p-Schatten norm sequential generalized fractional Ostrowski and Grüss type inequalities for several functions

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Abstract

Using sequential generalized Caputo fractional left and right vectorial Taylor formulae we establish sequential generalized fractional Ostrowski and Grüss type inequalities for several functions that take values in the von Neumann-Schatten class $\mathcal{B}_p(H)$, $1 \le p < \infty$. The estimates are given for all p-Schatten norms, $1 \le p < \infty$. We finish with applications.

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1 Introduction

The following results inspire our work.

Theorem 1 *(1938, Ostrowski [14]) Let $f : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b) whose derivative $f' : (a,b) \to \mathbb{R}$ is bounded on (a,b), i.e., $\|f'\|_\infty^{\sup} := \sup_{t \in (a,b)} |f'(t)| < +\infty$. Then*

$$\left| \frac{1}{b-a} \int_a^b f(t)\, dt - f(x) \right| \le \left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right] (b-a) \|f'\|_\infty^{\sup}, \qquad (1)$$

for any $x \in [a,b]$. The constant $\frac{1}{4}$ is the best possible.

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Ostrowski type inequalities have great applications to integral approximations in Numerical Analysis.

Theorem 2 *(1882, Čebyšev [7]) Let $f, g : [a, b] \to \mathbb{R}$ be absolutely continuous functions with $f', g' \in L_\infty([a, b])$. Then*

$$\left| \frac{1}{b-a} \int_a^b f(x) g(x) dx - \left(\frac{1}{b-a} \int_a^b f(x) dx \right) \left(\frac{1}{b-a} \int_a^b g(x) dx \right) \right|$$

$$\leq \frac{1}{12} (b-a)^2 \|f'\|_\infty \|g'\|_\infty . \quad (2)$$

The above integrals are assumed to exist.

The related Grüss type inequalities have many applications to Probability Theory. We presented also ([4], Ch. 8,9) mixed fractional Ostrowski and Grüss-Cebysev type inequalities for several functions, acting to all possible directions. The estimates involve the left and right Caputo fractional derivatives. See also the monographs written by the author [2], Chapters 24-26 and [3], Chapters 2-6.

We are motivated also by S. Dragomir [10] recent work:

An operator $A \in \mathcal{B}(H)$ is said to belong to the von Neumann-Schatten class $\mathcal{B}_p(H)$, $1 \leq p < \infty$ if the p-Schatten norm is finite

$$\|A\|_p := [tr(|A|^p)]^{\frac{1}{p}} < \infty.$$

Assume that $A : [a, b] \to \mathcal{B}_p(H)$, $B : [a, b] \to \mathcal{B}_q(H)$, $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, are continuous and B is strongly differentiable on (a, b), then

$$\left\| \int_a^b A(t) B(t) dt - \left(\int_a^b A(s) ds \right) B(u) \right\|_1 \leq$$

$$\sup_{t \in [a,b]} \|B'(t)\|_q \times \begin{cases} \left[\frac{1}{2}(b-a) + \left| u - \frac{a+b}{2} \right| \right] \int_a^b \|A(t)\|_p dt, \\ \left[\frac{(u-a)^{\beta+1} + (b-u)^{\beta+1}}{\beta+1} \right]^{\frac{1}{\beta}} \left(\int_a^b \|A(t)\|_p^\alpha \right)^{\frac{1}{\alpha}}, \\ \text{for } \alpha, \beta > 1 \text{ with } \frac{1}{\alpha} + \frac{1}{\beta} = 1 \\ \left[\frac{1}{4}(b-a)^2 + \left(u - \frac{a+b}{2} \right)^2 \right] \sup_{t \in [a,b]} \|A(t)\|_p , \end{cases} \quad (3)$$

for all $u \in [a, b]$, an Ostrowski type inequality.

Further inspiration comes from S. Dragomir [11] recent work on Grüss inequalities:

For two continuous functions $A, B : [a,b] \to \mathcal{B}(H)$ we define the noncommutative Cebysev fractional

$$D(A,B) := (b-a)\int_a^b A(t)B(t)\,dt - \int_a^b A(t)\,dt\int_a^b B(t)\,dt. \tag{4}$$

If $p, q > 1$ with $\frac{1}{p}+\frac{1}{q}=1$, let $A : [a,b] \to \mathcal{B}_p(H)$, $B : [a,b] \to \mathcal{B}_q(H)$ be strongly differentiable functions on the interval (a,b), then

$$\|D(A,B)\|_1 \leq D\left(\int_a^b \|A'(u)\|_p\,du, \int_a^b \|B'(u)\|_q\,du\right) \leq$$

$$\frac{1}{4}(b-a)^2\int_a^b \|A'(u)\|_p\,du\int_a^b \|B'(u)\|_q\,du.$$

In this article we generalize [4], Ch. 8,9 for several Banach algebra $\mathcal{B}_p(H)$ valued functions, in the sense of developing sequential fractional Ostrowski and Grüss type inequalities. Now our left and right sequential generalized Caputo fractional derivatives are for Banach space valued functions and our integrals are of Bochner type [1], [12]. Applications finish the article.

2 Vectorial sequential generalized fractional calculus background

We need

Definition 3 *([5], p. 106) Let $0 < \alpha \leq 1$, $f \in C^1([a,b],X)$, where $[a,b] \subset \mathbb{R}$, and $(X, \|\cdot\|)$ is a Banach space. Let $g \in C^1([a,b])$, strictly increasing, such that $g^{-1} \in C^1([g(a), g(b)])$.*

We define the left generalized g-fractional derivative X-valued of f of order α as follows:

$$\left(D^{\alpha}_{a+;g}f\right)(x) := \frac{1}{\Gamma(1-\alpha)}\int_a^x (g(x)-g(t))^{-\alpha} g'(t)\left(f\circ g^{-1}\right)'(g(t))\,dt, \tag{5}$$

$\forall\, x \in [a,b]$, where Γ is the gamma function. The last integral is of Bochner type ([12]).

If $0 < \alpha < 1$, by Theorem 4.10, p. 98, [5], we have that $\left(D^{\alpha}_{a+;g}f\right) \in C([a,b],X)$.

We set

$$D^1_{a+;g}f(x) := \left(\left(f\circ g^{-1}\right)'\circ g\right)(x) \in C([a,b],X), \tag{6}$$

$$D^0_{a+;g}f(x) = f(x), \quad \forall\, x \in [a,b].$$

When $g = id$, then

$$D_{a+;g}^{\alpha} f = D_{a+;id}^{\alpha} f = D_{*a}^{\alpha} f, \tag{7}$$

the usual left X-valued Caputo fractional derivative, see [5], Ch. 1.

We make

Remark 4 *By (5) we have*

$$\begin{aligned}
\left\| \left(D_{a+;g}^{\alpha} f \right)(x) \right\| &\leq \frac{1}{\Gamma(1-\alpha)} \int_a^x (g(x) - g(t))^{-\alpha} g'(t) \left\| \left(f \circ g^{-1} \right)' (g(t)) \right\| dt \leq \\
&\frac{\left\| \left\| \left(f \circ g^{-1} \right)' \circ g \right\| \right\|_{\infty,[a,b]}}{\Gamma(1-\alpha)} \int_a^x (g(x) - g(t))^{-\alpha} g'(t)\, dt = \\
&\frac{\left\| \left\| \left(f \circ g^{-1} \right)' \circ g \right\| \right\|_{\infty,[a,b]}}{\Gamma(1-\alpha)} \frac{(g(x) - g(t))^{1-\alpha}}{1-\alpha} = \\
&\frac{\left\| \left\| \left(f \circ g^{-1} \right)' \circ g \right\| \right\|_{\infty,[a,b]}}{\Gamma(2-\alpha)} (g(x) - g(a))^{1-\alpha}, \quad \forall\, x \in [a,b].
\end{aligned} \tag{8}$$

Hence

$$\left(D_{a+;g}^{\alpha} f \right)(a) = 0. \tag{9}$$

We need

Definition 5 *([5], p. 107) Let $0 < \alpha \leq 1$, $f \in C^1([a,b], X)$, where $[a,b] \subset \mathbb{R}$, and $(X, \|\cdot\|)$ is a Banach space. Let $g \in C^1([a,b])$, strictly increasing, such that $g^{-1} \in C^1([g(a), g(b)])$.*

We define the right generalized g-fractional derivative X-valued of f of order α as follows:

$$\left(D_{b-;g}^{\alpha} f \right)(x) := \frac{-1}{\Gamma(1-\alpha)} \int_x^b (g(t) - g(x))^{-\alpha} g'(t) \left(f \circ g^{-1} \right)' (g(t))\, dt, \tag{10}$$

$\forall\, x \in [a,b]$. The last integral is of Bochner type.

If $0 < \alpha < 1$, by Theorem 4.11, p. 101 ([5]), we have that $\left(D_{b-;g}^{\alpha} f \right) \in C([a,b], X)$.

We set

$$D_{b-;g}^{1} f(x) := -\left(\left(f \circ g^{-1} \right)' \circ g \right)(x) \in C([a,b], X), \tag{11}$$

$$D_{b-;g}^{0} f(x) := f(x), \quad \forall\, x \in [a,b].$$

When $g = id$, then

$$D_{b-;g}^{\alpha} f(x) = D_{b-;id}^{\alpha} f(x) = D_{b-}^{\alpha} f, \tag{12}$$

the usual right X-valued Caputo fractional derivative, see [5], Ch. 2.

We make

Remark 6 *By (10) we have*

$$\left\|\left(D_{b-;g}^{\alpha}f\right)(x)\right\| \leq \frac{1}{\Gamma(1-\alpha)}\int_{x}^{b}(g(t)-g(x))^{-\alpha}g'(t)\left\|\left(f\circ g^{-1}\right)'(g(t))\right\|dt \leq$$

$$\frac{\left\|\left\|\left(f\circ g^{-1}\right)'\circ g\right\|\right\|_{\infty,[a,b]}}{\Gamma(1-\alpha)}\int_{x}^{b}(g(t)-g(x))^{-\alpha}g'(t)\,dt = \tag{13}$$

$$\frac{\left\|\left\|\left(f\circ g^{-1}\right)'\circ g\right\|\right\|_{\infty,[a,b]}}{\Gamma(1-\alpha)}\frac{(g(b)-g(x))^{1-\alpha}}{1-\alpha} =$$

$$\frac{\left\|\left\|\left(f\circ g^{-1}\right)'\circ g\right\|\right\|_{\infty,[a,b]}}{\Gamma(2-\alpha)}(g(b)-g(x))^{1-\alpha}, \quad \forall\, x\in[a,b].$$

Hence

$$\left(D_{b-;g}^{\alpha}f\right)(b) = 0. \tag{14}$$

We need

Definition 7 *([5], p. 115) Denote by* ($0<\alpha\leq 1$)

$$D_{a+;g}^{n\alpha} := D_{a+;g}^{\alpha}D_{a+;g}^{\alpha}...D_{a+;g}^{\alpha} \quad (n \text{ times}),\ n\in\mathbb{N} \tag{15}$$

and $D_{a+;g}^{0}=I$ *(identity operator).*

We also need

Definition 8 *([5], p. 118)*

$$D_{b-;g}^{n\alpha} := D_{b-;g}^{\alpha}D_{b-;g}^{\alpha}...D_{b-;g}^{\alpha} \quad (n \text{ times}),\ n\in\mathbb{N} \tag{16}$$

and $D_{b-;g}^{0}=I$ *(identity operator).*

Based on (9) and Theorem 4.30, p. 117, ([5]), we have the following g-left generalized modified X-valued Taylor's formula:

Theorem 9 *Let* $0<\alpha\leq 1$, $n\in\mathbb{N}$, $f\in C^{1}([a,b],X)$, $(X,\|\cdot\|)$ *a Banach space,* $g\in C^{1}([a,b])$, *strictly increasing, such that* $g^{-1}\in C^{1}([g(a),g(b)])$. *Let* $F_k := D_{a+;g}^{k\alpha}f$, $k=1,...,n$, *that fulfill* $F_k\in C^{1}([a,b],X)$, *and* $F_{n+1}\in C([a,b],X)$. *Then*

$$f(x)-f(a) = \sum_{i=2}^{n}\frac{(g(x)-g(a))^{i\alpha}}{\Gamma(i\alpha+1)}\left(D_{a+;g}^{i\alpha}f\right)(a) +$$

$$\frac{1}{\Gamma((n+1)\alpha)}\int_{a}^{x}(g(x)-g(t))^{(n+1)\alpha-1}g'(t)\left(D_{a+;g}^{(n+1)\alpha}f\right)(t)\,dt, \tag{17}$$

$\forall\, x\in[a,b]$.

When $n=1$ we obtain

Corollary 10 *Let $0<\alpha\leq 1$, $f\in C^{1}([a,b],X)$, $(X,\|\cdot\|)$ is a Banach space, $g\in C^{1}([a,b])$, strictly increasing, such that $g^{-1}\in C^{1}([g(a),g(b)])$. Assume that $D_{a+;g}^{\alpha}f\in C^{1}([a,b],X)$, and $D_{a+;g}^{2\alpha}f\in C([a,b],X)$. Then*

$$f(x)-f(a)=\frac{1}{\Gamma(2\alpha)}\int_{a}^{x}(g(x)-g(t))^{2\alpha-1}g'(t)\left(D_{a+;g}^{2\alpha}f\right)(t)\,dt, \tag{18}$$

$\forall\ x\in[a,b]$.

Based on (14) and Theorem 4.33, p. 120, ([5]), we have the following g-right generalized modified X-valued Taylor's formula:

Theorem 11 *Let $f\in C^{1}([a,b],X)$, $(X,\|\cdot\|)$ a Banach space, $g\in C^{1}([a,b])$, strictly increasing, such that $g^{-1}\in C^{1}([g(a),g(b)])$. Suppose $F_{k}:=D_{b-;g}^{k\alpha}f$, $k=1,...,n$, fulfill $F_{k}\in C^{1}([a,b],X)$, and $F_{n+1}\in C([a,b],X)$, where $0<\alpha\leq 1$, $n\in\mathbb{N}$. Then*

$$f(x)-f(b)=\sum_{i=2}^{n}\frac{(g(b)-g(x))^{i\alpha}}{\Gamma(i\alpha+1)}\left(D_{b-;g}^{i\alpha}f\right)(b)+$$

$$\frac{1}{\Gamma((n+1)\alpha)}\int_{x}^{b}(g(t)-g(x))^{(n+1)\alpha-1}g'(t)\left(D_{b-;g}^{(n+1)\alpha}f\right)(t)\,dt, \tag{19}$$

$\forall\ x\in[a,b]$.

When $n=1$ we obtain

Corollary 12 *Let $0<\alpha\leq 1$, $f\in C^{1}([a,b],X)$, $(X,\|\cdot\|)$ is a Banach space, $g\in C^{1}([a,b])$, strictly increasing, such that $g^{-1}\in C^{1}([g(a),g(b)])$. Assume that $D_{b-;g}^{\alpha}f\in C^{1}([a,b],X)$, and $D_{b-;g}^{2\alpha}f\in C([a,b],X)$. Then*

$$f(x)-f(b)=\frac{1}{\Gamma(2\alpha)}\int_{x}^{b}(g(t)-g(x))^{2\alpha-1}g'(t)\left(D_{b-;g}^{2\alpha}f\right)(t)\,dt, \tag{20}$$

$\forall\ x\in[a,b]$.

We are greatly motivated by the following sequential generalized fractional Ostrowski type inequality:

Theorem 13 *(p. 140, [5]) Let $g\in C^{1}([a,b])$ and strictly increasing, such that $g^{-1}\in C^{1}([g(a),g(b)])$, and $0<\alpha<1$, $n\in\mathbb{N}$, $f\in C^{1}([a,b],X)$, where $(X,\|\cdot\|)$ is a Banach space. Let $x_{0}\in[a,b]$ be fixed. Assume that $F_{k}^{x_{0}}:=D_{x_{0}-;g}^{k\alpha}f$, for $k=1,...,n$, fulfill $F_{k}^{x_{0}}\in C^{1}([a,b],X)$ and $F_{n+1}^{x_{0}}\in C([a,x_{0}],X)$ and $\left(D_{x_{0}-;g}^{i\alpha}f\right)(x_{0})=0$, $i=1,...,n$.*

Similarly, we assume that $G_k^{x_0} := D_{x_0+;g}^{k\alpha} f$, *for* $k = 1, ..., n$, *fulfill* $G_k^{x_0} \in C^1([x_0, b], X)$ *and* $G_{n+1}^{x_0} \in ([x_0, b], X)$ *and* $\left(D_{x_0+;g}^{i\alpha} f\right)(x_0) = 0$, $i = 1, ..., n$.

Then

$$\left\| \frac{1}{b-a} \int_a^b f(x)\, dx - f(x_0) \right\| \leq \frac{1}{(b-a)\,\Gamma((n+1)\alpha+1)} \cdot$$

$$\left\{ (g(b) - g(x_0))^{(n+1)\alpha} (b - x_0) \left\| D_{x_0+;g}^{(n+1)\alpha} f \right\|_{\infty,[x_0,b]} + \right. \tag{21}$$

$$\left. (g(x_0) - g(a))^{(n+1)\alpha} (x_0 - a) \left\| D_{x_0-;g}^{(n+1)\alpha} f \right\|_{\infty,[a,x_0]} \right\}.$$

3 Banach Algebras basic background

All here come from [15].

