DETERMINANT INEQUALITIES FOR POSITIVE DEFINITE MATRICES VIA CARTWRIGHT-FIELD'S RESULT FOR ARITHMETIC AND GEOMETRIC WEIGHTED MEANS

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Abstract. Assume that A_j, $j \in \{1, ..., m\}$ are positive definite matrices of order n. In this paper we prove among others that, if $0 < \ell I_n \leq A_j$, $j \in \{1, ..., m\}$ in the operator order, for some positive constant ℓ, and I_n is the unity matrix of order n, then

$$0 \leq \frac{1}{2}\left[\sum_{k=1}^{m} p_k (1 - p_k) [\det (2A_j - \ell I_n)]^{-1/2} - 2\sum_{1\leq j<k\leq m} p_j p_k [\det (A_j + A_k - \ell I_n)]^{-1/2}\right] \leq \sum_{j=1}^{m} p_j [\det (A_j)]^{-1/2} - \left[\det \left(\sum_{k=1}^{m} p_k A_k\right)\right]^{-1/2},$$

where $p_k \geq 0$ for $k \in \{1, ..., m\}$ and $\sum_{j=1}^{m} p_j = 1$.

1. Introduction

A real square matrix $A = (a_{ij})$, $i, j = 1, ..., n$ is *symmetric* provided $a_{ij} = a_{ji}$ for all $i, j = 1, ..., n$. A real symmetric matrix is said to be *positive definite* provided the quadratic form $Q(x) = \sum_{i,j=1}^{n} a_{ij}x_i x_j$ is positive for all $x = (x_1, ..., x_n) \in \mathbb{R}^n \setminus \{0\}$. It is well known that a necessary and sufficient condition for the symmetric matrix A to be positive definite, and we write $A > 0$, is that all determinants

$$\det(A_k) = \det (a_{ij}),\ i,\ j = 1, ..., k;\ k = 1, ..., n$$

are positive.

It is know that the following integral representation is valid, see [1, pp. 61-62] or [9, pp. 211-212]

$$J_n (A) := \int_{\mathbb{R}^n} \exp(-\langle Ax, x\rangle)dx := \int_{-\infty}^{\infty} ... \int_{-\infty}^{\infty} \exp(-\langle Ax, x\rangle)dx = \frac{\pi^{n/2}}{[\det (A)]^{1/2}}, \tag{1.1}$$

where A is a positive definite matrix of order n and $\langle\cdot,\cdot\rangle$ is the usual inner product on $\mathbb{R}^n$.

By utilizing the representation (1.1) and Hölder's integral inequality for multiple integrals one can prove the *logarithmic concavity* of the determinant that is due to

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Ky Fan ([1, p. 63] or [9, p. 212]), namely

$$\det\left(\left(1-\lambda\right)A+\lambda B\right)\geq\left[\det\left(A\right)\right]^{1-\lambda}\left[\det\left(B\right)\right]^{\lambda} \tag{1.2}$$

for any positive definite matrices A, B and $\lambda\in[0,1]$.

By mathematical induction we can get a generalization of (1.2) which was obtained by L. Mirsky in [8], see also [9, p. 212]

$$\det\left(\sum_{j=1}^{m}\lambda_{j}A_{j}\right)\geq\prod_{j=1}^{m}\left[\det\left(A_{j}\right)\right]^{\lambda_{j}},\ m\geq2, \tag{1.3}$$

where $\lambda_j>0$, $j=1,...,m$ with $\sum_{j=1}^{m}\lambda_j=1$ and $A_j>0$, $j=1,...,m$.

If we write (1.3) for $A_j=B_j^{-1}$ we get

$$\det\left(\sum_{j=1}^{m}\lambda_{j}B_{j}^{-1}\right)\geq\prod_{j=1}^{m}\left[\det\left(B_{j}^{-1}\right)\right]^{\lambda_{j}}=\left(\prod_{j=1}^{m}\left[\det\left(B_{j}\right)\right]^{\lambda_{j}}\right)^{-1},$$

which also gives

$$\prod_{j=1}^{m}\left[\det\left(A_{j}\right)\right]^{\lambda_{j}}\geq\det\left[\left(\sum_{j=1}^{m}\lambda_{j}A_{j}^{-1}\right)^{-1}\right], \tag{1.4}$$

where $\lambda_j>0$, $j=1,...,m$ with $\sum_{j=1}^{m}\lambda_j=1$ and $A_j>0$, $j=1,...,m$.

