8\pi^2} \|y-x\|^2 \int_\gamma \frac{|f(\xi)|}{\min\left\{ (|\xi|-\|x\|)^2, (|\xi|-\|y\|)^2 \right\}} |d\xi| \\ &\times \begin{cases} \frac{1}{2} \int_\gamma \frac{|g(\xi)|}{(|\xi|-\|y\|)(|\xi|-\|x\|)} |d\xi| \\ \\ [B(q+1,q+1)]^{1/q} \left(\int_\gamma |g(\xi)|^q |d\xi| \right)^{1/q} \\ \times \left(\frac{1}{(2p-1)(\|y\|-\|x\|)} \int_\gamma \frac{(|\xi|-\|x\|)^{2p-1} - (|\xi|-\|y\|)^{2p-1}}{(|\xi|-\|x\|)^{2p-1}(|\xi|-\|y\|)^{2p-1}} |d\xi| \right)^{1/p}, \\ \\ \frac{1}{6} \int_\gamma \frac{|g(\xi)|}{\min\left\{ (|\xi|-\|x\|)^2, (|\xi|-\|y\|)^2 \right\}} |d\xi|, \end{cases} \end{aligned}$$

where $p, q > 1$ *with* $\frac{1}{p} + \frac{1}{q} = 1$.

The proof follows by Corollary 1 and Proposition 1 for $x(t) = f((1-t)x+ty)$, $y(t) = g((1-t)x+ty)$, $t \in [0,1]$.

4. The Case of Circular Paths

We consider the circular path $\xi(s) = Re^{2\pi i s}$ where $s \in [0,1]$, then $d\xi(s) = 2\pi i Re^{2\pi i s} ds$, $|d\xi(s)| = 2\pi R ds$ and $|\xi| = R$.

Assume that $f,\ g: G \to \mathbb{C}$ are analytic on the convex domain G and $x,\ y \in \mathcal{B}$, $x \neq y$ with $\sigma(x),\ \sigma(y) \subset D(0,R) \subset G$. Then by Theorem 4 we derive

$$
\begin{aligned}
&\|D(f,g,x,y)\| \\
&\le \frac{1}{2}R^2 \frac{\|y-x\|^2}{\min\left\{(R-\|x\|)^2, (R-\|y\|)^2\right\}} \int_0^1 \left|f\left(Re^{2\pi is}\right)\right| ds \\
&\times \begin{cases}
\frac{1}{2}\frac{1}{(R-\|y\|)(R-\|x\|)} \int_0^1 \left|g\left(Re^{2\pi is}\right)\right| ds \\
\\
[B(q+1,q+1)]^{1/q}\left(\int_0^1 \left|g\left(Re^{2\pi is}\right)\right|^q ds\right)^{1/q} \\
\times\left(\frac{1}{(2p-1)(\|y\|-\|x\|)}\frac{(R-\|x\|)^{2p-1}-(R-\|y\|)^{2p-1}}{(R-\|x\|)^{2p-1}(R-\|y\|)^{2p-1}}\right)^{1/p}, \\
\\
\frac{1}{6}\frac{1}{\min\left\{(R-\|x\|)^2,(R-\|y\|)^2\right\}} \int_0^1 \left|g\left(Re^{2\pi is}\right)\right| ds.
\end{cases}
\end{aligned}
\tag{4.1}
$$

The *modified Bessel function of the first kind* $I_\nu(z)$ for real number ν can be defined by the power series as [1, p. 376]

$$I_\nu(z) = \left(\frac{1}{2}z\right)^\nu \sum_{k=0}^{\infty} \frac{\left(\frac{1}{4}z^2\right)^k}{k!\Gamma(\nu+k+1)},$$

where Γ is the *gamma function.* For $n=0$ we have $I_0(z)$ given by

$$I_0(z) = \sum_{k=0}^{\infty} \frac{\left(\frac{1}{4}z^2\right)^k}{(k!)^2}.$$

An integral formula for real number ν is

$$I_\nu(z) = \frac{1}{\pi}\int_0^\pi e^{z\cos\theta}\cos(\nu\theta)\,d\theta - \frac{\sin(\nu\pi)}{\pi}\int_0^\infty e^{-z\cosh t-\nu t}dt,$$

which simplifies for ν an integer n to

$$I_n(z) = \frac{1}{\pi}\int_0^\pi e^{z\cos\theta}\cos(n\theta)\,d\theta.$$

For $n=0$ we have

$$I_0(z) = \frac{1}{\pi}\int_0^\pi e^{z\cos\theta}d\theta.$$

Consider the exponential function $f(a) = \exp a$, $a \in \mathcal{B}$. Assume that $x,\ y \in \mathcal{B}$ and $\|x\|,\ \|y\| < R$ for some $R > 0$. Observe that

$$\left|\exp\left(Re^{2\pi it}\right)\right| = \left|\exp\left[R\left(\cos(2\pi t) + i\sin(2\pi t)\right)\right]\right| = \exp\left[R\cos(2\pi t)\right]$$

and, by changing the variable $\theta = 2\pi t$, we get $dt = \frac{1}{2\pi} d\theta$ and

$$\begin{aligned} \int_0^1 \exp\left[R\cos\left(2\pi t\right)\right] dt \\ &= \frac{1}{2\pi}\int_0^{2\pi} \exp\left[R\cos\theta\right] d\theta \\ &= \frac{1}{2}\left(\frac{1}{\pi}\int_0^{\pi} \exp\left[R\cos\theta\right] d\theta + \frac{1}{\pi}\int_{\pi}^{2\pi} \exp\left[R\cos\theta\right] d\theta\right) \\ &= \frac{1}{2}\left(I_0(R) + I_0(-R)\right) = I_0(R). \end{aligned} \tag{4.2}$$

If we apply the inequalities (4.1) to the functions $f(z) = \exp z$ and $g(z) = z^n$ with n a natural number, then we get for $x, y \in \mathcal{B}$ with $\|x\|, \|y\| < R$ that

$$\begin{aligned} &\Big\| \int_0^1 \exp\left(\left(1-t\right)x + ty\right)\left(\left(1-t\right)x + ty\right)^n dt \\ &- \int_0^1 \exp\left(\left(1-t\right)x + ty\right) dt \int_0^1 \left(\left(1-t\right)x + ty\right)^n dt \Big\| \\ &\leq \frac{1}{2} R^{n+2} I_0(R) \frac{\|y-x\|^2}{\min\left\{\left(R-\|x\|\right)^2, \left(R-\|y\|\right)^2\right\}} \\ &\times \begin{cases} \frac{1}{2}\frac{1}{(R-\|y\|)(R-\|x\|)} \\ \\ \left[B\left(q+1,q+1\right)\right]^{1/q} \\ \times\left(\frac{1}{(2p-1)(\|y\|-\|x\|)}\frac{(R-\|x\|)^{2p-1}-(R-\|y\|)^{2p-1}}{(R-\|x\|)^{2p-1}(R-\|y\|)^{2p-1}}\right)^{1/p}, \\ \\ \frac{1}{6}\frac{1}{\min\left\{(R-\|x\|)^2,(R-\|y\|)^2\right\}}. \end{cases} \end{aligned} \tag{4.3}$$

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