We need

Definition 14 *([15], p. 245) A complex algebra is a vector space* A *over the complex filed* $\mathbb{C}$ *in which a multiplication is defined that satisfies*

$$x(yz) = (xy)z, \tag{22}$$

$$(x+y)z = xz + yz, \quad x(y+z) = xy + xz, \tag{23}$$

and

$$\alpha(xy) = (\alpha x)y = x(\alpha y), \tag{24}$$

for all x, y *and* z *in* A *and for all scalars* α.

Additionally if A *is a Banach space with respect to a norm that satisfies the multiplicative inequality*

$$\|xy\| \leq \|x\| \|y\| \quad (x \in A,\ y \in A) \tag{25}$$

and if A *contains a unit element* e *such that*

$$xe = ex = x \quad (x \in A) \tag{26}$$

and

$$\|e\| = 1, \tag{27}$$

then A *is called a Banach algebra.*

A *is commutative iff* $xy = yx$ *for all* $x, y \in A$.

We make

Remark 15 *There exists at most one $e \in A$ that satisfies (26).*

Inequality (25) makes multiplication to be continuous, more precisely left and right continuous, see [15], p. 246.

Multiplication in A is not necessarily the numerical multiplication, it is something more general and it is defined abstractly, that is for $x, y \in A$ we have $xy \in A$, e.g. composition or convolution, etc.

For nice examples about Banach algebras see [15], p. 247-248, § 10.3.

We also make

Remark 16 *Next we mention about integration of A-valued functions, see [15], p. 259, § 10.22:*

If A is a Banach algebra and f is a continuous A-valued function on some compact Hausdorff space Q on which a complex Borel measure μ is defined, then $\int f d\mu$ exists and has all the properties that were discussed in Chapter 3 of [15], simply because A is a Banach space. However, an additional property can be added to these, namely: If $x \in A$, then

$$x \int_Q f \; d\mu = \int_Q x f(p) \; d\mu(p) \tag{28}$$

and

$$\left(\int_Q f \; d\mu \right) x = \int_Q f(p) \, x \; d\mu(p) . \tag{29}$$

The Bochner integrals we will involve in our article follow (28) and (29). Also, let $f \in C([a,b], X)$, where $[a,b] \subset \mathbb{R}$, $(X, \|\cdot\|)$ is a Banach space. By [5], p. 3, f is Bochner integrable.

4 p-Schatten norms background

In this advanced section all come from [10].

Let $(H, \langle \cdot, \cdot \rangle)$ be a complex Hilbert space and $\mathcal{B}(H)$ the Banach algebra of all bounded linear operators on H. If $\{e_i\}_{i \in I}$ an orthonormal basis of H, we say that $A \in \mathcal{B}(H)$ is of trace class if

$$\|A\|_1 := \sum_{i \in I} \langle |A| \, e_i, e_i \rangle < \infty. \tag{30}$$

The definition of $\|A\|_1$ does not depend on the choice of the orthornormal basis $\{e_i\}_{i \in I}$. We denote by $\mathcal{B}_1(H)$ the set of trace class operators in $\mathcal{B}(H)$.

We define the trace of a trace class operator $A \in \mathcal{B}_1(H)$ to be

$$tr(A) := \sum_{i \in I} \langle A e_i, e_i \rangle , \tag{31}$$

where $\{e_i\}_{i\in I}$ an orthonormal basis of H. Note that this coincides with the usual definition of the trace if H is finite-dimensional. We observe that the series (31) converges absolutely and it is independent from the choice of basis.

The following result collects some properties of the trace:

Theorem 17 *We have:*

(i) If $A \in \mathcal{B}_1(H)$ then $A^* \in \mathcal{B}_1(H)$ and

$$tr(A^*) = \overline{tr(A)}; \tag{32}$$

(ii) If $A \in \mathcal{B}_1(H)$ and $T \in \mathcal{B}(H)$, then $AT, TA \in \mathcal{B}_1(H)$ and

$$tr(AT) = tr(TA) \quad and \quad |tr(AT)| \leq \|A\|_1 \|T\|; \tag{33}$$

(iii) $tr(\cdot)$ is a bounded linear functional on $\mathcal{B}_1(H)$ with $\|tr\| = 1$;

(iv) If $A, B \in \mathcal{B}_2(H)$ then $AB,\ BA \in \mathcal{B}_1(H)$ and $tr(AB) = tr(BA)$;

(v) $\mathcal{B}_{fin}(H)$ (finite rank operators) is a dense subspace of $\mathcal{B}_1(H)$.

An operator $A \in \mathcal{B}(H)$ is said to belong to the von Neumann-Schatten class $\mathcal{B}_p(H)$, $1 \leq p < \infty$ if the p-Schatten norm is finite [17, p. 60-64]

$$\|A\|_p := [tr(|A|^p)]^{\frac{1}{p}} < \infty,$$

$|A|^p$ is an operator notation and not a power.

For $1 < p < q < \infty$ we have that

$$\mathcal{B}_1(H) \subset \mathcal{B}_p(H) \subset \mathcal{B}_q(H) \subset \mathcal{B}(H) \tag{34}$$

and

$$\|A\|_1 \geq \|A\|_p \geq \|A\|_q \geq \|A\|. \tag{35}$$

For $p \geq 1$ the functional $\|\cdot\|_p$ is a norm on the $*$-ideal $\mathcal{B}_p(H)$, which is a Banach algebra, and $\left(\mathcal{B}_p(H), \|\cdot\|_p\right)$ is a Banach space.

Also, see for instance [17, p. 60-64], for $p \geq 1$,

$$\|A\|_p = \|A^*\|_p, \ \ A \in \mathcal{B}_p(H) \tag{36}$$

$$\|AB\|_p \leq \|A\|_p \|B\|_p, \ \ A, B \in \mathcal{B}_p(H) \tag{37}$$

and

$$\|AB\|_p \leq \|A\|_p \|B\|, \ \ \|BA\|_p \leq \|B\| \|A\|_p, \ \ A \in \mathcal{B}_p(H),\ B \in \mathcal{B}(H). \tag{38}$$

This implies that

$$\|CAB\|_p \leq \|C\| \|A\|_p \|B\|, \ \ A \in \mathcal{B}_p(H),\ B, C \in \mathcal{B}(H). \tag{39}$$

In terms of p-Schatten norm we have the Hölder inequality for $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$:

$$(|tr(AB)| \leq) \|AB\|_1 \leq \|A\|_p \|B\|_q, \quad A \in \mathcal{B}_p(H), \ B \in \mathcal{B}_q(H). \tag{40}$$

For the theory of trace functionals and their applications the interested reader is referred to [16] and [17].

For some classical trace inequalities see [8], [9] and [13], which are continuations of the work of Bellman [6].

5 Main Results

We start with 1-Schatten norm weighted mixed sequential generalized fractional Ostrowski type inequalities for several functions taking values in the Banach algebra $\mathcal{B}_2(H) \subset \mathcal{B}(H)$:

Theorem 18 *Let the $*$-ideal $\mathcal{B}_2(H)$, which $(\mathcal{B}_2(H), \|\cdot\|_2)$ is a Banach algebra; $x_0 \in [a,b] \subset \mathbb{R}$, $0 < \alpha < 1$; $A_i \in C^1([a,b], \mathcal{B}_2(H))$, $i = 1,...,r \in \mathbb{N} - \{1\}$; $g \in C^1([a,b])$, strictly increasing, such that $g^{-1} \in C^1([g(a), g(b)])$. Assume that $F_{ki}^{x_0} := D_{x_0-;g}^{k\alpha} A_i$, for $k = 1,...,n \in \mathbb{N}$, fulfill $F_{ki}^{x_0} \in C^1([a,x_0], \mathcal{B}_2(H))$ and $F_{(n+1)i}^{x_0} \in C([a,x_0], \mathcal{B}_2(H))$, and $\left(D_{x_0-;g}^{j\alpha} A_i\right)(x_0) = 0$, $j = 2,...,n$; $i = 1,...,r$. Similarly, we assume that $G_{ki}^{x_0} := D_{x_0+;g}^{k\alpha} A_i$, $k = 1,...,n$, fulfill $G_{ki}^{x_0} \in C^1([x_0,b], \mathcal{B}_2(H))$ and $G_{(n+1)i}^{x_0} \in C([x_0,b], \mathcal{B}_2(H))$, and $\left(D_{x_0+;g}^{j\alpha} A_i\right)(x_0) = 0$, $j = 2,...,n$; $i = 1,...,r$.*

Denote by

$$\Omega(A_1,...,A_r)(x_0) :=$$
$$\sum_{i=1}^{r} \left[\int_a^b \left(\prod_{\substack{j=1 \\ j \neq i}}^{r} A_j(x) \right) A_i(x)\, dx - \left(\int_a^b \left(\prod_{\substack{j=1 \\ j \neq i}}^{r} A_j(x) \right) dx \right) A_i(x_0) \right]. \tag{41}$$

Then

$$\|\Omega(A_1,...,A_r)(x_0)\|_1 \leq \frac{1}{\Gamma((n+1)\alpha+1)} \sum_{i=1}^{r} \left[\left[\left\| \left\| \left(D_{x_0-;g}^{(n+1)\alpha} A_i \right) \right\|_2 \right\|_{\infty,[a,x_0]} \right. \right.$$
$$(g(x_0) - g(a))^{(n+1)\alpha} \left(\int_a^{x_0} \left(\prod_{\substack{j=1 \\ j \neq i}}^{r} \|A_j(x)\|_2 \right) dx \right) \Bigg] + \tag{42}$$
$$\left[\left\| \left\| \left(D_{x_0+;g}^{(n+1)\alpha} A_i \right) \right\|_2 \right\|_{\infty,[x_0,b]} (g(b) - g(x_0))^{(n+1)\alpha} \left(\int_{x_0}^{b} \left(\prod_{\substack{j=1 \\ j \neq i}}^{r} \|A_j(x)\|_2 \right) dx \right) \right] \Bigg].$$

Proof. By Theorem 11 we obtain

$$A_i(x)-A_i(x_0)=\frac{1}{\Gamma((n+1)\alpha)}\int_x^{x_0}(g(t)-g(x))^{(n+1)\alpha-1}g'(t)\left(D_{x_0-;g}^{(n+1)\alpha}A_i\right)(t)\,dt, \tag{43}$$

$\forall\ x\in[a,x_0],\ i=1,...,r.$

Also, by Theorem 9, we get

$$A_i(x)-A_i(x_0)=\frac{1}{\Gamma((n+1)\alpha)}\int_{x_0}^{x}(g(x)-g(t))^{(n+1)\alpha-1}g'(t)\left(D_{x_0+;g}^{(n+1)\alpha}A_i\right)(t)\,dt, \tag{44}$$

$\forall\ x\in[x_0,b],\ i=1,...,r.$

Left multiplying (43) and (44) with $\left(\prod_{\substack{j=1\\j\neq i}}^{r}A_j(x)\right)$ we get, respectively,

$$\left(\prod_{\substack{j=1\\j\neq i}}^{r}A_j(x)\right)A_i(x)-\left(\prod_{\substack{j=1\\j\neq i}}^{r}A_j(x)\right)A_i(x_0)= \tag{45}$$

$$\frac{\left(\prod_{\substack{j=1\\j\neq i}}^{r}A_j(x)\right)}{\Gamma((n+1)\alpha)}\int_x^{x_0}(g(t)-g(x))^{(n+1)\alpha-1}g'(t)\left(D_{x_0-;g}^{(n+1)\alpha}A_i\right)(t)\,dt,$$

$\forall\ x\in[a,x_0],\ i=1,...,r,$

and

$$\left(\prod_{\substack{j=1\\j\neq i}}^{r}A_j(x)\right)A_i(x)-\left(\prod_{\substack{j=1\\j\neq i}}^{r}A_j(x)\right)A_i(x_0)=$$

$$\frac{\left(\prod_{\substack{j=1\\j\neq i}}^{r}A_j(x)\right)}{\Gamma((n+1)\alpha)}\int_{x_0}^{x}(g(x)-g(t))^{(n+1)\alpha-1}g'(t)\left(D_{x_0+;g}^{(n+1)\alpha}A_i\right)(t)\,dt, \tag{46}$$

$\forall\ x\in[x_0,b],\ i=1,...,r.$

Adding (45) and (46) as separate groups, we obtain

$$\sum_{i=1}^{r}\left(\prod_{\substack{j=1\\j\neq i}}^{r}A_j(x)\right)A_i(x)-\sum_{i=1}^{r}\left(\prod_{\substack{j=1\\j\neq i}}^{r}A_j(x)\right)A_i(x_0)=$$

$$\frac{\sum_{i=1}^{r}\left(\prod_{\substack{j=1\\j\neq i}}^{r}A_j(x)\right)}{\Gamma((n+1)\alpha)}\int_x^{x_0}(g(t)-g(x))^{(n+1)\alpha-1}g'(t)\left(D_{x_0-;g}^{(n+1)\alpha}A_i\right)(t)\,dt, \tag{47}$$

$\forall\ x \in [a, x_0]$,

and

$$\sum_{i=1}^{r}\left(\prod_{\substack{j=1\\ j\neq i}}^{r} A_j(x)\right) A_i(x) - \sum_{i=1}^{r}\left(\prod_{\substack{j=1\\ j\neq i}}^{r} A_j(x)\right) A_i(x_0) =$$

$$\frac{\sum_{i=1}^{r}\left(\prod_{\substack{j=1\\ j\neq i}}^{r} A_j(x)\right)}{\Gamma((n+1)\alpha)} \int_{x_0}^{x} (g(x) - g(t))^{(n+1)\alpha-1} g'(t) \left(D_{x_0+;g}^{(n+1)\alpha} A_i\right)(t)\, dt, \tag{48}$$

$\forall\ x \in [x_0, b]$.

Next, we integrate (47) and (48) with respect to $x \in [a, b]$. We have

$$\sum_{i=1}^{r}\int_{a}^{x_0}\left(\prod_{\substack{j=1\\ j\neq i}}^{r} A_j(x)\right) A_i(x)\, dx - \sum_{i=1}^{r}\left(\int_{a}^{x_0}\left(\prod_{\substack{j=1\\ j\neq i}}^{r} A_j(x)\right) dx\right) A_i(x_0) =$$

$$\frac{1}{\Gamma((n+1)\alpha)} \sum_{i=1}^{r}\left[\int_{a}^{x_0}\left(\prod_{\substack{j=1\\ j\neq i}}^{r} A_j(x)\right)\right.$$

$$\left.\left(\int_{x}^{x_0} (g(t) - g(x))^{(n+1)\alpha-1} g'(t) \left(D_{x_0-;g}^{(n+1)\alpha} A_i\right)(t)\, dt\right) dx\right], \tag{49}$$

and

$$\sum_{i=1}^{r}\int_{x_0}^{b}\left(\prod_{\substack{j=1\\ j\neq i}}^{r} A_j(x)\right) A_i(x)\, dx - \sum_{i=1}^{r}\left(\int_{x_0}^{b}\left(\prod_{\substack{j=1\\ j\neq i}}^{r} A_j(x)\right) dx\right) A_i(x_0) =$$

$$\frac{1}{\Gamma((n+1)\alpha)} \sum_{i=1}^{r}\left[\int_{x_0}^{b}\left(\prod_{\substack{j=1\\ j\neq i}}^{r} A_j(x)\right)\right.$$

$$\left.\left(\int_{x_0}^{x} (g(x) - g(t))^{(n+1)\alpha-1} g'(t) \left(D_{x_0+;g}^{(n+1)\alpha} A_i\right)(t)\, dt\right) dx\right]. \tag{50}$$

Finally, adding (49) and (50) we obtain the useful identity

$$\Omega(A_1, ..., A_r)(x_0) :=$$

$$\sum_{i=1}^{r}\left[\int_{a}^{b}\left(\prod_{\substack{j=1\\ j\neq i}}^{r} A_j(x)\right) A_i(x)\, dx - \left(\int_{a}^{b}\left(\prod_{\substack{j=1\\ j\neq i}}^{r} A_j(x)\right) dx\right) A_i(x_0)\right] =$$

$$\frac{1}{\Gamma\left(\left(n+1\right)\alpha\right)}\sum_{i=1}^{r}\left[\left[\int_{a}^{x_{0}}\left(\prod_{\substack{j=1\\ j\neq i}}^{r}A_{j}\left(x\right)\right)\right.\right.$$

$$\left.\left(\int_{x}^{x_{0}}\left(g\left(t\right)-g\left(x\right)\right)^{\left(n+1\right)\alpha-1}g'\left(t\right)\left(D_{x_{0}-;g}^{\left(n+1\right)\alpha}A_{i}\right)\left(t\right)dt\right)dx\right]$$

$$\left.\left.+\left[\int_{x_{0}}^{b}\left(\prod_{\substack{j=1\\ j\neq i}}^{r}A_{j}\left(x\right)\right)\left(\int_{x_{0}}^{x}\left(g\left(x\right)-g\left(t\right)\right)^{\left(n+1\right)\alpha-1}g'\left(t\right)\left(D_{x_{0}+;g}^{\left(n+1\right)\alpha}A_{i}\right)\left(t\right)dt\right)dx\right]\right]. \tag{51}$$