Using the representation (1.1) one can also prove the result, see [9, p. 212],

$$\det\left(A\right)=\det\left(A_{1n}\right)\leq\det\left(A_{1k}\right)\det\left(A_{(k+1)n}\right),\ k=1,...,n; \tag{1.5}$$

where the determinant $\det\left(A_{rs}\right)$ is defined by

$$\det\left(A_{rs}\right)=\det\left(a_{ij}\right),i,\ j=r,...,s.$$

In particular,

$$\det\left(A\right)\leq a_{11}a_{22}...a_{nn}. \tag{1.6}$$

We recall also the Minkowski's type inequality,

$$\left[\det\left(A+B\right)\right]^{1/n}\geq\left[\det\left(A\right)\right]^{1/n}+\left[\det\left(B\right)\right]^{1/n} \tag{1.7}$$

for A, B positive definite matrices of order n. For other determinant inequalities see Chapter VIII of the classic book [9]. For some recent results see [3]-[7].

2. One Integral Representation

In [2], D. I. Cartwright and M. J. Field obtained the following lower and upper bounds for the difference between the *weighted arithmetic* and *geometric means*

$$\begin{aligned}0&\leq\frac{1}{2b}\sum_{k=1}^{m}p_{k}\left(x_{k}-\sum_{j=1}^{m}p_{j}x_{j}\right)^{2}\leq\sum_{j=1}^{m}p_{j}x_{j}-\prod_{k=1}^{m}x_{k}^{p_{k}}\\&\leq\frac{1}{2a}\sum_{k=1}^{m}p_{k}\left(x_{k}-\sum_{j=1}^{m}p_{j}x_{j}\right)^{2},\end{aligned} \tag{2.1}$$

where $p_k\geq0$, $0<a\leq x_k\leq b$ for $k\in\{1,...,m\}$ and $\sum_{j=1}^{m}p_j=1$.

Theorem 1. *Assume that A_j, $j \in \{1,...,m\}$ are positive definite matrices. If $0 < \ell I_n \leq A_j$, $j \in \{1,...,m\}$ in the operator order, for some positive constant ℓ, and I_n is the unity matrix of order n, then*

$$0 \leq \frac{1}{2}\left[\sum_{k=1}^{m} p_k (1-p_k) [\det(2A_j - \ell I_n)]^{-1/2} \right. \tag{2.2}$$
$$\left. -2 \sum_{1\leq j<k\leq m} p_j p_k [\det(A_j + A_k - \ell I_n)]^{-1/2} \right]$$
$$\leq \sum_{j=1}^{m} p_j [\det(A_j)]^{-1/2} - \left[\det\left(\sum_{k=1}^{m} p_k A_k\right)\right]^{-1/2},$$

where $p_k \geq 0$ for $k \in \{1,...,m\}$ and $\sum_{j=1}^{m} p_j = 1$.

If $\frac{M}{2} I_n < A_j \leq M I_n$ for $j \in \{1,...,m\}$, where $M > 0$, then

$$\sum_{j=1}^{m} p_j [\det(A_j)]^{-1/2} - \left[\det\left(\sum_{k=1}^{m} p_k A_k\right)\right]^{-1/2} \tag{2.3}$$
$$\leq \frac{1}{2}\left[\sum_{k=1}^{m} p_k (1-p_k) [\det(2A_j - M I_n)]^{-1/2} \right.$$
$$\left. -2 \sum_{1\leq j<k\leq m} p_j p_k [\det(A_j + A_k - M I_n)]^{-1/2} \right].$$

Proof. Observe that

$$\sum_{k=1}^{m} p_k \left(x_k - \sum_{j=1}^{m} p_j x_j\right)^2 \tag{2.4}$$
$$= \sum_{k=1}^{m} p_k \left(x_k^2 - 2x_k \sum_{j=1}^{m} p_j x_j + \left(\sum_{j=1}^{m} p_j x_j\right)^2\right)$$
$$= \sum_{k=1}^{m} p_k x_k^2 - 2\sum_{k=1}^{m} p_k x_k \sum_{j=1}^{m} p_j x_j + \sum_{k=1}^{m} p_k \left(\sum_{j=1}^{m} p_j x_j\right)^2$$
$$= \sum_{k=1}^{m} p_k x_k^2 - \left(\sum_{j=1}^{m} p_j x_j\right)^2 = \sum_{k=1}^{m} p_k x_k^2 - \left(\sum_{k=1}^{m} p_k x_k\right)\left(\sum_{j=1}^{m} p_j x_j\right)$$
$$= \sum_{k=1}^{m} p_k x_k^2 - \sum_{k=1}^{m}\sum_{j=1}^{m} p_j p_k x_j x_k = \sum_{k=1}^{m} p_k (1-p_k) x_k^2 - 2\sum_{1\leq j<k\leq m} p_j p_k x_j x_k$$