Therefore, we get (by using the p-Schatten norm and Hölder's type inequality (40) for $p=q=2$)

$$\left\|\Omega\left(A_{1},...,A_{r}\right)\left(x_{0}\right)\right\|_{1}\leq\frac{1}{\Gamma\left(\left(n+1\right)\alpha\right)}\sum_{i=1}^{r}\left[\left\|\left[\int_{a}^{x_{0}}\left(\prod_{\substack{j=1\\ j\neq i}}^{r}A_{j}\left(x\right)\right)\right.\right.\right.$$

$$\left.\left.\left(\int_{x}^{x_{0}}\left(g\left(t\right)-g\left(x\right)\right)^{\left(n+1\right)\alpha-1}g'\left(t\right)\left(D_{x_{0}-;g}^{\left(n+1\right)\alpha}A_{i}\right)\left(t\right)dt\right)dx\right]\right\|_{1}+$$

$$\left.\left\|\left[\int_{x_{0}}^{b}\left(\prod_{\substack{j=1\\ j\neq i}}^{r}A_{j}\left(x\right)\right)\left(\int_{x_{0}}^{x}\left(g\left(x\right)-g\left(t\right)\right)^{\left(n+1\right)\alpha-1}g'\left(t\right)\left(D_{x_{0}+;g}^{\left(n+1\right)\alpha}A_{i}\right)\left(t\right)dt\right)dx\right]\right\|_{1}\right] \tag{52}$$

$$\leq\frac{1}{\Gamma\left(\left(n+1\right)\alpha\right)}\sum_{i=1}^{r}\left[\left[\int_{a}^{x_{0}}\left\|\left(\prod_{\substack{j=1\\ j\neq i}}^{r}A_{j}\left(x\right)\right)\right.\right.\right.$$

$$\left.\left.\left(\int_{x}^{x_{0}}\left(g\left(t\right)-g\left(x\right)\right)^{\left(n+1\right)\alpha-1}g'\left(t\right)\left(D_{x_{0}-;g}^{\left(n+1\right)\alpha}A_{i}\right)\left(t\right)dt\right)\right\|_{1}dx\right]+$$

$$\left.\left[\int_{x_{0}}^{b}\left\|\left(\prod_{\substack{j=1\\ j\neq i}}^{r}A_{j}\left(x\right)\right)\left(\int_{x_{0}}^{x}\left(g\left(x\right)-g\left(t\right)\right)^{\left(n+1\right)\alpha-1}g'\left(t\right)\left(D_{x_{0}+;g}^{\left(n+1\right)\alpha}A_{i}\right)\left(t\right)dt\right)\right\|_{1}dx\right]\right]$$

(by (37), (40))

$$\leq\frac{1}{\Gamma\left(\left(n+1\right)\alpha\right)}\sum_{i=1}^{r}\left[\left[\int_{a}^{x_{0}}\left(\prod_{\substack{j=1\\ j\neq i}}^{r}\left\|A_{j}\left(x\right)\right\|_{2}\right)\right.\right.$$

$$\left(\int_x^{x_0} (g(t)-g(x))^{(n+1)\alpha-1} g'(t) \left\| \left(D_{x_0-;g}^{(n+1)\alpha} A_i\right)(t)\right\|_2 dt\right) dx\Bigg] +$$

$$\left[\int_{x_0}^b \left(\prod_{\substack{j=1\\ j\neq i}}^r \|A_j(x)\|_2\right)\left(\int_{x_0}^x (g(x)-g(t))^{(n+1)\alpha-1} g'(t) \left\| \left(D_{x_0+;g}^{(n+1)\alpha} A_i\right)(t)\right\|_2 dt\right) dx\right]\Bigg] \tag{53}$$

$$=: (*).$$

Hence it holds

$$\|\Omega(A_1,...,A_r)(x_0)\|_1 \leq (*). \tag{54}$$

We have that

$$(*) \leq \frac{1}{\Gamma((n+1)\alpha+1)}$$

$$\sum_{i=1}^r \Bigg[\Bigg[\left\| \left\| \left(D_{x_0-;g}^{(n+1)\alpha} A_i\right)\right\|_2\right\|_{\infty,[a,x_0]} \int_a^{x_0} \left(\prod_{\substack{j=1\\ j\neq i}}^r \|A_j(x)\|_2\right)(g(x_0)-g(x))^{(n+1)\alpha} dx\Bigg]$$

$$+\left[\left\| \left\| \left(D_{x_0+;g}^{(n+1)\alpha} A_i\right)\right\|_2\right\|_{\infty,[x_0,b]} \int_{x_0}^b \left(\prod_{\substack{j=1\\ j\neq i}}^r \|A_j(x)\|_2\right)(g(x)-g(x_0))^{(n+1)\alpha} dx\right]\Bigg] \leq$$

$$\frac{1}{\Gamma((n+1)\alpha+1)} \sum_{i=1}^r \Bigg[\Bigg[\left\| \left\| \left(D_{x_0-;g}^{(n+1)\alpha} A_i\right)\right\|_2\right\|_{\infty,[a,x_0]}$$

$$(g(x_0)-g(a))^{(n+1)\alpha} \left(\int_a^{x_0} \left(\prod_{\substack{j=1\\ j\neq i}}^r \|A_j(x)\|_2\right) dx\right)\Bigg] + \tag{55}$$

$$\left[\left\| \left\| \left(D_{x_0+;g}^{(n+1)\alpha} A_i\right)\right\|_2\right\|_{\infty,[x_0,b]} (g(b)-g(x_0))^{(n+1)\alpha} \left(\int_{x_0}^b \left(\prod_{\substack{j=1\\ j\neq i}}^r \|A_j(x)\|_2\right) dx\right)\right]\Bigg],$$

proving (42). ∎

Next comes an L_1 estimate.

Theorem 19 *All as in Theorem 18, plus* $\frac{1}{(n+1)} \leq \alpha < 1$, $n \in \mathbb{N}$. *Then*

$$\|\Omega(A_1,...,A_r)(x_0)\|_1 \leq \frac{\|g'\|_{\infty,[a,b]}}{\Gamma((n+1)\alpha)}$$

$$\sum_{i=1}^{r}\left[\left[\left\|\left\|\left(D_{x_0-;g}^{(n+1)\alpha}A_i\right)\right\|_2\right\|_{L_1([a,x_0])}(g(x_0)-g(a))^{(n+1)\alpha-1}\left(\int_a^{x_0}\left(\prod_{\substack{j=1\\j\neq i}}^{r}\|A_j(x)\|_2\right)dx\right)\right]\right.$$

$$\left.+\left[\left\|\left\|\left(D_{x_0+;g}^{(n+1)\alpha}A_i\right)\right\|_2\right\|_{L_1([x_0,b])}(g(b)-g(x_0))^{(n+1)\alpha-1}\int_{x_0}^{b}\left(\prod_{\substack{j=1\\j\neq i}}^{r}\|A_j(x)\|_2\right)dx\right]\right]. \tag{56}$$

Proof. We have that (by (53), (54))

$$(*)\leq\frac{\|g'\|_{\infty,[a,b]}}{\Gamma((n+1)\alpha)}$$

$$\sum_{i=1}^{r}\left[\left[\left\|\left\|\left(D_{x_0-;g}^{(n+1)\alpha}A_i\right)\right\|_2\right\|_{L_1([a,x_0])}\int_a^{x_0}\left(\prod_{\substack{j=1\\j\neq i}}^{r}\|A_j(x)\|_2\right)(g(x_0)-g(x))^{(n+1)\alpha-1}dx\right]\right.$$

$$\left.+\left[\left\|\left\|\left(D_{x_0+;g}^{(n+1)\alpha}A_i\right)\right\|_2\right\|_{L_1([x_0,b])}\int_{x_0}^{b}\left(\prod_{\substack{j=1\\j\neq i}}^{r}\|A_j(x)\|_2\right)(g(x)-g(x_0))^{(n+1)\alpha-1}dx\right]\right]$$

$$\leq\frac{\|g'\|_{\infty,[a,b]}}{\Gamma((n+1)\alpha)}$$

$$\sum_{i=1}^{r}\left[\left[\left\|\left\|\left(D_{x_0-;g}^{(n+1)\alpha}A_i\right)\right\|_2\right\|_{L_1([a,x_0])}(g(x_0)-g(a))^{(n+1)\alpha-1}\left(\int_a^{x_0}\left(\prod_{\substack{j=1\\j\neq i}}^{r}\|A_j(x)\|_2\right)dx\right)\right]\right. \tag{57}$$

$$\left.+\left[\left\|\left\|\left(D_{x_0+;g}^{(n+1)\alpha}A_i\right)\right\|_2\right\|_{L_1([x_0,b])}(g(b)-g(x_0))^{(n+1)\alpha-1}\left(\int_{x_0}^{b}\left(\prod_{\substack{j=1\\j\neq i}}^{r}\|A_j(x)\|_2\right)dx\right)\right]\right],$$

proving the claim. ■

An L_γ estimate follows.

Theorem 20 *All as in Theorem 18, plus* $\gamma,\delta>1:\frac{1}{\gamma}+\frac{1}{\delta}=1$, *and* $\frac{1}{\delta(n+1)}<\alpha<1$. *Then*

$$\|\Omega(A_1,...,A_r)(x_0)\|_1\leq\frac{1}{\Gamma((n+1)\alpha)(\gamma((n+1)\alpha-1)+1)^{\frac{1}{\gamma}}}$$

$$\sum_{i=1}^{r}\left[\left[\left\|\left\|\left(D_{x_0-;g}^{(n+1)\alpha}A_i\right)\circ g^{-1}\right\|_2\right\|_{\delta,[g(a),g(x_0)]}\right.\right.$$

$$\left.(g(x_0)-g(a))^{(n+1)\alpha-\frac{1}{\delta}}\left(\int_a^{x_0}\left(\prod_{\substack{j=1\\j\neq i}}^{r}\|A_j(x)\|_2\right)dx\right)\right]+$$

$$\left[\left\|\left\|\left(D_{x_0+;g}^{(n+1)\alpha}A_i\right)\circ g^{-1}\right\|_2\right\|_{\delta,[g(x_0),g(b)]}(g(b)-g(x_0))^{(n+1)\alpha-\frac{1}{\delta}}\right.$$

$$\left.\left.\left(\int_{x_0}^{b}\left(\prod_{\substack{j=1\\j\neq i}}^{r}\|A_j(x)\|_2\right)dx\right)\right]\right]. \tag{58}$$

Proof. By (53) and (54) we obtain

$$(*)=\frac{1}{\Gamma((n+1)\alpha)}\sum_{i=1}^{r}\left[\left[\int_a^{x_0}\left(\prod_{\substack{j=1\\j\neq i}}^{r}\|A_j(x)\|_2\right)\right.\right.$$

$$\left.\left(\int_x^{x_0}(g(t)-g(x))^{(n+1)\alpha-1}\left\|\left(\left(D_{x_0-;g}^{(n+1)\alpha}A_i\right)\circ g^{-1}\right)(g(t))\right\|_2dg(t)\right)dx\right]+$$

$$\left[\int_{x_0}^{b}\left(\prod_{\substack{j=1\\j\neq i}}^{r}\|A_j(x)\|_2\right)\right.$$

$$\left.\left.\left(\int_{x_0}^{x}(g(x)-g(t))^{(n+1)\alpha-1}\left\|\left(\left(D_{x_0+;g}^{(n+1)\alpha}A_i\right)\circ g^{-1}\right)(g(t))\right\|_2dg(t)\right)dx\right]\right] \tag{59}$$

$$=\frac{1}{\Gamma((n+1)\alpha)}\sum_{i=1}^{r}\left[\left[\int_a^{x_0}\left(\prod_{\substack{j=1\\j\neq i}}^{r}\|A_j(x)\|_2\right)\right.\right.$$

$$\left.\left(\int_{g(x)}^{g(x_0)}(z-g(x))^{(n+1)\alpha-1}\left\|\left(\left(D_{x_0-;g}^{(n+1)\alpha}A_i\right)\circ g^{-1}\right)(z)\right\|_2dz\right)dx\right]+$$

$$\left.\left.\left[\int_{x_0}^{b}\left(\prod_{\substack{j=1\\j\neq i}}^{r}\|A_j(x)\|_2\right)\left(\int_{g(x_0)}^{g(x)}(g(x)-z)^{(n+1)\alpha-1}\left\|\left(\left(D_{x_0+;g}^{(n+1)\alpha}A_i\right)\circ g^{-1}\right)(z)\right\|_2dz\right)dx\right]\right]$$

$$\leq \frac{1}{\Gamma((n+1)\alpha)} \sum_{i=1}^{r} \left[\left[\int_a^{x_0} \left(\prod_{\substack{j=1 \\ j \neq i}}^{r} \|A_j(x)\|_2 \right) \left(\int_{g(x)}^{g(x_0)} (z - g(x))^{\gamma((n+1)\alpha - 1)} dz \right)^{\frac{1}{\gamma}} \right.\right.$$

$$\left. \left(\int_{g(x)}^{g(x_0)} \left\| \left(\left(D_{x_0-;g}^{(n+1)\alpha} A_i \right) \circ g^{-1} \right)(z) \right\|_2^{\delta} dz \right)^{\frac{1}{\delta}} dx \right] +$$

$$\left[\int_{x_0}^{b} \left(\prod_{\substack{j=1 \\ j \neq i}}^{r} \|A_j(x)\|_2 \right) \left(\int_{g(x_0)}^{g(x)} (g(x) - z)^{\gamma((n+1)\alpha - 1)} dz \right)^{\frac{1}{\gamma}} \right.$$

$$\left.\left. \left(\int_{g(x_0)}^{g(x)} \left\| \left(\left(D_{x_0+;g}^{(n+1)\alpha} A_i \right) \circ g^{-1} \right)(z) \right\|_2^{\delta} dz \right)^{\frac{1}{\delta}} dx \right] \right] \leq \tag{60}$$

$$\frac{1}{\Gamma((n+1)\alpha)(\gamma((n+1)\alpha - 1) + 1)^{\frac{1}{\gamma}}}$$

$$\sum_{i=1}^{r} \left[\left[\left\| \left\| \left(\left(D_{x_0-;g}^{(n+1)\alpha} A_i \right) \circ g^{-1} \right) \right\|_2 \right\|_{\delta,[g(a),g(x_0)]} \right.\right.$$

$$\left. \left(\int_a^{x_0} \left(\prod_{\substack{j=1 \\ j \neq i}}^{r} \|A_j(x)\|_2 \right) (g(x_0) - g(x))^{\frac{\gamma((n+1)\alpha-1)+1}{\gamma}} dx \right) \right] +$$

$$\left[\left\| \left\| \left(\left(D_{x_0+;g}^{(n+1)\alpha} A_i \right) \circ g^{-1} \right) \right\|_2 \right\|_{\delta,[g(x_0),g(b)]} \right.$$

$$\left.\left. \left(\int_{x_0}^{b} \left(\prod_{\substack{j=1 \\ j \neq i}}^{r} \|A_j(x)\|_2 \right) (g(x) - g(x_0))^{\frac{\gamma((n+1)\alpha-1)+1}{\gamma}} dx \right) \right] \right]$$

$$\leq \frac{1}{\Gamma((n+1)\alpha)(\gamma((n+1)\alpha - 1) + 1)^{\frac{1}{\gamma}}}$$

$$\sum_{i=1}^{r} \left[\left\| \left\| \left(D_{x_0-;g}^{(n+1)\alpha} A_i \right) \circ g^{-1} \right\|_2 \right\|_{\delta,[g(a),g(x_0)]} \right.$$

$$(g(x_0) - g(a))^{(n+1)\alpha - \frac{1}{\delta}} \left(\int_a^{x_0} \left(\prod_{\substack{j=1 \\ j \neq i}}^{r} \|A_j(x)\|_2 \right) dx \right) \tag{61}$$

$$+ \left\| \left\| \left(D_{x_0+;g}^{(n+1)\alpha} A_i \right) \circ g^{-1} \right\|_2 \right\|_{\delta,[g(x_0),g(b)]} (g(b) - g(x_0))^{(n+1)\alpha - \frac{1}{\delta}}$$

$$\left(\int_{x_0}^{b}\left(\prod_{\substack{j=1\\ j\neq i}}^{r}\left\|A_j\left(x\right)\right\|_2\right)dx\right)\Bigg],$$

proving the claim. ■

When $n=1$ we get the following results:

Corollary 21 *(to Theorem 18) Let the $*$-ideal $\mathcal{B}_2\left(H\right)$; $x_0\in[a,b]\subset\mathbb{R}$, $0<\alpha<1$; $A_i\in C^1\left(\left[a,b\right],\mathcal{B}_2\left(H\right)\right)$, $i=1,...,r\in\mathbb{N}-\{1\}$; $g\in C^1\left(\left[a,b\right]\right)$, strictly increasing, such that $g^{-1}\in C^1\left(\left[g\left(a\right),g\left(b\right)\right]\right)$. Assume that $D^{\alpha}_{x_0-;g}A_i\in C^1\left(\left[a,x_0\right],\mathcal{B}_2\left(H\right)\right)$ and $D^{2\alpha}_{x_0-;g}A_i\in C\left(\left[a,x_0\right],\mathcal{B}_2\left(H\right)\right)$, $i=1,...,r$. Similarly, assume that $D^{\alpha}_{x_0+;g}A_i\in C^1\left(\left[x_0,b\right],\mathcal{B}_2\left(H\right)\right)$ and $D^{2\alpha}_{x_0+;g}A_i\in C\left(\left[x_0,b\right],\mathcal{B}_2\left(H\right)\right)$; $i=1,...,r$.*

Denote by

$$\Omega\left(A_1,...,A_r\right)\left(x_0\right):=$$

$$\sum_{i=1}^{r}\left[\int_a^b\left(\prod_{\substack{j=1\\ j\neq i}}^{r}A_j\left(x\right)\right)A_i\left(x\right)dx-\left(\int_a^b\left(\prod_{\substack{j=1\\ j\neq i}}^{r}A_j\left(x\right)\right)dx\right)A_i\left(x_0\right)\right]. \quad (62)$$

Then

$$\left\|\Omega\left(A_1,...,A_r\right)\left(x_0\right)\right\|_1\leq\frac{1}{\Gamma\left(2\alpha+1\right)}\sum_{i=1}^{r}\Bigg[\Bigg[\left\|\left\|\left(D^{2\alpha}_{x_0-;g}A_i\right)\right\|_2\right\|_{\infty,[a,x_0]}$$

$$\left(g\left(x_0\right)-g\left(a\right)\right)^{2\alpha}\left(\int_a^{x_0}\left(\prod_{\substack{j=1\\ j\neq i}}^{r}\left\|A_j\left(x\right)\right\|_2\right)dx\right)\Bigg]+ \quad (63)$$

$$\Bigg[\left\|\left\|\left(D^{2\alpha}_{x_0+;g}A_i\right)\right\|_2\right\|_{\infty,[x_0,b]}\left(g\left(b\right)-g\left(x_0\right)\right)^{2\alpha}\left(\int_{x_0}^{b}\left(\prod_{\substack{j=1\\ j\neq i}}^{r}\left\|A_j\left(x\right)\right\|_2\right)dx\right)\Bigg]\Bigg]$$

$$=:\Lambda_1\left(x_0\right).$$

Proof. By Theorem 18 and Corollaries 10, 12. ■

We continue with

Corollary 22 *(to Theorem 19) All as in Corollary 21, plus $\frac{1}{2}\leq\alpha<1$. Then*

$$\left\|\Omega\left(A_1,...,A_r\right)\left(x_0\right)\right\|_1\leq\frac{\left\|g'\right\|_{\infty,[a,b]}}{\Gamma\left(2\alpha\right)}\sum_{i=1}^{r}\Bigg[\Bigg[\left\|\left\|\left(D^{2\alpha}_{x_0-;g}A_i\right)\right\|_2\right\|_{L_1([a,x_0])}$$

$$\left(g\left(x_0\right)-g\left(a\right)\right)^{2\alpha-1}\left(\int_a^{x_0}\left(\prod_{\substack{j=1\\ j\neq i}}^{r}\left\|A_j\left(x\right)\right\|_2\right)dx\right)\Bigg]+$$

$$\left[\left\|\left\|\left(D^{2\alpha}_{x_0+;g}A_i\right)\right\|_2\right\|_{L_1([x_0,b])}(g(b)-g(x_0))^{2\alpha-1}\left(\int_{x_0}^{b}\left(\prod_{\substack{j=1\\j\neq i}}^{r}\|A_j(x)\|_2\right)dx\right)\right]\Bigg] \tag{64}$$

$$=:\Lambda_2(x_0).$$

Proof. By Theorem 19. ■

We also give

Corollary 23 *(to Theorem 20) All as in Corollary 21, plus* $\gamma,\delta>1:\frac{1}{\gamma}+\frac{1}{\delta}=1$, *and* $\frac{1}{2\delta}<\alpha<1$. *Then*

$$\|\Omega(A_1,...,A_r)(x_0)\|_1\leq\frac{1}{\Gamma(2\alpha)(\gamma(2\alpha-1)+1)^{\frac{1}{\gamma}}}$$

$$\sum_{i=1}^{r}\left[\left[\left\|\left\|\left(D^{2\alpha}_{x_0-;g}A_i\right)\circ g^{-1}\right\|_2\right\|_{\delta,[g(a),g(x_0)]}\right.\right.$$

$$\left.(g(x_0)-g(a))^{2\alpha-\frac{1}{\delta}}\left(\int_a^{x_0}\left(\prod_{\substack{j=1\\j\neq i}}^{r}\|A_j(x)\|_2\right)dx\right)\right]+$$

$$\left[\left\|\left\|\left(D^{2\alpha}_{x_0+;g}A_i\right)\circ g^{-1}\right\|_2\right\|_{\delta,[g(x_0),g(b)]}(g(b)-g(x_0))^{2\alpha-\frac{1}{\delta}}\right.$$

$$\left.\left.\left(\int_{x_0}^{b}\left(\prod_{\substack{j=1\\j\neq i}}^{r}\|A_j(x)\|_2\right)dx\right)\right]\right]=\Lambda_3(x_0). \tag{65}$$

Proof. By Theorem 20. ■

Next come p-Schatten norm, $p>1$, fractional Ostrowski type inequalities for $\mathcal{B}_p(H)$ valued functions, $\mathcal{B}_p(H)\subset\mathcal{B}(H)$:

Theorem 24 *Let* $p>1$, *the* $*$-*ideal* $\mathcal{B}_p(H)$, *which* $\left(\mathcal{B}_p(H),\|\cdot\|_p\right)$ *is a Banach algebra;* $x_0\in[a,b]\subset\mathbb{R}$, $0<\alpha<1$; $A_i\in C^1([a,b],\mathcal{B}_p(H))$, $i=1,...,r\in\mathbb{N}-\{1\}$; $g\in C^1([a,b])$, *strictly increasing, such that* $g^{-1}\in C^1([g(a),g(b)])$. *Assume that* $F^{x_0}_{ki}:=D^{k\alpha}_{x_0-;g}A_i$, *for* $k=1,...,n\in\mathbb{N}$, *fulfill* $F^{x_0}_{ki}\in C^1([a,x_0],\mathcal{B}_p(H))$ *and* $F^{x_0}_{(n+1)i}\in C([a,x_0],\mathcal{B}_p(H))$, *and* $\left(D^{j\alpha}_{x_0-;g}A_i\right)(x_0)=0$, $j=2,...,n$; $i=1,...,r$. *Similarly, we assume that* $G^{x_0}_{ki}:=D^{k\alpha}_{x_0+;g}A_i$, $k=1,...,n$, *fulfill* $G^{x_0}_{ki}\in C^1([x_0,b],\mathcal{B}_p(H))$ *and* $G^{x_0}_{(n+1)i}\in C([x_0,b],\mathcal{B}_p(H))$, *and* $\left(D^{j\alpha}_{x_0+;g}A_i\right)(x_0)=0$, $j=2,...,n$; $i=1,...,r$.

Denote by

$$\Omega\left(A_1,...,A_r\right)\left(x_0\right) :=$$

$$\sum_{i=1}^{r}\left[\int_a^b \left(\prod_{\substack{j=1\\ j\neq i}}^{r} A_j\left(x\right)\right) A_i\left(x\right)dx - \left(\int_a^b \left(\prod_{\substack{j=1\\ j\neq i}}^{r} A_j\left(x\right)\right)dx\right) A_i\left(x_0\right)\right]. \quad (66)$$

Then

$$\left\|\Omega\left(A_1,...,A_r\right)\left(x_0\right)\right\|_p \leq \frac{1}{\Gamma\left(\left(n+1\right)\alpha+1\right)}\sum_{i=1}^{r}\left[\left[\left\|\left\|\left(D_{x_0-;g}^{(n+1)\alpha}A_i\right)\right\|_p\right\|_{\infty,[a,x_0]}\right.\right.$$

$$\left.\left(g\left(x_0\right)-g\left(a\right)\right)^{(n+1)\alpha}\left(\int_a^{x_0}\left(\prod_{\substack{j=1\\ j\neq i}}^{r}\left\|A_j\left(x\right)\right\|_p\right)dx\right)\right] + \quad (67)$$

$$\left.\left[\left\|\left\|\left(D_{x_0+;g}^{(n+1)\alpha}A_i\right)\right\|_p\right\|_{\infty,[x_0,b]}\left(g\left(b\right)-g\left(x_0\right)\right)^{(n+1)\alpha}\left(\int_{x_0}^{b}\left(\prod_{\substack{j=1\\ j\neq i}}^{r}\left\|A_j\left(x\right)\right\|_p\right)dx\right)\right]\right].$$

Proof. As similar to Theorem 18 is omitted. Use of (37). ■

Next comes an L_1 estimate.

Theorem 25 *All as in Theorem 24, plus* $\frac{1}{(n+1)} \leq \alpha < 1$, $n \in \mathbb{N}$. *Then*

$$\left\|\Omega\left(A_1,...,A_r\right)\left(x_0\right)\right\|_p \leq \frac{\left\|g'\right\|_{\infty,[a,b]}}{\Gamma\left(\left(n+1\right)\alpha\right)}$$

$$\sum_{i=1}^{r}\left[\left[\left\|\left\|\left(D_{x_0-;g}^{(n+1)\alpha}A_i\right)\right\|_p\right\|_{L_1([a,x_0])}\left(g\left(x_0\right)-g\left(a\right)\right)^{(n+1)\alpha-1}\left(\int_a^{x_0}\left(\prod_{\substack{j=1\\ j\neq i}}^{r}\left\|A_j\left(x\right)\right\|_p\right)dx\right)\right]\right. \quad (68)$$

$$\left.+\left[\left\|\left\|\left(D_{x_0+;g}^{(n+1)\alpha}A_i\right)\right\|_p\right\|_{L_1([x_0,b])}\left(g\left(b\right)-g\left(x_0\right)\right)^{(n+1)\alpha-1}\int_{x_0}^{b}\left(\prod_{\substack{j=1\\ j\neq i}}^{r}\left\|A_j\left(x\right)\right\|_p\right)dx\right]\right].$$

Proof. As similar to the proof of Theorem 19 is omitted. ■

An L_γ estimate follows.

Theorem 26 *All as in Theorem 24, plus* $\gamma,\delta > 1 : \frac{1}{\gamma}+\frac{1}{\delta} = 1$, *and* $\frac{1}{\delta(n+1)} < \alpha < 1$. *Then*

$$\left\|\Omega\left(A_1,...,A_r\right)\left(x_0\right)\right\|_p \leq \frac{1}{\Gamma\left(\left(n+1\right)\alpha\right)\left(\gamma\left(\left(n+1\right)\alpha-1\right)+1\right)^{\frac{1}{\gamma}}}$$

$$
\sum_{i=1}^{r}\left[\left[\left\|\left\|\left(D_{x_0-;g}^{(n+1)\alpha}A_i\right)\circ g^{-1}\right\|_p\right\|_{\delta,[g(a),g(x_0)]}\right.\right.
$$

$$
(g(x_0)-g(a))^{(n+1)\alpha-\frac{1}{\delta}}\left(\int_a^{x_0}\left(\prod_{\substack{j=1\\ j\neq i}}^{r}\|A_j(x)\|_p\right)dx\right)\Bigg]+
$$

$$
\left[\left\|\left\|\left(D_{x_0+;g}^{(n+1)\alpha}A_i\right)\circ g^{-1}\right\|_p\right\|_{\delta,[g(x_0),g(b)]}(g(b)-g(x_0))^{(n+1)\alpha-\frac{1}{\delta}}\right. \tag{69}
$$

$$
\left.\left.\left(\int_{x_0}^{b}\left(\prod_{\substack{j=1\\ j\neq i}}^{r}\|A_j(x)\|_p\right)dx\right)\right]\right].
$$

Proof. As similar to Theorem 20 is omitted. ■

When $n=1$ we get the following results:

Corollary 27 *(to Theorem 24) Let $p>1$, the $*$-ideal $\mathcal{B}_p(H)$; $x_0\in[a,b]\subset\mathbb{R}$, $0<\alpha<1$; $A_i\in C^1([a,b],\mathcal{B}_p(H))$, $i=1,...,r\in\mathbb{N}-\{1\}$; $g\in C^1([a,b])$, strictly increasing, such that $g^{-1}\in C^1([g(a),g(b)])$. Assume that $D_{x_0-;g}^{\alpha}A_i\in C^1([a,x_0],\mathcal{B}_p(H))$ and $D_{x_0-;g}^{2\alpha}A_i\in C([a,x_0],\mathcal{B}_p(H))$, $i=1,...,r$. Similarly, assume that $D_{x_0+;g}^{\alpha}A_i\in C^1([x_0,b],\mathcal{B}_p(H))$ and $D_{x_0+;g}^{2\alpha}A_i\in C([x_0,b],\mathcal{B}_p(H))$; $i=1,...,r$; $\Omega(A_1,...,A_r)(x_0)$ is as in (66).*

Then

$$
\|\Omega(A_1,...,A_r)(x_0)\|_p\leq\frac{1}{\Gamma(2\alpha+1)}\sum_{i=1}^{r}\left[\left[\left\|\left\|\left(D_{x_0-;g}^{2\alpha}A_i\right)\right\|_p\right\|_{\infty,[a,x_0]}\right.\right.
$$

$$
(g(x_0)-g(a))^{2\alpha}\left(\int_a^{x_0}\left(\prod_{\substack{j=1\\ j\neq i}}^{r}\|A_j(x)\|_p\right)dx\right)\Bigg]+ \tag{70}
$$

$$
\left.\left.\left[\left\|\left\|\left(D_{x_0+;g}^{2\alpha}A_i\right)\right\|_p\right\|_{\infty,[x_0,b]}(g(b)-g(x_0))^{2\alpha}\left(\int_{x_0}^{b}\left(\prod_{\substack{j=1\\ j\neq i}}^{r}\|A_j(x)\|_p\right)dx\right)\right]\right]
$$

$$
=:\Lambda_4(x_0).
$$

Proof. By Theorem 24 and Corollaries 10, 12. ■

We continue with

Corollary 28 *(to Theorem 25) All as in Corollary 27, plus* $\frac{1}{2} \leq \alpha < 1$*. Then*

$$\left\| \Omega \left(A_1, ..., A_r \right) (x_0) \right\|_p \leq \frac{\|g'\|_{\infty,[a,b]}}{\Gamma(2\alpha)} \sum_{i=1}^{r} \left[\left[\left\| \left\| \left(D_{x_0-;g}^{2\alpha} A_i \right) \right\|_p \right\|_{L_1([a,x_0])} \right. \right.$$

$$\left. (g(x_0) - g(a))^{2\alpha-1} \left(\int_a^{x_0} \left(\prod_{\substack{j=1 \\ j \neq i}}^{r} \|A_j(x)\|_p \right) dx \right) \right] + \tag{71}$$

$$\left[\left\| \left\| \left(D_{x_0+;g}^{2\alpha} A_i \right) \right\|_p \right\|_{L_1([x_0,b])} (g(b) - g(x_0))^{2\alpha-1} \left(\int_{x_0}^{b} \left(\prod_{\substack{j=1 \\ j \neq i}}^{r} \|A_j(x)\|_p \right) dx \right) \right] \Bigg]$$

$$= \Lambda_5 (x_0).$$

Proof. By Theorem 25. ∎

We also give

Corollary 29 *(to Theorem 26) All as in Corollary 27, plus* $\gamma, \delta > 1 : \frac{1}{\gamma} + \frac{1}{\delta} = 1$*, and* $\frac{1}{2\delta} < \alpha < 1$*. Then*

$$\left\| \Omega \left(A_1, ..., A_r \right) (x_0) \right\|_p \leq \frac{1}{\Gamma(2\alpha) \left(\gamma (2\alpha - 1) + 1 \right)^{\frac{1}{\gamma}}}$$

$$\sum_{i=1}^{r} \left[\left[\left\| \left\| \left(D_{x_0-;g}^{2\alpha} A_i \right) \circ g^{-1} \right\|_p \right\|_{\delta,[g(a),g(x_0)]} \right. \right.$$

$$\left. (g(x_0) - g(a))^{2\alpha - \frac{1}{\delta}} \left(\int_a^{x_0} \left(\prod_{\substack{j=1 \\ j \neq i}}^{r} \|A_j(x)\|_p \right) dx \right) \right] + \tag{72}$$

$$\left[\left\| \left\| \left(D_{x_0+;g}^{2\alpha} A_i \right) \circ g^{-1} \right\|_p \right\|_{\delta,[g(x_0),g(b)]} (g(b) - g(x_0))^{2\alpha - \frac{1}{\delta}} \right.$$

$$\left. \left(\int_{x_0}^{b} \left(\prod_{\substack{j=1 \\ j \neq i}}^{r} \|A_j(x)\|_p \right) dx \right) \right] \Bigg] = \Lambda_6 (x_0).$$

Proof. By Theorem 26. ∎

When $r = 2$ we obtain the following operator related sequential fractional Ostrowski type inequalities.