and the inequality (2.1) can be written in a more convenient form for what follows

$$\begin{aligned}
&\frac{1}{2b}\left[\sum_{k=1}^{m} p_k\left(1-p_k\right)x_k^2-2\sum_{1\leq j<k\leq m} p_j p_k x_j x_k\right] \\
&\leq \sum_{j=1}^{m} p_j x_j-\prod_{k=1}^{m} x_k^{p_k} \\
&\leq \frac{1}{2a}\left[\sum_{k=1}^{m} p_k\left(1-p_k\right)x_k^2-2\sum_{1\leq j<k\leq m} p_j p_k x_j x_k\right].
\end{aligned} \tag{2.5}$$

Assume that $0<\ell I_n\leq A_j\leq MI_n$, where $j\in\{1,...,m\}$. For $x\in\mathbb{R}^n$ we have $-\ell\|x\|^2\geq-\langle A_j x,x\rangle\geq-M\|x\|^2$, which implies that

$$\exp\left(-\ell\|x\|^2\right)\geq\exp\left(-\langle A_j x,x\rangle\right)\geq\exp\left(-M\|x\|^2\right)$$

for $j\in\{1,...,m\}$.

If we take $a=\exp\left(-M\|x\|^2\right)$, $x_j=\exp\left(-\langle A_j x,x\rangle\right)$, $j\in\{1,...,m\}$ and $b=\exp\left(-\ell\|x\|^2\right)$ in (2.5), then we get

$$\begin{aligned}
0&\leq\frac{1}{2\exp\left(-\ell\|x\|^2\right)} \\
&\times\left[\sum_{k=1}^{m} p_k\left(1-p_k\right)\exp\left(-\langle 2A_j x,x\rangle\right)-2\sum_{1\leq j<k\leq m} p_j p_k\exp\left(-\langle\left(A_j+A_k\right)x,x\rangle\right)\right] \\
&\leq\sum_{j=1}^{m} p_j\exp\left(-\langle A_j x,x\rangle\right)-\exp\left(-\left\langle\sum_{k=1}^{m} p_k A_k x,x\right\rangle\right) \\
&\leq\frac{1}{2\exp\left(-M\|x\|^2\right)} \\
&\times\left[\sum_{k=1}^{m} p_k\left(1-p_k\right)\exp\left(-\langle 2A_j x,x\rangle\right)-2\sum_{1\leq j<k\leq m} p_j p_k\exp\left(-\langle\left(A_j+A_k\right)x,x\rangle\right)\right],
\end{aligned}$$

namely

$$0 \le \frac{1}{2}\left[\sum_{k=1}^{m} p_k (1-p_k) \exp\left(-\langle 2A_j x, x\rangle + \ell \|x\|^2\right) - 2\sum_{1\le j<k\le m} p_j p_k \exp\left(-\langle (A_j + A_k) x, x\rangle + \ell \|x\|^2\right)\right]$$
$$\le \sum_{j=1}^{m} p_j \exp(-\langle A_j x, x\rangle) - \exp\left(-\left\langle \sum_{k=1}^{m} p_k A_k x, x\right\rangle\right)$$
$$\le \frac{1}{2}\left[\sum_{k=1}^{m} p_k (1-p_k) \exp\left(-\langle 2A_j x, x\rangle + M \|x\|^2\right) - 2\sum_{1\le j<k\le m} p_j p_k \exp\left(-\langle (A_j + A_k) x, x\rangle + M \|x\|^2\right)\right],$$

for $x \in \mathbb{R}^n$.

This is equivalent to

$$0 \le \frac{1}{2}\left[\sum_{k=1}^{m} p_k (1-p_k) \exp(-\langle (2A_j - \ell I_n) x, x\rangle) - 2\sum_{1\le j<k\le m} p_j p_k \exp(-\langle (A_j + A_k - \ell I_n) x, x\rangle)\right]$$
$$\le \sum_{j=1}^{m} p_j \exp(-\langle A_j x, x\rangle) - \exp\left(-\left\langle \sum_{k=1}^{m} p_k A_k x, x\right\rangle\right)$$
$$\le \frac{1}{2}\left[\sum_{k=1}^{m} p_k (1-p_k) \exp(-\langle (2A_j - M I_n) x, x\rangle) - 2\sum_{1\le j<k\le m} p_j p_k \exp(-\langle (A_j + A_k - M I_n) x, x\rangle)\right], \tag{2.6}$$

for $x \in \mathbb{R}^n$.