Theorem 30 *Let* $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$, *and let the* $*$*-ideals* $\mathcal{B}_p(H)$, $\mathcal{B}_q(H)$, *for which* $\left(\mathcal{B}_p(H), \|\cdot\|_p\right)$, $\left(\mathcal{B}_q(H), \|\cdot\|_q\right)$ *are Banach algebras;* $x_0 \in [a,b] \subset \mathbb{R}$, $0 < \alpha < 1$, $A_1 \in C^1([a,b], \mathcal{B}_p(H))$, $A_2 \in C^1([a,b], \mathcal{B}_q(H))$; $g \in C^1([a,b])$, *strictly increasing, such that* $g^{-1} \in C^1([g(a), g(b)])$. *Assume that* $F_{ki}^{x_0} := D_{x_0-;g}^{k\alpha} A_i$, *for* $k = 1, ..., n \in \mathbb{N}$, $i = 1, 2$, *fulfill* $F_{k1}^{x_0} \in C^1([a, x_0], \mathcal{B}_p(H))$, $F_{(n+1)1}^{x_0} \in C([a, x_0], \mathcal{B}_p(H))$; $F_{k2}^{x_0} \in C^1([a, x_0], \mathcal{B}_q(H))$, $F_{(n+1)2}^{x_0} \in C([a, x_0], \mathcal{B}_q(H))$, *and* $\left(D_{x_0-;g}^{j\alpha} A_i\right)(x_0) = 0$, $j = 2, ..., n$; $i = 1, 2$. *Similarly, we assume that* $G_{ki}^{x_0} := D_{x_0+;g}^{k\alpha} A_i$, $k = 1, ..., n$, $i = 1, 2$, *fulfill* $G_{k1}^{x_0} \in C^1([x_0, b], \mathcal{B}_p(H))$, $G_{(n+1)1}^{x_0} \in C([x_0, b], \mathcal{B}_p(H))$, $G_{k2}^{x_0} \in C^1([x_0, b], \mathcal{B}_q(H))$, $G_{(n+1)2}^{x_0} \in C([x_0, b], \mathcal{B}_q(H))$, *and* $\left(D_{x_0+;g}^{j\alpha} A_i\right)(x_0) = 0$, $j = 2, ..., n$; $i = 1, 2$.

Then

1) it holds

$$\Omega(A_1, A_2)(x_0) := \int_a^b A_2(x) A_1(x) + \int_a^b A_1(x) A_2(x)\, dx -$$

$$\left(\int_a^b A_2(x)\, dx\right) A_1(x_0) - \left(\int_a^b A_1(x)\, dx\right) A_2(x_0) =$$

$$\frac{1}{\Gamma((n+1)\alpha)} \left\{ \left[\int_a^{x_0} A_2(x) \left(\int_{g(x)}^{g(x_0)} (z - g(x))^{(n+1)\alpha - 1} \left(\left(D_{x_0-;g}^{(n+1)\alpha} A_1 \right) \circ g^{-1} \right)(z)\, dz \right) dx \right] + \right.$$

$$\left[\int_{x_0}^b A_2(x) \left(\int_{g(x_0)}^{g(x)} (g(x) - z)^{(n+1)\alpha - 1} \left(\left(D_{x_0+;g}^{(n+1)\alpha} A_1 \right) \circ g^{-1} \right)(z)\, dz \right) dx \right] + \tag{73}$$

$$\left[\int_a^{x_0} A_1(x) \left(\int_{g(x)}^{g(x_0)} (z - g(x))^{(n+1)\alpha - 1} \left(\left(D_{x_0-;g}^{(n+1)\alpha} A_2 \right) \circ g^{-1} \right)(z)\, dz \right) dx \right] +$$

$$\left. \left[\int_{x_0}^b A_1(x) \left(\int_{g(x_0)}^{g(x)} (g(x) - z)^{(n+1)\alpha - 1} \left(\left(D_{x_0+;g}^{(n+1)\alpha} A_2 \right) \circ g^{-1} \right)(z)\, dz \right) dx \right] \right\},$$

2) for $\gamma, \delta > 1 : \frac{1}{\gamma} + \frac{1}{\delta} = 1$, *with* $\frac{1}{\delta(n+1)} < \alpha < 1$, *we have that*

$$\|\Omega(A_1, A_2)(x_0)\|_1 \leq \frac{1}{\Gamma((n+1)\alpha) (\gamma((n+1)\alpha - 1) + 1)^{\frac{1}{\gamma}}}$$

$$\left\{ \left[\left\| \left\| \left(D_{x_0-;g}^{(n+1)\alpha} A_1 \right) \circ g^{-1} \right\|_p \right\|_{\delta, [g(a), g(x_0)]} \int_a^{x_0} \|A_2(x)\|_q (g(x_0) - g(x))^{(n+1)\alpha - \frac{1}{\delta}}\, dx \right] + \right.$$

$$\left[\left\| \left\| \left(D_{x_0+;g}^{(n+1)\alpha} A_1 \right) \circ g^{-1} \right\|_p \right\|_{\delta, [g(x_0), g(b)]} \int_{x_0}^b \|A_2(x)\|_q (g(x) - g(x_0))^{(n+1)\alpha - \frac{1}{\delta}}\, dx \right] + \tag{74}$$

$$\left[\left\|\left\|\left(D_{x_0-;g}^{(n+1)\alpha}A_2\right)\circ g^{-1}\right\|_q\right\|_{\delta,[g(a),g(x_0)]}\int_a^{x_0}\|A_1(x)\|_p(g(x_0)-g(x))^{(n+1)\alpha-\frac{1}{\delta}}dx\right]+$$

$$\left[\left\|\left\|\left(D_{x_0+;g}^{(n+1)\alpha}A_2\right)\circ g^{-1}\right\|_q\right\|_{\delta,[g(x_0),g(b)]}\int_{x_0}^{b}\|A_1(x)\|_p(g(x)-g(x_0))^{(n+1)\alpha-\frac{1}{\delta}}dx\right]\Bigg\},$$

3) if $(n+1)\alpha\geq 1$, we obtain

$$\|\Omega(A_1,A_2)(x_0)\|_1\leq\frac{1}{\Gamma((n+1)\alpha)}$$

$$\Bigg\{\left[\left\|\left\|\left(D_{x_0-;g}^{(n+1)\alpha}A_1\right)\circ g^{-1}\right\|_p\right\|_{L_1([g(a),g(x_0)])}\int_a^{x_0}\|A_2(x)\|_q(g(x_0)-g(x))^{(n+1)\alpha-1}dx\right]+$$

$$\left[\left\|\left\|\left(D_{x_0+;g}^{(n+1)\alpha}A_1\right)\circ g^{-1}\right\|_p\right\|_{L_1([g(x_0),g(b)])}\int_{x_0}^{b}\|A_2(x)\|_q(g(x)-g(x_0))^{(n+1)\alpha-1}dx\right]+$$

$$\left[\left\|\left\|\left(D_{x_0-;g}^{(n+1)\alpha}A_2\right)\circ g^{-1}\right\|_q\right\|_{L_1([g(a),g(x_0)])}\int_a^{x_0}\|A_1(x)\|_p(g(x_0)-g(x))^{(n+1)\alpha-1}dx\right]+ \tag{75}$$

$$\left[\left\|\left\|\left(D_{x_0+;g}^{(n+1)\alpha}A_2\right)\circ g^{-1}\right\|_q\right\|_{L_1([g(x_0),g(b)])}\int_{x_0}^{b}\|A_1(x)\|_p(g(x)-g(x_0))^{(n+1)\alpha-1}dx\right]\Bigg\},$$

and

4)

$$\|\Omega(A_1,A_2)(x_0)\|_1\leq\frac{1}{\Gamma((n+1)\alpha+1)}$$

$$\Bigg\{\left[\left\|\left\|\left(D_{x_0-;g}^{(n+1)\alpha}A_1\right)\circ g^{-1}\right\|_p\right\|_{\infty,[g(a),g(x_0)]}\int_a^{x_0}\|A_2(x)\|_q(g(x_0)-g(x))^{(n+1)\alpha}dx\right]+$$

$$\left[\left\|\left\|\left(D_{x_0+;g}^{(n+1)\alpha}A_1\right)\circ g^{-1}\right\|_p\right\|_{\infty,[g(x_0),g(b)]}\int_{x_0}^{b}\|A_2(x)\|_q(g(x)-g(x_0))^{(n+1)\alpha}dx\right]+ \tag{76}$$

$$\left[\left\|\left\|\left(D_{x_0-;g}^{(n+1)\alpha}A_2\right)\circ g^{-1}\right\|_q\right\|_{\infty,[g(a),g(x_0)]}\int_a^{x_0}\|A_1(x)\|_p(g(x_0)-g(x))^{(n+1)\alpha}dx\right]+$$

$$\left[\left\|\left\|\left(D_{x_0+;g}^{(n+1)\alpha}A_2\right)\circ g^{-1}\right\|_q\right\|_{\infty,[g(x_0),g(b)]}\int_{x_0}^{b}\|A_1(x)\|_p(g(x)-g(x_0))^{(n+1)\alpha}dx\right]\Bigg\}.$$

Proof. Here we have that (acting as in the proof of Theorem 18 for $r=2$)

$$\Omega(A_1,A_2)(x_0):=\int_a^b A_2(x)A_1(x)+\int_a^b A_1(x)A_2(x)\,dx-$$

$$\left(\int_a^b A_2(x)\,dx\right) A_1(x_0) - \left(\int_a^b A_1(x)\,dx\right) A_2(x_0) \overset{(51)}{=}$$

$$\frac{1}{\Gamma((n+1)\alpha)} \left\{ \left[\int_a^{x_0} A_2(x) \left(\int_{g(x)}^{g(x_0)} (z-g(x))^{(n+1)\alpha-1} \left(\left(D_{x_0-;g}^{(n+1)\alpha} A_1 \right) \circ g^{-1} \right)(z)\,dz \right) dx \right] + \right.$$

$$\left[\int_{x_0}^{b} A_2(x) \left(\int_{g(x_0)}^{g(x)} (g(x)-z)^{(n+1)\alpha-1} \left(\left(D_{x_0+;g}^{(n+1)\alpha} A_1 \right) \circ g^{-1} \right)(z)\,dz \right) dx \right] +$$

$$\left[\int_a^{x_0} A_1(x) \left(\int_{g(x)}^{g(x_0)} (z-g(x))^{(n+1)\alpha-1} \left(\left(D_{x_0-;g}^{(n+1)\alpha} A_2 \right) \circ g^{-1} \right)(z)\,dz \right) dx \right] + \tag{77}$$

$$\left. \left[\int_{x_0}^{b} A_1(x) \left(\int_{g(x_0)}^{g(x)} (g(x)-z)^{(n+1)\alpha-1} \left(\left(D_{x_0+;g}^{(n+1)\alpha} A_2 \right) \circ g^{-1} \right)(z)\,dz \right) dx \right] \right\}.$$

Therefore it holds by taking the 1-Schatten norm that

$$\|\Omega(A_1,A_2)(x_0)\|_1 = \left\| \int_a^b A_2(x)\,A_1(x) + \int_a^b A_1(x)\,A_2(x)\,dx - \right.$$

$$\left. \left(\int_a^b A_2(x)\,dx\right) A_1(x_0) - \left(\int_a^b A_1(x)\,dx\right) A_2(x_0) \right\|_1 \leq$$

$$\frac{1}{\Gamma((n+1)\alpha)} \left\{ \left[\left\| \int_a^{x_0} A_2(x) \left(\int_{g(x)}^{g(x_0)} (z-g(x))^{(n+1)\alpha-1} \left(\left(D_{x_0-;g}^{(n+1)\alpha} A_1 \right) \circ g^{-1} \right)(z)\,dz \right) dx \right\|_1 \right] + \right.$$

$$\left[\left\| \int_{x_0}^{b} A_2(x) \left(\int_{g(x_0)}^{g(x)} (g(x)-z)^{(n+1)\alpha-1} \left(\left(D_{x_0+;g}^{(n+1)\alpha} A_1 \right) \circ g^{-1} \right)(z)\,dz \right) dx \right\|_1 \right] +$$

$$\left[\left\| \int_a^{x_0} A_1(x) \left(\int_{g(x)}^{g(x_0)} (z-g(x))^{(n+1)\alpha-1} \left(\left(D_{x_0-;g}^{(n+1)\alpha} A_2 \right) \circ g^{-1} \right)(z)\,dz \right) dx \right\|_1 \right] +$$

$$\left. \left[\left\| \int_{x_0}^{b} A_1(x) \left(\int_{g(x_0)}^{g(x)} (g(x)-z)^{(n+1)\alpha-1} \left(\left(D_{x_0+;g}^{(n+1)\alpha} A_2 \right) \circ g^{-1} \right)(z)\,dz \right) dx \right\|_1 \right] \right\} \leq$$

$$\frac{1}{\Gamma((n+1)\alpha)} \left\{ \left[\int_a^{x_0} \left\| A_2(x) \left(\int_{g(x)}^{g(x_0)} (z-g(x))^{(n+1)\alpha-1} \left(\left(D_{x_0-;g}^{(n+1)\alpha} A_1 \right) \circ g^{-1} \right)(z)\,dz \right) \right\|_1 dx \right] + \right.$$

$$\left[\int_{x_0}^{b} \left\| A_2(x) \left(\int_{g(x_0)}^{g(x)} (g(x)-z)^{(n+1)\alpha-1} \left(\left(D_{x_0+;g}^{(n+1)\alpha} A_1 \right) \circ g^{-1} \right)(z)\,dz \right) \right\|_1 dx \right] + \tag{78}$$

$$\left[\int_a^{x_0} \left\| A_1(x) \left(\int_{g(x)}^{g(x_0)} (z-g(x))^{(n+1)\alpha-1} \left(\left(D_{x_0-;g}^{(n+1)\alpha} A_2 \right) \circ g^{-1} \right)(z)\,dz \right) \right\|_1 dx \right] +$$

$$\left[\int_{x_0}^{b}\left\|A_1(x)\left(\int_{g(x_0)}^{g(x)}(g(x)-z)^{(n+1)\alpha-1}\left(\left(D_{x_0+;g}^{(n+1)\alpha}A_2\right)\circ g^{-1}\right)(z)\,dz\right)\right\|_1 dx\right]\right\}\leq \tag{79}$$

(by using the p-Schatten norm and Hölder's type inequality (40) for $p,q>1:$ $\frac{1}{p}+\frac{1}{q}=1$)

$$\frac{1}{\Gamma((n+1)\alpha)}\left\{\left[\int_{a}^{x_0}\|A_2(x)\|_q\left\|\left(\int_{g(x)}^{g(x_0)}(z-g(x))^{(n+1)\alpha-1}\left(\left(D_{x_0-;g}^{(n+1)\alpha}A_1\right)\circ g^{-1}\right)(z)\,dz\right)\right\|_p dx\right]+\right.$$
$$\left[\int_{x_0}^{b}\|A_2(x)\|_q\left\|\left(\int_{g(x_0)}^{g(x)}(g(x)-z)^{(n+1)\alpha-1}\left(\left(D_{x_0+;g}^{(n+1)\alpha}A_1\right)\circ g^{-1}\right)(z)\,dz\right)\right\|_p dx\right]+$$
$$\left[\int_{a}^{x_0}\|A_1(x)\|_p\left\|\left(\int_{g(x)}^{g(x_0)}(z-g(x))^{(n+1)\alpha-1}\left(\left(D_{x_0-;g}^{(n+1)\alpha}A_2\right)\circ g^{-1}\right)(z)\,dz\right)\right\|_q dx\right]+$$
$$\left.\left[\int_{x_0}^{b}\|A_1(x)\|_p\left\|\left(\int_{g(x_0)}^{g(x)}(g(x)-z)^{(n+1)\alpha-1}\left(\left(D_{x_0+;g}^{(n+1)\alpha}A_2\right)\circ g^{-1}\right)(z)\,dz\right)\right\|_q dx\right]\right\}\leq \tag{80}$$

$$\frac{1}{\Gamma((n+1)\alpha)}\left\{\left[\int_{a}^{x_0}\|A_2(x)\|_q\left(\int_{g(x)}^{g(x_0)}(z-g(x))^{(n+1)\alpha-1}\left\|\left(\left(D_{x_0-;g}^{(n+1)\alpha}A_1\right)\circ g^{-1}\right)(z)\right\|_p dz\right)dx\right]+\right.$$
$$\left[\int_{x_0}^{b}\|A_2(x)\|_q\left(\int_{g(x_0)}^{g(x)}(g(x)-z)^{(n+1)\alpha-1}\left(\left(D_{x_0+;g}^{(n+1)\alpha}A_1\right)\circ g^{-1}\right)(z)\,\|_p dz\right)dx\right]+$$
$$\left[\int_{a}^{x_0}\|A_1(x)\|_p\left(\int_{g(x)}^{g(x_0)}(z-g(x))^{(n+1)\alpha-1}\left\|\left(\left(D_{x_0-;g}^{(n+1)\alpha}A_2\right)\circ g^{-1}\right)(z)\right\|_q dz\right)dx\right]+$$
$$\left.\left[\int_{x_0}^{b}\|A_1(x)\|_p\left(\int_{g(x_0)}^{g(x)}(g(x)-z)^{(n+1)\alpha-1}\left\|\left(\left(D_{x_0+;g}^{(n+1)\alpha}A_2\right)\circ g^{-1}\right)(z)\right\|_q dz\right)dx\right]\right\}. \tag{81}$$