Since $0 < \ell I_n \le A_j$, hence $2A_j - \ell I_n > 0$, $A_j + A_k - \ell I_n > 0$, $k, j \in \{1, ..., m\}$ and by taking the integral $\int_{\mathbb{R}^n}$ in the first inequality in (2.6), then

$$0 \le \frac{1}{2}\left[\sum_{k=1}^{m} p_k (1-p_k) \int_{\mathbb{R}^n} \exp(-\langle (2A_j - \ell I_n) x, x\rangle)\, dx - 2\sum_{1\le j<k\le m} p_j p_k \int_{\mathbb{R}^n} \exp(-\langle (A_j + A_k - \ell I_n) x, x\rangle)\, dx\right]$$
$$\le \sum_{j=1}^{m} p_j \int_{\mathbb{R}^n} \exp(-\langle A_j x, x\rangle)\, dx - \int_{\mathbb{R}^n} \exp\left(-\left\langle \sum_{k=1}^{m} p_k A_k x, x\right\rangle\right) dx. \tag{2.7}$$

By utilizing (1.1), we have

$$\frac{1}{2}\left[\sum_{k=1}^{m}p_k\left(1-p_k\right)J_n\left(2A_j-\ell I_n\right)-2\sum_{1\le j<k\le m}p_jp_kJ_n\left(A_j+A_k-\ell I_n\right)\right]$$
$$\le\sum_{j=1}^{m}p_jJ_n\left(A_j\right)-J_n\left(\sum_{k=1}^{m}p_kA_k\right),$$

namely

$$\frac{1}{2}\left[\sum_{k=1}^{m}\frac{p_k\left(1-p_k\right)}{\left[\det\left(2A_j-\ell I_n\right)\right]^{1/2}}-2\sum_{1\le j<k\le m}\frac{p_jp_k}{\left[\det\left(A_j+A_k-\ell I_n\right)\right]^{1/2}}\right]$$
$$\le\sum_{j=1}^{m}\frac{p_j}{\left[\det\left(A_j\right)\right]^{1/2}}-\frac{1}{\left[\det\left(\sum_{k=1}^{m}p_kA_k\right)\right]^{1/2}},$$

which is equivalent to (2.2).

If $\frac{M}{2}I_n<A_j\le MI_n$ for $j\in\{1,...,m\}$, then $2A_j-MI_n>0$ and $A_j+A_k-MI_n>0$ for $j,\ k\in\{1,...,m\}$. By making use of the second inequality in (2.6) and take the integral $\int_{\mathbb{R}^n}$, then we get

$$\sum_{j=1}^{m}p_j\int_{\mathbb{R}^n}\exp\left(-\left\langle A_jx,x\right\rangle\right)dx-\int_{\mathbb{R}^n}\exp\left(-\left\langle\sum_{k=1}^{m}p_kA_kx,x\right\rangle\right)dx$$
$$\le\frac{1}{2}\left[\sum_{k=1}^{m}p_k\left(1-p_k\right)\int_{\mathbb{R}^n}\exp\left(-\left\langle\left(2A_j-MI_n\right)x,x\right\rangle\right)dx\right.$$
$$\left.-2\sum_{1\le j<k\le m}p_jp_k\int_{\mathbb{R}^n}\exp\left(-\left\langle\left(A_j+A_k-MI_n\right)x,x\right\rangle\right)dx\right].$$

By utilizing (1.1) we get

$$\sum_{j=1}^{m}p_jJ_n\left(A_j\right)-J_n\left(\sum_{k=1}^{m}p_kA_k\right)$$
$$\le\frac{1}{2}\left[\sum_{k=1}^{m}p_k\left(1-p_k\right)J_n\left(2A_j-MI_n\right)-2\sum_{1\le j<k\le m}p_jp_kJ_n\left(A_j+A_k-MI_n\right)\right],$$

which gives (2.3). □

Remark 1. *If we take* $p_k=\frac{1}{m}$, $k\in\{1,...,m\}$ *in (2.2), then we get*

$$0\le\frac{1}{2}\frac{1}{m^2}\left[(m-1)\sum_{k=1}^{m}\left[\det\left(2A_j-\ell I_n\right)\right]^{-1/2}\right.$$
$$\left.-2\sum_{1\le j<k\le m}\left[\det\left(A_j+A_k-\ell I_n\right)\right]^{-1/2}\right]$$
$$\le\frac{1}{m}\sum_{j=1}^{m}\left[\det\left(A_j\right)\right]^{-1/2}-\left[\det\left(\frac{1}{m}\sum_{k=1}^{m}A_k\right)\right]^{-1/2},\tag{2.8}$$

provided that $0 < \ell I_n \leq A_j,\ j \in \{1, ..., m\}$.

From (2.3) we get

$$\frac{1}{m}\sum_{j=1}^{m}[\det(A_j)]^{-1/2} - \left[\det\left(\frac{1}{m}\sum_{k=1}^{m}A_k\right)\right]^{-1/2} \leq \frac{1}{2m^2}\left[(m-1)\sum_{k=1}^{m}[\det(2A_j - MI_n)]^{-1/2} - 2\sum_{1\leq j<k\leq m}[\det(A_j + A_k - MI_n)]^{-1/2}\right], \tag{2.9}$$

provided that $\frac{M}{2}I_n < A_j \leq MI_n$ *for* $j \in \{1, ..., m\}$.

Corollary 1. *Assume that* A_1, A_2 *are positive definite matrices. If* $0 < \ell I_n \leq A_1$, A_2 *in the operator order, then*

$$0 \leq \frac{1}{2}t(1-t)\left[[\det(2A_1 - \ell I_n)]^{-1/2} + [\det(2A_2 - \ell I_n)]^{-1/2} - 2[\det(A_1 + A_2 - \ell I_n)]^{-1/2}\right] \leq (1-t)[\det(A_1)]^{-1/2} + t[\det(A_2)]^{-1/2} - [\det((1-t)A_1 + tA_2)]^{-1/2}, \tag{2.10}$$

where $t \in (0,1)$.

If $\frac{M}{2}I_n < A_1, A_2 \leq MI_n$ *for some* $M > 0$, *then*

$$(1-t)[\det(A_1)]^{-1/2} + t[\det(A_2)]^{-1/2} - [\det((1-t)A_1 + tA_2)]^{-1/2} \leq \frac{1}{2}t(1-t)\left[[\det(2A_1 - MI_n)]^{-1/2} + [\det(2A_2 - MI_n)]^{-1/2} - 2[\det(A_1 + A_2 - MI_n)]^{-1/2}\right], \tag{2.11}$$

where $t \in (0,1)$.

The proof follows by Theorem 1 on taking $m = 2$, $q_1 = 1 - t$, $q_2 = t$ where $t \in (0,1)$.

Corollary 2. *With the assumptions of Corollary 1 we have*

$$0 \leq \frac{1}{12}\left[[\det(2A_1 - \ell I_n)]^{-1/2} + [\det(2A_2 - \ell I_n)]^{-1/2} - 2[\det(A_1 + A_2 - \ell I_n)]^{-1/2}\right] \leq \frac{[\det(A_1)]^{-1/2} + [\det(A_2)]^{-1/2}}{2} - \int_0^1 [\det((1-t)A_1 + tA_2)]^{-1/2}\,dt, \tag{2.12}$$

provided that $0 < \ell I_n \leq A_1,\ A_2$ *and*

$$\frac{[\det(A_1)]^{-1/2} + [\det(A_2)]^{-1/2}}{2} - \int_0^1 [\det((1-t)A_1 + tA_2)]^{-1/2} dt \tag{2.13}$$
$$\leq \frac{1}{12}\Big[[\det(2A_1 - MI_n)]^{-1/2} + [\det(2A_2 - MI_n)]^{-1/2}$$
$$-2[\det(A_1 + A_2 - MI_n)]^{-1/2}\Big],$$

provided that $\frac{M}{2} I_n < A_1,\ A_2 \leq MI_n$.

3. Two Integrals Representation

If we take the square in the representation (1.1), then we get

$$\left(\int_{\mathbb{R}^n} \exp(-\langle x, Ax\rangle) dx\right)^2 = \frac{\pi^n}{\det(A)}.$$

Since

$$\left(\int_{\mathbb{R}^n} \exp(-\langle Ax, x\rangle) dx\right)^2 = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \exp(-\langle Ax, x\rangle)\exp(-\langle Ay, y\rangle) dxdy$$
$$= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \exp(-\langle Ax, x\rangle - \langle Ay, y\rangle)) dxdy,$$

hence

$$K_n(A) := \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \exp(-\langle Ax, x\rangle - \langle Ay, y\rangle)) dxdy = \frac{\pi^n}{\det(A)} \tag{3.1}$$

for A a positive definite matrix of order n and $\langle \cdot, \cdot \rangle$ is the usual inner product on $\mathbb{R}^n$.