We have proved, so far, that

$$\|\Omega(A_1,A_2)(x_0)\|_1\leq$$
$$\frac{1}{\Gamma((n+1)\alpha+1)}\left\{\left[\int_{a}^{x_0}\|A_2(x)\|_q\left(\int_{g(x)}^{g(x_0)}(z-g(x))^{(n+1)\alpha-1}\left\|\left(\left(D_{x_0-;g}^{(n+1)\alpha}A_1\right)\circ g^{-1}\right)(z)\right\|_p dz\right)dx\right]+\right.$$
$$\left[\int_{x_0}^{b}\|A_2(x)\|_q\left(\int_{g(x_0)}^{g(x)}(g(x)-z)^{(n+1)\alpha-1}\left\|\left(\left(D_{x_0+;g}^{(n+1)\alpha}A_1\right)\circ g^{-1}\right)(z)\right\|_p dz\right)dx\right]+$$
$$\left[\int_{a}^{x_0}\|A_1(x)\|_p\left(\int_{g(x)}^{g(x_0)}(z-g(x))^{(n+1)\alpha-1}\left\|\left(\left(D_{x_0-;g}^{n+1)\alpha}A_2\right)\circ g^{-1}\right)(z)\right\|_q dz\right)dx\right]+$$

$$\left[\int_{x_0}^{b}\|A_1(x)\|_p\left(\int_{g(x_0)}^{g(x)}(g(x)-z)^{(n+1)\alpha-1}\left\|\left(\left(D_{x_0+;g}^{(n+1)\alpha}A_2\right)\circ g^{-1}\right)(z)\right\|_q dz\right)dx\right]\right\} =: (\lambda). \tag{82}$$

Let now $\gamma,\delta>1$ such that $\frac{1}{\gamma}+\frac{1}{\delta}=1$, and we apply the usual Hölder's inequality in (82); $\frac{1}{\delta(n+1)}<\alpha<1$. Then we have that

$$\|\Omega(A_1,A_2)(x_0)\|_1 \leq (\lambda) \leq \frac{1}{\Gamma((n+1)\alpha)(\gamma((n+1)\alpha-1)+1)^{\frac{1}{\gamma}}}$$

$$\left\{\left[\int_a^{x_0}\|A_2(x)\|_q(g(x_0)-g(x))^{\frac{\gamma((n+1)\alpha-1)+1}{\gamma}}\left(\int_{g(x)}^{g(x_0)}\left\|\left(\left(D_{x_0-;g}^{(n+1)\alpha}A_1\right)\circ g^{-1}\right)(z)\right\|_p^{\delta}dz\right)^{\frac{1}{\delta}}dx\right]+\right.$$

$$\left[\int_{x_0}^{b}\|A_2(x)\|_q(g(x)-g(x_0))^{\frac{\gamma((n+1)\alpha-1)+1}{\gamma}}\left(\int_{g(x_0)}^{g(x)}\left\|\left(\left(D_{x_0+;g}^{(n+1)\alpha}A_1\right)\circ g^{-1}\right)(z)\right\|_p^{\delta}dz\right)^{\frac{1}{\delta}}dx\right]+$$

$$\left[\int_a^{x_0}\|A_1(x)\|_p(g(x_0)-g(x))^{\frac{\gamma((n+1)\alpha-1)+1}{\gamma}}\left(\int_{g(x)}^{g(x_0)}\left\|\left(\left(D_{x_0-;g}^{(n+1)\alpha}A_2\right)\circ g^{-1}\right)(z)\right\|_q^{\delta}dz\right)^{\frac{1}{\delta}}dx\right]+ \tag{83}$$

$$\left.\left[\int_{x_0}^{b}\|A_1(x)\|_p(g(x)-g(x_0))^{\frac{\gamma((n+1)\alpha-1)+1}{\gamma}}\left(\int_{g(x_0)}^{g(x)}\left\|\left(\left(D_{x_0+;g}^{(n+1)\alpha}A_2\right)\circ g^{-1}\right)(z)\right\|_q^{\delta}dz\right)^{\frac{1}{\delta}}dx\right]\right\}$$

$$\leq \frac{1}{\Gamma((n+1)\alpha)(\gamma((n+1)\alpha-1)+1)^{\frac{1}{\gamma}}}$$

$$\left\{\left[\left\|\left\|\left(D_{x_0-;g}^{(n+1)\alpha}A_1\right)\circ g^{-1}\right\|_p\right\|_{\delta,[g(a),g(x_0)]}\int_a^{x_0}\|A_2(x)\|_q(g(x_0)-g(x))^{(n+1)\alpha-\frac{1}{\delta}}dx\right]+\right.$$

$$\left[\left\|\left\|\left(D_{x_0+;g}^{(n+1)\alpha}A_1\right)\circ g^{-1}\right\|_p\right\|_{\delta,[g(x_0),g(b)]}\int_{x_0}^{b}\|A_2(x)\|_q(g(x)-g(x_0))^{(n+1)\alpha-\frac{1}{\delta}}dx\right]+ \tag{84}$$

$$\left[\left\|\left\|\left(D_{x_0-;g}^{(n+1)\alpha}A_2\right)\circ g^{-1}\right\|_q\right\|_{\delta,[g(a),g(x_0)]}\int_a^{x_0}\|A_1(x)\|_p(g(x_0)-g(x))^{(n+1)\alpha-\frac{1}{\delta}}dx\right]+$$

$$\left.\left[\left\|\left\|\left(D_{x_0+;g}^{(n+1)\alpha}A_2\right)\circ g^{-1}\right\|_q\right\|_{\delta,[g(x_0),g(b)]}\int_{x_0}^{b}\|A_1(x)\|_p(g(x)-g(x_0))^{(n+1)\alpha-\frac{1}{\delta}}dx\right]\right\},$$

proving (74).

If $(n+1)\alpha\geq 1$, we obtain

$$\|\Omega(A_1,A_2)(x_0)\|_1 \leq (\lambda) \leq \frac{1}{\Gamma((n+1)\alpha)}$$

$$\left\{\left[\left\|\left\|\left(D_{x_0-;g}^{(n+1)\alpha}A_1\right)\circ g^{-1}\right\|_p\right\|_{L_1([g(a),g(x_0)])}\int_a^{x_0}\|A_2(x)\|_q(g(x_0)-g(x))^{(n+1)\alpha-1}dx\right]+\right.$$

$$\left[\left\|\left\|\left(D_{x_0+;g}^{(n+1)\alpha}A_1\right)\circ g^{-1}\right\|_p\right\|_{L_1([g(x_0),g(b)])}\int_{x_0}^{b}\|A_2(x)\|_q(g(x)-g(x_0))^{(n+1)\alpha-1}dx\right]+$$

$$\left[\left\|\left\|\left(D_{x_0-;g}^{(n+1)\alpha}A_2\right)\circ g^{-1}\right\|_q\right\|_{L_1([g(a),g(x_0)])}\int_a^{x_0}\|A_1(x)\|_p(g(x_0)-g(x))^{(n+1)\alpha-1}dx\right]+ \tag{85}$$

$$\left.\left[\left\|\left\|\left(D_{x_0+;g}^{(n+1)\alpha}A_2\right)\circ g^{-1}\right\|_q\right\|_{L_1([g(x_0),g(b)])}\int_{x_0}^{b}\|A_1(x)\|_p(g(x)-g(x_0))^{(n+1)\alpha-1}dx\right]\right\},$$

proving (75).

At last we derive

$$\|\Omega(A_1,A_2)(x_0)\|_1\leq(\lambda)\leq\frac{1}{\Gamma((n+1)\alpha+1)}$$

$$\left\{\left[\left\|\left\|\left(D_{x_0-;g}^{(n+1)\alpha}A_1\right)\circ g^{-1}\right\|_p\right\|_{\infty,[g(a),g(x_0)]}\int_a^{x_0}\|A_2(x)\|_q(g(x_0)-g(x))^{(n+1)\alpha}dx\right]+\right.$$

$$\left[\left\|\left\|\left(D_{x_0+;g}^{(n+1)\alpha}A_1\right)\circ g^{-1}\right\|_p\right\|_{\infty,[g(x_0),g(b)]}\int_{x_0}^{b}\|A_2(x)\|_q(g(x)-g(x_0))^{(n+1)\alpha}dx\right]+$$

$$\left[\left\|\left\|\left(D_{x_0-;g}^{(n+1)\alpha}A_2\right)\circ g^{-1}\right\|_q\right\|_{\infty,[g(a),g(x_0)]}\int_a^{x_0}\|A_1(x)\|_p(g(x_0)-g(x))^{(n+1)\alpha}dx\right]+$$

$$\left.\left[\left\|\left\|\left(D_{x_0+;g}^{(n+1)\alpha}A_2\right)\circ g^{-1}\right\|_q\right\|_{\infty,[g(x_0),g(b)]}\int_{x_0}^{b}\|A_1(x)\|_p(g(x)-g(x_0))^{(n+1)\alpha}dx\right]\right\}, \tag{86}$$

proving (76).

The theorem is proved. ■

When $r=2$ and $n=1$ we obtain the following special operator related sequential fractional Ostrowski type inequalities.

Corollary 31 *(to Theorem 30) Let $p,q>1:\frac{1}{p}+\frac{1}{q}=1$, and let the $*$-ideals $\mathcal{B}_p(H)$, $\mathcal{B}_q(H)$, for which $\left(\mathcal{B}_p(H),\|\cdot\|_p\right)$, $\left(\mathcal{B}_q(H),\|\cdot\|_q\right)$ are Banach algebras; $x_0\in[a,b]\subset\mathbb{R}$, $0<\alpha<1$, $A_1\in C^1([a,b],\mathcal{B}_p(H))$, $A_2\in C^1([a,b],\mathcal{B}_q(H))$; $g\in C^1([a,b])$, strictly increasing, such that $g^{-1}\in C^1([g(a),g(b)])$. Assume that $F_{ki}^{x_0}:=D_{x_0-;g}^{k\alpha}A_i$, for $k=1,2$, $i=1,2$, fulfill $F_{1,1}^{x_0}\in C^1([a,x_0],\mathcal{B}_p(H))$, $F_{2,1}^{x_0}\in C([a,x_0],\mathcal{B}_p(H))$; $F_{1,2}^{x_0}\in C^1([a,x_0],\mathcal{B}_q(H))$, $F_{2,2}^{x_0}\in C([a,x_0],\mathcal{B}_q(H))$. Similarly, we assume that $G_{ki}^{x_0}:=D_{x_0+;g}^{k\alpha}A_i$, $k=1,2$, $i=1,2$, fulfill $G_{1,1}^{x_0}\in C^1([x_0,b],\mathcal{B}_p(H))$, $G_{2,1}^{x_0}\in C([x_0,b],\mathcal{B}_p(H))$, $G_{1,2}^{x_0}\in C^1([x_0,b],\mathcal{B}_q(H))$, $G_{2,2}^{x_0}\in C([x_0,b],\mathcal{B}_q(H))$.*

Then
1) it holds

$$\Omega\left(A_1,A_2\right)(x_0):=\int_a^b A_2(x)\,A_1(x)+\int_a^b A_1(x)\,A_2(x)\,dx-$$

$$\left(\int_a^b A_2(x)\,dx\right)A_1(x_0)-\left(\int_a^b A_1(x)\,dx\right)A_2(x_0)=$$

$$\frac{1}{\Gamma(2\alpha)}\left\{\left[\int_a^{x_0} A_2(x)\left(\int_{g(x)}^{g(x_0)}(z-g(x))^{2\alpha-1}\left(\left(D_{x_0-;g}^{2\alpha}A_1\right)\circ g^{-1}\right)(z)\,dz\right)dx\right]+\right.$$

$$\left[\int_{x_0}^b A_2(x)\left(\int_{g(x_0)}^{g(x)}(g(x)-z)^{2\alpha-1}\left(\left(D_{x_0+;g}^{2\alpha}A_1\right)\circ g^{-1}\right)(z)\,dz\right)dx\right]+$$

$$\left[\int_a^{x_0} A_1(x)\left(\int_{g(x)}^{g(x_0)}(z-g(x))^{2\alpha-1}\left(\left(D_{x_0-;g}^{2\alpha}A_2\right)\circ g^{-1}\right)(z)\,dz\right)dx\right]+$$

$$\left.\left[\int_{x_0}^b A_1(x)\left(\int_{g(x_0)}^{g(x)}(g(x)-z)^{2\alpha-1}\left(\left(D_{x_0+;g}^{2\alpha}A_2\right)\circ g^{-1}\right)(z)\,dz\right)dx\right]\right\}, \quad (87)$$

2) for $\gamma,\delta>1:\frac{1}{\gamma}+\frac{1}{\delta}=1$, *with* $\frac{1}{2\delta}<\alpha<1$, *we have that*

$$\left\|\Omega\left(A_1,A_2\right)(x_0)\right\|_1\le\frac{1}{\Gamma(2\alpha)\left(\gamma(2\alpha-1)+1\right)^{\frac{1}{\gamma}}}$$

$$\left\{\left[\left\|\left\|\left(D_{x_0-;g}^{2\alpha}A_1\right)\circ g^{-1}\right\|_p\right\|_{\delta,[g(a),g(x_0)]}\int_a^{x_0}\|A_2(x)\|_q\,(g(x_0)-g(x))^{2\alpha-\frac{1}{\delta}}\,dx\right]+\right.$$

$$\left[\left\|\left\|\left(D_{x_0+;g}^{2\alpha}A_1\right)\circ g^{-1}\right\|_p\right\|_{\delta,[g(x_0),g(b)]}\int_{x_0}^b\|A_2(x)\|_q\,(g(x)-g(x_0))^{2\alpha-\frac{1}{\delta}}\,dx\right]+ \quad (88)$$

$$\left[\left\|\left\|\left(D_{x_0-;g}^{2\alpha}A_2\right)\circ g^{-1}\right\|_q\right\|_{\delta,[g(a),g(x_0)]}\int_a^{x_0}\|A_1(x)\|_p\,(g(x_0)-g(x))^{2\alpha-\frac{1}{\delta}}\,dx\right]+$$

$$\left.\left[\left\|\left\|\left(D_{x_0+;g}^{2\alpha}A_2\right)\circ g^{-1}\right\|_q\right\|_{\delta,[g(x_0),g(b)]}\int_{x_0}^b\|A_1(x)\|_p\,(g(x)-g(x_0))^{2\alpha-\frac{1}{\delta}}\,dx\right]\right\}$$

$$=:\Lambda_7(x_0),$$

3) if $\alpha\ge\frac{1}{2}$, *we obtain*

$$\left\|\Omega\left(A_1,A_2\right)(x_0)\right\|_1\le\frac{1}{\Gamma(2\alpha)}$$

$$\left\{\left[\left\|\left\|\left(D_{x_0-;g}^{2\alpha}A_1\right)\circ g^{-1}\right\|_p\right\|_{L_1([g(a),g(x_0)])}\int_a^{x_0}\|A_2(x)\|_q\,(g(x_0)-g(x))^{2\alpha-1}\,dx\right]+\right.$$

$$\left[\left\|\left\|\left(D^{2\alpha}_{x_0+;g}A_1\right)\circ g^{-1}\right\|_p\right\|_{L_1([g(x_0),g(b)])}\int_{x_0}^{b}\|A_2(x)\|_q\,(g(x)-g(x_0))^{2\alpha-1}\,dx\right]+$$

$$\left[\left\|\left\|\left(D^{2\alpha}_{x_0-;g}A_2\right)\circ g^{-1}\right\|_q\right\|_{L_1([g(a),g(x_0)])}\int_{a}^{x_0}\|A_1(x)\|_p\,(g(x_0)-g(x))^{2\alpha-1}\,dx\right]+$$

$$\left.\left[\left\|\left\|\left(D^{2\alpha}_{x_0+;g}A_2\right)\circ g^{-1}\right\|_q\right\|_{L_1([g(x_0),g(b)])}\int_{x_0}^{b}\|A_1(x)\|_p\,(g(x)-g(x_0))^{2\alpha-1}\,dx\right]\right\} \tag{89}$$

$$=:\Lambda_8(x_0),$$

and

4)

$$\|\Omega(A_1,A_2)(x_0)\|_1\leq\frac{1}{\Gamma(2\alpha+1)}$$

$$\left\{\left[\left\|\left\|\left(D^{2\alpha}_{x_0-;g}A_1\right)\circ g^{-1}\right\|_p\right\|_{\infty,[g(a),g(x_0)]}\int_{a}^{x_0}\|A_2(x)\|_q\,(g(x_0)-g(x))^{2\alpha}\,dx\right]+\right.$$

$$\left[\left\|\left\|\left(D^{2\alpha}_{x_0+;g}A_1\right)\circ g^{-1}\right\|_p\right\|_{\infty,[g(x_0),g(b)]}\int_{x_0}^{b}\|A_2(x)\|_q\,(g(x)-g(x_0))^{2\alpha}\,dx\right]+$$

$$\left[\left\|\left\|\left(D^{2\alpha}_{x_0-;g}A_2\right)\circ g^{-1}\right\|_q\right\|_{\infty,[g(a),g(x_0)]}\int_{a}^{x_0}\|A_1(x)\|_p\,(g(x_0)-g(x))^{2\alpha}\,dx\right]+$$

$$\left.\left[\left\|\left\|\left(D^{2\alpha}_{x_0+;g}A_2\right)\circ g^{-1}\right\|_q\right\|_{\infty,[g(x_0),g(b)]}\int_{x_0}^{b}\|A_1(x)\|_p\,(g(x)-g(x_0))^{2\alpha}\,dx\right]\right\} \tag{90}$$

$$=:\Lambda_9(x_0).$$

Proof. By Theorem 30. ■

We make

Remark 32 *Let* $\Omega(A_1,...,A_r)(x_0)$ *as in (41). Denote by*

$$\Delta(A_1,...,A_r):=\int_a^b\Omega(A_1,...,A_r)(x_0)\,dx_0=$$

$$\sum_{i=1}^{r}\left[(b-a)\int_a^b\left(\prod_{\substack{j=1\\j\neq i}}^{r}A_j(x)\right)A_i(x)\,dx-\left(\int_a^b\left(\prod_{\substack{j=1\\j\neq i}}^{r}A_j(x)\right)dx\right)\left(\int_a^bA_i(x)\,dx\right)\right], \tag{91}$$

$r\in\mathbb{N}-\{1\}$.

In particular, we have that

$$\Delta(A_1,A_2):=\int_a^b\Omega(A_1,A_2)(x_0)\,dx_0=$$

$$(b-a)\left(\int_a^b A_2(x)A_1(x)\,dx+\int_a^b A_1(x)A_2(x)\,dx\right)- \tag{92}$$

$$\left(\int_a^b A_2(x)\,dx\right)\left(\int_a^b A_1(x)\,dx\right)-\left(\int_a^b A_1(x)\,dx\right)\left(\int_a^b A_2(x)\,dx\right),$$

Clearly, it holds that

$$\|\Delta(A_1,...,A_r)\|_p\leq\int_a^b\|\Omega(A_1,...,A_r)(x_0)\|_p\,dx_0, \tag{93}$$

$\forall\ p\geq 1.$

We need

Remark 33 *i) Call and assume*

$$W_1(A_1,...,A_r):=\max_{i=1,\dots,r}\left\{\sup_{x_0\in[a,b]}\left\|\left\|\left(D^{2\alpha}_{x_0-;g}A_i\right)\right\|_2\right\|_{\infty,[a,x_0]},\right.$$

$$\left.\sup_{x_0\in[a,b]}\left\|\left\|\left(D^{2\alpha}_{x_0+;g}A_i\right)\right\|_2\right\|_{\infty,[x_0,b]}\right\}<\infty. \tag{94}$$

Hence by (63) we obtain

$$\|\Omega(A_1,...,A_r)(x_0)\|_1\leq\Lambda_1(x_0)\leq \tag{95}$$

$$\frac{W_1(A_1,...,A_r)(g(b)-g(a))^{2\alpha}}{\Gamma(2\alpha+1)}\sum_{i=1}^r\left[\int_a^b\left(\prod_{\substack{j=1\\j\neq i}}^r\|A_j(x)\|_2\right)dx\right],$$

and

$$\|\Delta(A_1,...,A_r)\|_1\leq$$

$$\frac{W_1(A_1,...,A_r)(g(b)-g(a))^{2\alpha}(b-a)}{\Gamma(2\alpha+1)}\sum_{i=1}^r\left[\int_a^b\left(\prod_{\substack{j=1\\j\neq i}}^r\|A_j(x)\|_2\right)dx\right]. \tag{96}$$

ii) Here $\frac{1}{2}\leq\alpha<1$. Call and assume

$$W_2(A_1,...,A_r):=\max_{i=1,\dots,r}\left\{\sup_{x_0\in[a,b]}\left\|\left\|\left(D^{2\alpha}_{x_0-;g}A_i\right)\right\|_2\right\|_{L_1([a,x_0])},\right.$$

$$\left.\sup_{x_0\in[a,b]}\left\|\left\|\left(D^{2\alpha}_{x_0+;g}A_i\right)\right\|_2\right\|_{L_1([x_0,b])}\right\}<\infty. \tag{97}$$

Hence by (64) we obtain

$$\|\Omega(A_1,...,A_r)(x_0)\|_1 \leq \Lambda_2(x_0) \leq$$

$$\frac{\|g'\|_{\infty,[a,b]}}{\Gamma(2\alpha)} W_2(A_1,...,A_r)(g(b)-g(a))^{2\alpha-1} \sum_{i=1}^{r} \left[\int_a^b \left(\prod_{\substack{j=1 \\ j\neq i}}^{r} \|A_j(x)\|_2 \right) dx \right], \tag{98}$$

and

$$\|\Delta(A_1,...,A_r)\|_1 \leq \frac{\|g'\|_{\infty,[a,b]}}{\Gamma(2\alpha)}$$

$$W_2(A_1,...,A_r)(g(b)-g(a))^{2\alpha-1}(b-a) \sum_{i=1}^{r} \left[\int_a^b \left(\prod_{\substack{j=1 \\ j\neq i}}^{r} \|A_j(x)\|_2 \right) dx \right]. \tag{99}$$

iii) Here $\gamma, \delta > 1 : \frac{1}{\gamma} + \frac{1}{\delta} = 1$, *and* $\frac{1}{2\delta} < \alpha < 1$. *Call and assume*

$$W_3(A_1,...,A_r) := \max_{i=1,...,r} \left\{ \sup_{x_0\in[a,b]} \left\| \left\| \left(D^{2\alpha}_{x_0-;g} A_i \right) \circ g^{-1} \right\|_2 \right\|_{\delta,[g(a),g(x_0)]}, \right.$$

$$\left. \sup_{x_0\in[a,b]} \left\| \left\| \left(D^{2\alpha}_{x_0+;g} A_i \right) \circ g^{-1} \right\|_2 \right\|_{\delta,[g(x_0),g(b)]} \right\} < \infty. \tag{100}$$

Therefore by (65) we get

$$\|\Omega(A_1,...,A_r)(x_0)\|_1 \leq \Lambda_3(x_0) \leq$$

$$\frac{W_3(A_1,...,A_r)(g(b)-g(a))^{2\alpha-\frac{1}{\delta}}}{\Gamma(2\alpha)(\gamma(2\alpha-1)+1)^{\frac{1}{\gamma}}} \sum_{i=1}^{r} \left[\int_a^b \left(\prod_{\substack{j=1 \\ j\neq i}}^{r} \|A_j(x)\|_2 \right) dx \right], \tag{101}$$

and

$$\|\Delta(A_1,...,A_r)\|_1 \leq$$

$$\frac{W_3(A_1,...,A_r)(g(b)-g(a))^{2\alpha-\frac{1}{\delta}}(b-a)}{\Gamma(2\alpha)(\gamma(2\alpha-1)+1)^{\frac{1}{\gamma}}} \sum_{i=1}^{r} \left[\int_a^b \left(\prod_{\substack{j=1 \\ j\neq i}}^{r} \|A_j(x)\|_2 \right) dx \right]. \tag{102}$$

We need

Remark 34 *i) Call and assume*

$$W_4(A_1,...,A_r) := \max_{i=1,...,r}\left\{ \sup_{x_0\in[a,b]} \left\| \left\| \left(D^{2\alpha}_{x_0-;g}A_i\right)\right\|_p \right\|_{\infty,[a,x_0]}, \right.$$

$$\left. \sup_{x_0\in[a,b]} \left\| \left\| \left(D^{2\alpha}_{x_0+;g}A_i\right)\right\|_p \right\|_{\infty,[x_0,b]} \right\} < \infty. \tag{103}$$

Hence by (70) we obtain

$$\|\Omega(A_1,...,A_r)(x_0)\|_p \leq \Lambda_4(x_0) \leq \tag{104}$$

$$\frac{W_4(A_1,...,A_r)(g(b)-g(a))^{2\alpha}}{\Gamma(2\alpha+1)} \sum_{i=1}^{r}\left[\int_a^b \left(\prod_{\substack{j=1\\ j\neq i}}^{r} \|A_j(x)\|_p\right) dx\right],$$

and

$$\|\Delta(A_1,...,A_r)\|_p \leq$$

$$\frac{W_4(A_1,...,A_r)(g(b)-g(a))^{2\alpha}(b-a)}{\Gamma(2\alpha+1)} \sum_{i=1}^{r}\left[\int_a^b \left(\prod_{\substack{j=1\\ j\neq i}}^{r} \|A_j(x)\|_p\right) dx\right], \tag{105}$$

ii) Here $\frac{1}{2} \leq \alpha < 1$*. Call and assume*

$$W_5(A_1,...,A_r) := \max_{i=1,...,r}\left\{ \sup_{x_0\in[a,b]} \left\| \left\| \left(D^{2\alpha}_{x_0-;g}A_i\right)\right\|_p \right\|_{L_1([a,x_0])}, \right.$$

$$\left. \sup_{x_0\in[a,b]} \left\| \left\| \left(D^{2\alpha}_{x_0+;g}A_i\right)\right\|_p \right\|_{L_1([x_0,b])} \right\} < \infty. \tag{106}$$

Hence by (71) we obtain

$$\|\Omega(A_1,...,A_r)(x_0)\|_p \leq \Lambda_5(x_0) \leq$$

$$\frac{\|g'\|_{\infty,[a,b]}}{\Gamma(2\alpha)} W_5(A_1,...,A_r)(g(b)-g(a))^{2\alpha-1} \sum_{i=1}^{r}\left[\int_a^b \left(\prod_{\substack{j=1\\ j\neq i}}^{r} \|A_j(x)\|_p\right) dx\right], \tag{107}$$

and

$$\|\Delta(A_1,...,A_r)\|_p \leq \frac{\|g'\|_{\infty,[a,b]}}{\Gamma(2\alpha)}$$

$$W_5(A_1,...,A_r)(g(b)-g(a))^{2\alpha-1}(b-a)\sum_{i=1}^{r}\left[\int_a^b\left(\prod_{\substack{j=1\\j\neq i}}^{r}\|A_j(x)\|_p\right)dx\right]. \tag{108}$$

iii) Here $\gamma,\delta>1:\frac{1}{\gamma}+\frac{1}{\delta}=1$, *and* $\frac{1}{2\delta}<\alpha<1$. *Call and assume*

$$W_6(A_1,...,A_r):=\max_{i=1,...,r}\left\{\sup_{x_0\in[a,b]}\left\|\left\|\left(D^{2\alpha}_{x_0-;g}A_i\right)\circ g^{-1}\right\|_p\right\|_{\delta,[g(a),g(x_0)]},\right.$$

$$\left.\sup_{x_0\in[a,b]}\left\|\left\|\left(D^{2\alpha}_{x_0+;g}A_i\right)\circ g^{-1}\right\|_p\right\|_{\delta,[g(x_0),g(b)]}\right\}<\infty. \tag{109}$$

Therefore by (72) we get

$$\|\Omega(A_1,...,A_r)(x_0)\|_p\leq\Lambda_6(x_0)\leq$$

$$\frac{W_6(A_1,...,A_r)(g(b)-g(a))^{2\alpha-\frac{1}{\delta}}}{\Gamma(2\alpha)(\gamma(2\alpha-1)+1)^{\frac{1}{\gamma}}}\sum_{i=1}^{r}\left[\int_a^b\left(\prod_{\substack{j=1\\j\neq i}}^{r}\|A_j(x)\|_p\right)dx\right], \tag{110}$$

and

$$\|\Delta(A_1,...,A_r)\|_p\leq$$

$$\frac{W_6(A_1,...,A_r)(g(b)-g(a))^{2\alpha-\frac{1}{\delta}}(b-a)}{\Gamma(2\alpha)(\gamma(2\alpha-1)+1)^{\frac{1}{\gamma}}}\sum_{i=1}^{r}\left[\int_a^b\left(\prod_{\substack{j=1\\j\neq i}}^{r}\|A_j(x)\|_p\right)dx\right]. \tag{111}$$

We also need

Remark 35 *i) Call and assume*

$$W_7(A_1,A_2):=$$

$$\max\left\{\sup_{x_0\in[a,b]}\left\|\left\|\left(D^{2\alpha}_{x_0-;g}A_1\right)\circ g^{-1}\right\|_p\right\|_{\delta,[g(a),g(x_0)]},\right.$$

$$\sup_{x_0\in[a,b]}\left\|\left\|\left(D^{2\alpha}_{x_0+;g}A_1\right)\circ g^{-1}\right\|_p\right\|_{\delta,[g(x_0),g(b)]},$$

$$\sup_{x_0\in[a,b]}\left\|\left\|\left(D^{2\alpha}_{x_0-;g}A_2\right)\circ g^{-1}\right\|_q\right\|_{\delta,[g(a),g(x_0)]},$$

$$\left.\sup_{x_0\in[a,b]}\left\|\left\|\left(D^{2\alpha}_{x_0+;g}A_2\right)\circ g^{-1}\right\|_q\right\|_{\delta,[g(x_0),g(b)]}\right\}<\infty. \tag{112}$$

Let $\gamma, \delta > 1 : \frac{1}{\gamma} + \frac{1}{\delta} = 1$, *with* $\frac{1}{2\delta} < \alpha < 1$.

Then, by (88), we get

$$\left\|\Omega\left(A_1, A_2\right)\left(x_0\right)\right\|_1 \leq \Lambda_7\left(x_0\right) \leq$$

$$\frac{W_7\left(A_1, A_2\right)\left(g\left(b\right) - g\left(a\right)\right)^{2\alpha - \frac{1}{\delta}}}{\Gamma\left(2\alpha\right)\left(\gamma\left(2\alpha - 1\right) + 1\right)^{\frac{1}{\gamma}}}\left[\int_a^b \left(\left\|A_1\left(x\right)\right\|_p + \left\|A_2\left(x\right)\right\|_q\right) dx\right], \tag{113}$$

and

$$\left\|\Delta\left(A_1, A_2\right)\right\|_1 \leq$$

$$\frac{W_7\left(A_1, A_2\right)\left(g\left(b\right) - g\left(a\right)\right)^{2\alpha - \frac{1}{\delta}}\left(b - a\right)}{\Gamma\left(2\alpha\right)\left(\gamma\left(2\alpha - 1\right) + 1\right)^{\frac{1}{\gamma}}}\left[\int_a^b \left(\left\|A_1\left(x\right)\right\|_p + \left\|A_2\left(x\right)\right\|_q\right) dx\right]. \tag{114}$$

ii) Call and assume

$$W_8\left(A_1, A_2\right) :=$$

$$\max\left\{\sup_{x_0 \in [a,b]}\left\|\left\|\left(D_{x_0-;g}^{2\alpha}A_1\right) \circ g^{-1}\right\|_p\right\|_{L_1([g(a),g(x_0)])},\right.$$

$$\sup_{x_0 \in [a,b]}\left\|\left\|\left(D_{x_0+;g}^{2\alpha}A_1\right) \circ g^{-1}\right\|_p\right\|_{L_1([g(x_0),g(b)])},$$

$$\sup_{x_0 \in [a,b]}\left\|\left\|\left(D_{x_0-;g}^{2\alpha}A_2\right) \circ g^{-1}\right\|_q\right\|_{L_1([g(a),g(x_0)])},$$

$$\left.\sup_{x_0 \in [a,b]}\left\|\left\|\left(D_{x_0+;g}^{2\alpha}A_2\right) \circ g^{-1}\right\|_q\right\|_{L_1([g(x_0),g(b)])}\right\} < \infty. \tag{115}$$

If $\alpha \geq \frac{1}{2}$, *by (89), we get*

$$\left\|\Omega\left(A_1, A_2\right)\left(x_0\right)\right\|_1 \leq \Lambda_8\left(x_0\right) \leq$$

$$\frac{W_8\left(A_1, A_2\right)\left(g\left(b\right) - g\left(a\right)\right)^{2\alpha - 1}}{\Gamma\left(2\alpha\right)}\left[\int_a^b \left(\left\|A_1\left(x\right)\right\|_p + \left\|A_2\left(x\right)\right\|_q\right) dx\right], \tag{116}$$

and

$$\left\|\Delta\left(A_1, A_2\right)\right\|_1 \leq$$

$$\frac{W_8\left(A_1, A_2\right)\left(g\left(b\right) - g\left(a\right)\right)^{2\alpha - 1}\left(b - a\right)}{\Gamma\left(2\alpha\right)}\left[\int_a^b \left(\left\|A_1\left(x\right)\right\|_p + \left\|A_2\left(x\right)\right\|_q\right) dx\right]. \tag{117}$$

iii) Call and assume

$$W_9\left(A_1, A_2\right) :=$$

$$\max\left\{\sup_{x_0 \in [a,b]}\left\|\left\|\left(D_{x_0-;g}^{2\alpha}A_1\right) \circ g^{-1}\right\|_p\right\|_{\infty,[g(a),g(x_0)]},\right.$$

$$\sup_{x_0\in[a,b]}\left\|\left\|\left(D^{2\alpha}_{x_0+;g}A_1\right)\circ g^{-1}\right\|_p\right\|_{\infty,[g(x_0),g(b)]},$$
$$\sup_{x_0\in[a,b]}\left\|\left\|\left(D^{2\alpha}_{x_0-;g}A_2\right)\circ g^{-1}\right\|_q\right\|_{\infty,[g(a),g(x_0)]},$$
$$\left.\sup_{x_0\in[a,b]}\left\|\left\|\left(D^{2\alpha}_{x_0+;g}A_2\right)\circ g^{-1}\right\|_q\right\|_{\infty,[g(x_0),g(b)]}\right\}<\infty. \tag{118}$$

Therefore, by (90), we get

$$\|\Omega(A_1,A_2)(x_0)\|_1\leq\Lambda_9(x_0)\leq$$
$$\frac{W_9(A_1,A_2)(g(b)-g(a))^{2\alpha}}{\Gamma(2\alpha+1)}\left[\int_a^b\left(\|A_1(x)\|_p+\|A_2(x)\|_q\right)dx\right], \tag{119}$$

and

$$\|\Delta(A_1,A_2)\|_1\leq$$
$$\frac{W_9(A_1,A_2)(g(b)-g(a))^{2\alpha}(b-a)}{\Gamma(2\alpha+1)}\left[\int_a^b\left(\|A_1(x)\|_p+\|A_2(x)\|_q\right)dx\right]. \tag{120}$$

Based on the Remarks 32-35 we formulate the following generalized sequential fractional Grüss type inequalities results.