Theorem 2. *Assume that* $A_j,\ j \in \{1, ..., m\}$ *are positive definite matrices. If* $0 < \ell I_n \leq A_j,\ j \in \{1, ..., m\}$ *in the operator order, for some positive constant* ℓ, *and* I_n *is the unity matrix of order* n, *then*

$$0 \leq \frac{1}{2}\left[\sum_{k=1}^{m} p_k(1-p_k)[\det(2A_j - \ell I_n)]^{-1}\right. \tag{3.2}$$
$$\left. -2\sum_{1\leq j<k\leq m} p_j p_k [\det(A_j + A_k - \ell I_n)]^{-1}\right]$$
$$\leq \sum_{j=1}^{m} p_j [\det(A_j)]^{-1} - \left[\det\left(\sum_{k=1}^{m} p_k A_k\right)\right]^{-1},$$

where $p_k \geq 0$ *for* $k \in \{1, ..., m\}$ *and* $\sum_{j=1}^{m} p_j = 1$.

If $\frac{M}{2}I_n < A_j \leq MI_n$ *for* $j \in \{1, ..., m\}$, *where* $M > 0$, *then*

$$\sum_{j=1}^{m} p_j \left[\det(A_j)\right]^{-1} - \left[\det\left(\sum_{k=1}^{m} p_k A_k\right)\right]^{-1} \leq \frac{1}{2}\left[\sum_{k=1}^{m} p_k(1-p_k)\left[\det(2A_j - MI_n)\right]^{-1} - 2\sum_{1\leq j<k\leq m} p_j p_k \left[\det(A_j + A_k - MI_n)\right]^{-1}\right]. \tag{3.3}$$

Proof. Assume that $0 < \ell I_n \leq A_j \leq MI_n$, where $j \in \{1, ..., m\}$. For $x, y \in \mathbb{R}^n$ we have

$$-\ell\left(\|x\|^2 + \|y\|^2\right) \geq -\langle A_j x, x\rangle - \langle A_j y, y\rangle \geq -M\left(\|x\|^2 + \|y\|^2\right),$$

which implies that

$$\exp\left(-\ell\left(\|x\|^2 + \|y\|^2\right)\right) \geq \exp\left(-\langle A_j x, x\rangle - \langle A_j y, y\rangle\right) \geq \exp\left(-M\left(\|x\|^2 + \|y\|^2\right)\right)$$

for $j \in \{1, ..., m\}$.

If we take $a = \exp\left(-M\left(\|x\|^2 + \|y\|^2\right)\right)$, $x_j = \exp\left(-\langle A_j x, x\rangle - \langle A_j y, y\rangle\right)$, $j \in \{1, ..., m\}$ and $b = \exp\left(-\ell\left(\|x\|^2 + \|y\|^2\right)\right)$ in (2.5), then we get

$$\begin{aligned} 0 &\leq \frac{1}{2}\left[\sum_{k=1}^{m} p_k(1-p_k)\exp\left(-\langle(2A_j - \ell)x, x\rangle - \langle(2A_j - \ell)y, y\rangle\right)\right. \\ &\left.-2\sum_{1\leq j<k\leq m} p_j p_k \exp\left(-\langle(A_j + A_k - \ell)x, x\rangle - \langle(A_j + A_k - \ell)y, y\rangle\right)\right] \\ &\leq \sum_{j=1}^{m} p_j \exp\left(-\langle A_j x, x\rangle - \langle A_j y, y\rangle\right) \\ &- \exp\left(-\left\langle\sum_{k=1}^{m} p_k A_k x, x\right\rangle - \left\langle\sum_{k=1}^{m} p_k A_k y, y\right\rangle\right) \\ &\leq \frac{1}{2}\left[\sum_{k=1}^{m} p_k(1-p_k)\exp\left(-\langle(2A_j - M)x, x\rangle - \langle(2A_j - M)y, y\rangle\right)\right. \\ &\left.-2\sum_{1\leq j<k\leq m} p_j p_k \exp\left(-\langle(A_j + A_k - M)x, x\rangle - \langle(A_j + A_k - M)y, y\rangle\right)\right]. \end{aligned} \tag{3.4}$$

Since $0<\ell I_n\leq A_j$, hence $2A_j-\ell I_n>0$, $A_j+A_k-\ell I_n>0$, $k,j\in\{1,...,m\}$ and by taking the double integral $\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}$ in the first inequality in (3.4), then

$$\begin{aligned}
0 &\leq \frac{1}{2}\left[\sum_{k=1}^{m}p_k\left(1-p_k\right)\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\exp\left(-\left\langle\left(2A_j-\ell\right)x,x\right\rangle-\left\langle\left(2A_j-\ell\right)y,y\right\rangle\right)dxdy\right. \\
&-2\sum_{1\leq j<k\leq m}p_jp_k \\
&\left.\times\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\exp\left(-\left\langle\left(A_j+A_k-\ell\right)x,x\right\rangle-\left\langle\left(A_j+A_k-\ell\right)y,y\right\rangle\right)dxdy\right] \\
&\leq \sum_{j=1}^{m}p_j\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\exp\left(-\left\langle A_jx,x\right\rangle-\left\langle A_jy,y\right\rangle\right)dxdy \\
&-\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\exp\left(-\left\langle\sum_{k=1}^{m}p_kA_kx,x\right\rangle-\left\langle\sum_{k=1}^{m}p_kA_ky,y\right\rangle\right)dxdy,
\end{aligned}$$

namely, by (3.1)

$$\begin{aligned}
0 &\leq \frac{1}{2}\left[\sum_{k=1}^{m}p_k\left(1-p_k\right)K_n\left(2A_j-\ell I_n\right)-2\sum_{1\leq j<k\leq m}p_jp_kK_n\left(A_j+A_k-\ell I_n\right)\right] \\
&\leq \sum_{j=1}^{m}p_jJ_n\left(A_j\right)-J_n\left(\sum_{k=1}^{m}p_kA_k\right),
\end{aligned}$$

which, by (3.1) gives (3.2).

The inequality (3.3) follows by the second inequality in (3.4) and we omit the details. □

Remark 2. *If we take* $p_k=\frac{1}{m}$, $k\in\{1,...,m\}$ *in (3.2), then we get*

$$\begin{aligned}
0 &\leq \frac{1}{2}\frac{1}{m^2}\left[(m-1)\sum_{k=1}^{m}\left[\det\left(2A_j-\ell I_n\right)\right]^{-1}\right. \\
&\left.-2\sum_{1\leq j<k\leq m}\left[\det\left(A_j+A_k-\ell I_n\right)\right]^{-1}\right] \\
&\leq \frac{1}{m}\sum_{j=1}^{m}\left[\det\left(A_j\right)\right]^{-1}-\left[\det\left(\frac{1}{m}\sum_{k=1}^{m}A_k\right)\right]^{-1},
\end{aligned} \tag{3.5}$$

provided that $0<\ell I_n\leq A_j$, $j\in\{1,...,m\}$.

From (3.3) we obtain

$$\frac{1}{m}\sum_{j=1}^{m}[\det(A_j)]^{-1}-\left[\det\left(\frac{1}{m}\sum_{k=1}^{m}A_k\right)\right]^{-1} \tag{3.6}$$
$$\leq\frac{1}{2m^2}\left[(m-1)\sum_{k=1}^{m}[\det(2A_j-MI_n)]^{-1}\right.$$
$$\left.-2\sum_{1\leq j<k\leq m}[\det(A_j+A_k-MI_n)]^{-1}\right].$$

Corollary 3. *Assume that A_1, A_2 are positive definite matrices. If $0<\ell I_n\leq A_1$, A_2 in the operator order, then*

$$0\leq\frac{1}{2}t(1-t)\left[[\det(2A_1-\ell I_n)]^{-1}+[\det(2A_2-\ell I_n)]^{-1}\right. \tag{3.7}$$
$$\left.-2[\det(A_1+A_2-\ell I_n)]^{-1}\right]$$
$$\leq(1-t)[\det(A_1)]^{-1}+t[\det(A_2)]^{-1}$$
$$-[\det((1-t)A_1+tA_2)]^{-1},$$

where $t\in(0,1)$.

If $\frac{M}{2}I_n<A_1$, $A_2\leq MI_n$ for some $M>0$, then

$$(1-t)[\det(A_1)]^{-1}+t[\det(A_2)]^{-1} \tag{3.8}$$
$$-[\det((1-t)A_1+tA_2)]^{-1}$$
$$\leq\frac{1}{2}t(1-t)\left[[\det(2A_1-MI_n)]^{-1}+[\det(2A_2-MI_n)]^{-1}\right.$$
$$\left.-2[\det(A_1+A_2-MI_n)]^{-1/2}\right],$$

where $t\in(0,1)$.

The proof follows by Theorem 2 on taking $m=2$, $q_1=1-t$, $q_2=t$ where $t\in(0,1)$.

Corollary 4. *With the assumptions of Corollary 3 we have*

$$0\leq\frac{1}{12}\left[[\det(2A_1-\ell I_n)]^{-1}+[\det(2A_2-\ell I_n)]^{-1}\right. \tag{3.9}$$
$$\left.-2[\det(A_1+A_2-\ell I_n)]^{-1}\right]$$
$$\leq\frac{[\det(A_1)]^{-1}+[\det(A_2)]^{-1}}{2}-\int_0^1[\det((1-t)A_1+tA_2)]^{-1}dt,$$

provided that $0<\ell I_n\leq A_1$, A_2 and

$$\frac{[\det(A_1)]^{-1}+[\det(A_2)]^{-1}}{2}-\int_0^1[\det((1-t)A_1+tA_2)]^{-1}dt \tag{3.10}$$
$$\leq\frac{1}{12}\left[[\det(2A_1-MI_n)]^{-1}+[\det(2A_2-MI_n)]^{-1}\right.$$
$$\left.-2[\det(A_1+A_2-MI_n)]^{-1}\right],$$

provided that $\frac{M}{2}I_n < A_1,\ A_2 \le MI_n$.

4. The Case of Hermitian Matrices

A complex square matrix $H = (h_{ij})$, $i,j = 1,...,n$ is said to be Hermitian provided $h_{ij} = \overline{h_{ji}}$ for all $i,\ j = 1,...,n$. A Hermitian matrix is said to be positive definite if the Hermitian form $P(z) = \sum_{i,j=1}^{n} a_{ij} z_i \overline{z_j}$ is positive for all $z = (z_1,...,z_n) \in \mathbb{C}^n \setminus \{0\}$.

It is known that, see for instance [9, p. 215], for a positive definite Hermitian matrix H, we have

$$K_n(H) := \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \exp\left(-\langle \overline{z}, Hz\rangle\right) dxdy = \frac{\pi^n}{\det(H)}, \tag{4.1}$$

where $z = x + iy$ and dx and dy denote integration over real n-dimensional space $\mathbb{R}^n$. Here the inner product $\langle x,y\rangle$ is understood in the real sense, i.e. $\langle x,y\rangle = \sum_{k=1}^{n} x_k y_k$.

On making use of a similar argument to the one in Theorem 2 for the representation $K_n(\cdot)$ we can state the following result as well:

Theorem 3. *Assume that* H_j, $j \in \{1,...,m\}$ *are Hermitian positive definite matrices. If* $0 < \ell I_n \le H_j$, $j \in \{1,...,m\}$ *in the operator order, for some positive constant* ℓ, *and* I_n *is the unity matrix of order* n, *then*

$$\begin{aligned} 0 &\le \frac{1}{2}\left[\sum_{k=1}^{m} p_k(1-p_k)\left[\det(2H_j - \ell I_n)\right]^{-1}\right. \\ &\quad \left. -2\sum_{1\le j<k\le m} p_j p_k \left[\det(H_j + H_k - \ell I_n)\right]^{-1}\right] \\ &\le \sum_{j=1}^{m} p_j\left[\det(H_j)\right]^{-1} - \left[\det\left(\sum_{k=1}^{m} p_k H_k\right)\right]^{-1}, \end{aligned} \tag{4.2}$$

where $p_k \ge 0$ *for* $k \in \{1,...,m\}$ *and* $\sum_{j=1}^{m} p_j = 1$.

If $\frac{M}{2}I_n < H_j \le MI_n$ *for* $j \in \{1,...,m\}$, *where* $M > 0$, *then*

$$\begin{aligned} &\sum_{j=1}^{m} p_j\left[\det(H_j)\right]^{-1} - \left[\det\left(\sum_{k=1}^{m} p_k H_k\right)\right]^{-1} \\ &\le \frac{1}{2}\left[\sum_{k=1}^{m} p_k(1-p_k)\left[\det(2H_j - MI_n)\right]^{-1}\right. \\ &\quad \left. -2\sum_{1\le j<k\le m} p_j p_k \left[\det(H_j + H_k - MI_n)\right]^{-1}\right]. \end{aligned} \tag{4.3}$$

Similar particular cases may be stated to the ones from the previous section, however we do not present them here.

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