First come 1-Schatten norm generalized sequential fractional Grüss type inequalities involving several functions taking values in the Banach algebra $\mathcal{B}_2(H)\subset\mathcal{B}(H)$:

Theorem 36 *Let the $*$-ideal $\mathcal{B}_2(H)$, $0<\alpha<1$; $A_i\in C^1([a,b],\mathcal{B}_2(H))$, $i=1,...,r\in\mathbb{N}-\{1\}$; $g\in C^1([a,b])$, strictly increasing such that $g^{-1}\in C^1([g(a),g(b)])$. Assume $\forall\ x_0\in[a,b]$ that $D^{\alpha}_{x_0-;g}A_i\in C^1([a,x_0],\mathcal{B}_2(H))$ and $D^{2\alpha}_{x_0-;g}A_i\in C([a,x_0],\mathcal{B}_2(H))$; $i=1,...,r$. Similarly, assume $\forall\ x_0\in[a,b]$ that $D^{\alpha}_{x_0+;g}A_i\in C^1([x_0,b],\mathcal{B}_2(H))$ and $D^{2\alpha}_{x_0+;g}A_i\in C([x_0,b],\mathcal{B}_2(H))$; $i=1,...,r$. Here $\Delta(A_1,...,A_r)$ is as in (91), $W_1(A_1,...,A_r)$ as in (94), $W_2(A_1,...,A_r)$ is as in (97), and $W_3(A_1,...,A_r)$ is as in (100).*

Then

i)

$$\|\Delta(A_1,...,A_r)\|_1\leq\frac{W_1(A_1,...,A_r)(g(b)-g(a))^{2\alpha}(b-a)}{\Gamma(2\alpha+1)}$$
$$\sum_{i=1}^r\left[\int_a^b\left(\prod_{\substack{j=1\\j\neq i}}^r\|A_j(x)\|_2\right)dx\right], \tag{121}$$

ii) if $\frac{1}{2}\leq\alpha<1$, we have

$$\|\Delta(A_1,...,A_r)\|_1\leq\frac{\|g'\|_{\infty,[a,b]}}{\Gamma(2\alpha)}$$

$$W_2(A_1,...,A_r)(g(b)-g(a))^{2\alpha-1}(b-a)\sum_{i=1}^{r}\left[\int_a^b\left(\prod_{\substack{j=1\\j\neq i}}^{r}\|A_j(x)\|_2\right)dx\right], \tag{122}$$

iii) if $\gamma,\delta>1:\frac{1}{\gamma}+\frac{1}{\delta}=1$; $\frac{1}{2\delta}<\alpha<1$, *we have*

$$\|\Delta(A_1,...,A_r)\|_1\leq$$

$$\frac{W_3(A_1,...,A_r)(g(b)-g(a))^{2\alpha-\frac{1}{\delta}}(b-a)}{\Gamma(2\alpha)(\gamma(2\alpha-1)+1)^{\frac{1}{\gamma}}}\sum_{i=1}^{r}\left[\int_a^b\left(\prod_{\substack{j=1\\j\neq i}}^{r}\|A_j(x)\|_2\right)dx\right]. \tag{123}$$

We continue with p-Schatten norm generalized sequential fractional Grüss type inequalities involving several functions taking values in the Banach algebra $\mathcal{B}_p(H)\subset\mathcal{B}(H)$, $p>1$.

Theorem 37 *Let* $p>1$, *the* $*$*-ideal* $\mathcal{B}_p(H)$, $0<\alpha<1$; $A_i\in C^1([a,b],\mathcal{B}_p(H))$, $i=1,...,r\in\mathbb{N}-\{1\}$; $g\in C^1([a,b])$, *strictly increasing such that* $g^{-1}\in C^1([g(a),g(b)])$. *Assume* $\forall\, x_0\in[a,b]$ *that* $D^{\alpha}_{x_0-;g}A_i\in C^1([a,x_0],\mathcal{B}_p(H))$ *and* $D^{2\alpha}_{x_0-;g}A_i\in C([a,x_0],\mathcal{B}_p(H))$; $i=1,...,r$. *Similarly, assume* $\forall\, x_0\in[a,b]$ *that* $D^{\alpha}_{x_0+;g}A_i\in C^1([x_0,b],\mathcal{B}_p(H))$ *and* $D^{2\alpha}_{x_0+;g}A_i\in C([x_0,b],\mathcal{B}_p(H))$; $i=1,...,r$. *Here* $\Delta(A_1,...,A_r)$ *is as in (91),* $W_4(A_1,...,A_r)$ *as in (103),* $W_5(A_1,...,A_r)$ *is as in (106), and* $W_6(A_1,...,A_r)$ *is as in (109).*

Then

i)

$$\|\Delta(A_1,...,A_r)\|_p\leq\frac{W_4(A_1,...,A_r)(g(b)-g(a))^{2\alpha}(b-a)}{\Gamma(2\alpha+1)}$$

$$\sum_{i=1}^{r}\left[\int_a^b\left(\prod_{\substack{j=1\\j\neq i}}^{r}\|A_j(x)\|_p\right)dx\right], \tag{124}$$

ii) if $\frac{1}{2}\leq\alpha<1$, *we have*

$$\|\Delta(A_1,...,A_r)\|_p\leq\frac{\|g'\|_{\infty,[a,b]}}{\Gamma(2\alpha)}$$

$$W_5(A_1,...,A_r)(g(b)-g(a))^{2\alpha-1}(b-a)\sum_{i=1}^{r}\left[\int_a^b\left(\prod_{\substack{j=1\\j\neq i}}^{r}\|A_j(x)\|_p\right)dx\right], \tag{125}$$

iii) if $\gamma,\delta>1:\frac{1}{\gamma}+\frac{1}{\delta}=1$*;* $\frac{1}{2\delta}<\alpha<1$*, we have*

$$\left\Vert \Delta\left(A_1,...,A_r\right)\right\Vert_p\leq$$

$$\frac{W_6\left(A_1,...,A_r\right)\left(g\left(b\right)-g\left(a\right)\right)^{2\alpha-\frac{1}{\delta}}\left(b-a\right)}{\Gamma\left(2\alpha\right)\left(\gamma\left(2\alpha-1\right)+1\right)^{\frac{1}{\gamma}}}\sum_{i=1}^{r}\left[\int_a^b\left(\prod_{\substack{j=1\\j\neq i}}^{r}\left\Vert A_j\left(x\right)\right\Vert_p\right)dx\right]. \tag{126}$$

When $r=2$ we obtain the following operator related Grüss type sequential fractional inequalities.

Theorem 38 *Let* $p,q>1:\frac{1}{p}+\frac{1}{q}=1$*, and let the* $*$*-ideals* $\mathcal{B}_p\left(H\right)$*,* $\mathcal{B}_q\left(H\right)$*, for which* $\left(\mathcal{B}_p\left(H\right),\left\Vert\cdot\right\Vert_p\right)$*,* $\left(\mathcal{B}_q\left(H\right),\left\Vert\cdot\right\Vert_q\right)$ *are Banach algebras;* $0<\alpha<1$*,* $A_1\in C^1\left(\left[a,b\right],\mathcal{B}_p\left(H\right)\right)$*,* $A_2\in C^1\left(\left[a,b\right],\mathcal{B}_q\left(H\right)\right)$*;* $g\in C^1\left(\left[a,b\right]\right)$*, strictly increasing, such that* $g^{-1}\in C^1\left(\left[g\left(a\right),g\left(b\right)\right]\right)$*. Assume* $\forall\ x_0\in\left[a,b\right]$ *that* $F_{ki}^{x_0}:=D_{x_0-;g}^{k\alpha}A_i$*, for* $k=1,2$*,* $i=1,2$*, fulfill* $F_{1,1}^{x_0}\in C^1\left(\left[a,x_0\right],\mathcal{B}_p\left(H\right)\right)$*,* $F_{2,1}^{x_0}\in C\left(\left[a,x_0\right],\mathcal{B}_p\left(H\right)\right)$*;* $F_{1,2}^{x_0}\in C^1\left(\left[a,x_0\right],\mathcal{B}_q\left(H\right)\right)$*,* $F_{2,2}^{x_0}\in C\left(\left[a,x_0\right],\mathcal{B}_q\left(H\right)\right)$*. Similarly, we assume* $\forall\ x_0\in\left[a,b\right]$ *that* $G_{ki}^{x_0}:=D_{x_0+;g}^{k\alpha}A_i$*,* $k=1,2$*,* $i=1,2$*, fulfill* $G_{1,1}^{x_0}\in C^1\left(\left[x_0,b\right],\mathcal{B}_p\left(H\right)\right)$*,* $G_{2,1}^{x_0}\in C\left(\left[x_0,b\right],\mathcal{B}_p\left(H\right)\right)$*,* $G_{1,2}^{x_0}\in C^1\left(\left[x_0,b\right],\mathcal{B}_q\left(H\right)\right)$*,* $G_{2,2}^{x_0}\in C\left(\left[x_0,b\right],\mathcal{B}_q\left(H\right)\right)$*. Here* $\Delta\left(A_1,A_2\right)$ *is as in (92),* $W_7\left(A_1,A_2\right)$ *is as in (112),* $W_8\left(A_1,A_2\right)$ *is as in (115), and* $W_9\left(A_1,A_2\right)$ *as in (118).*

Then

i) if $\gamma,\delta>1:\frac{1}{\gamma}+\frac{1}{\delta}=1$*;* $\frac{1}{2\delta}<\alpha<1$*, we have*

$$\left\Vert \Delta\left(A_1,A_2\right)\right\Vert_1\leq$$

$$\frac{W_7\left(A_1,A_2\right)\left(g\left(b\right)-g\left(a\right)\right)^{2\alpha-\frac{1}{\delta}}\left(b-a\right)}{\Gamma\left(2\alpha\right)\left(\gamma\left(2\alpha-1\right)+1\right)^{\frac{1}{\gamma}}}\left[\int_a^b\left(\left\Vert A_1\left(x\right)\right\Vert_p+\left\Vert A_2\left(x\right)\right\Vert_q\right)dx\right], \tag{127}$$

ii) if $\frac{1}{2}\leq\alpha<1$*, we have*

$$\left\Vert \Delta\left(A_1,A_2\right)\right\Vert_1\leq$$

$$\frac{W_8\left(A_1,A_2\right)\left(g\left(b\right)-g\left(a\right)\right)^{2\alpha-1}\left(b-a\right)}{\Gamma\left(2\alpha\right)}\left[\int_a^b\left(\left\Vert A_1\left(x\right)\right\Vert_p+\left\Vert A_2\left(x\right)\right\Vert_q\right)dx\right], \tag{128}$$

and

iii)

$$\left\Vert \Delta\left(A_1,A_2\right)\right\Vert_1\leq$$

$$\frac{W_9\left(A_1,A_2\right)\left(g\left(b\right)-g\left(a\right)\right)^{2\alpha}\left(b-a\right)}{\Gamma\left(2\alpha+1\right)}\left[\int_a^b\left(\left\Vert A_1\left(x\right)\right\Vert_p+\left\Vert A_2\left(x\right)\right\Vert_q\right)dx\right]. \tag{129}$$

6 Applications

We give the following sequential fractional Ostrowski inequalities:

Corollary 39 *(to Corollary 31) All as in Corollary 31, with $g(t) = e^t$. Then*

1) for $\gamma, \delta > 1 : \frac{1}{\gamma} + \frac{1}{\delta} = 1$; $\frac{1}{2\delta} < \alpha < 1$, we have

$$\|\Omega(A_1, A_2)(x_0)\|_1 \leq \frac{1}{\Gamma(2\alpha)(\gamma(2\alpha - 1) + 1)^{\frac{1}{\gamma}}}$$

$$\left\{ \left[\left\| \left\| \left(D^{2\alpha}_{x_0-;e^t} A_1 \right) \circ \log \right\|_p \right\|_{\delta, [e^a, e^{x_0}]} \int_a^{x_0} \|A_2(x)\|_q (e^{x_0} - e^x)^{2\alpha - \frac{1}{\delta}} dx \right] + \right. \quad (130)$$

$$\left[\left\| \left\| \left(D^{2\alpha}_{x_0+;e^t} A_1 \right) \circ \log \right\|_p \right\|_{\delta, [e^{x_0}, e^b]} \int_{x_0}^b \|A_2(x)\|_q (e^x - e^{x_0})^{2\alpha - \frac{1}{\delta}} dx \right] +$$

$$\left[\left\| \left\| \left(D^{2\alpha}_{x_0-;e^t} A_2 \right) \circ \log \right\|_q \right\|_{\delta, [e^a, e^{x_0}]} \int_a^{x_0} \|A_1(x)\|_p (e^{x_0} - e^x)^{2\alpha - \frac{1}{\delta}} dx \right] +$$

$$\left. \left[\left\| \left\| \left(D^{2\alpha}_{x_0+;e^t} A_2 \right) \circ \log \right\|_q \right\|_{\delta, [e^{x_0}, e^b]} \int_{x_0}^b \|A_1(x)\|_p (e^x - e^{x_0})^{2\alpha - \frac{1}{\delta}} dx \right] \right\},$$

2) if $\frac{1}{2} \leq \alpha < 1$, we have

$$\|\Omega(A_1, A_2)(x_0)\|_1 \leq \frac{1}{\Gamma(2\alpha)}$$

$$\left\{ \left[\left\| \left\| \left(D^{2\alpha}_{x_0-;e^t} A_1 \right) \circ \log \right\|_p \right\|_{L_1([e^a, e^{x_0}])} \int_a^{x_0} \|A_2(x)\|_q (e^{x_0} - e^x)^{2\alpha - 1} dx \right] + \right. \quad (131)$$

$$\left[\left\| \left\| \left(D^{2\alpha}_{x_0+;e^t} A_1 \right) \circ \log \right\|_p \right\|_{L_1([e^{x_0}, e^b])} \int_{x_0}^b \|A_2(x)\|_q (e^x - e^{x_0})^{2\alpha - 1} dx \right] +$$

$$\left[\left\| \left\| \left(D^{2\alpha}_{x_0-;e^t} A_2 \right) \circ \log \right\|_q \right\|_{L_1([e^a, e^{x_0}])} \int_a^{x_0} \|A_1(x)\|_p (e^{x_0} - e^x)^{2\alpha - 1} dx \right] +$$

$$\left. \left[\left\| \left\| \left(D^{2\alpha}_{x_0+;e^t} A_2 \right) \circ \log \right\|_q \right\|_{L_1([e^{x_0}, e^b])} \int_{x_0}^b \|A_1(x)\|_p (e^x - e^{x_0})^{2\alpha - 1} dx \right] \right\},$$

and

3)

$$\|\Omega(A_1, A_2)(x_0)\|_1 \leq \frac{1}{\Gamma(2\alpha + 1)}$$

$$\left\{ \left[\left\| \left\| \left(D^{2\alpha}_{x_0-;e^t} A_1 \right) \circ \log \right\|_p \right\|_{\infty, [e^a, e^{x_0}]} \int_a^{x_0} \|A_2(x)\|_q (e^{x_0} - e^x)^{2\alpha} dx \right] + \right. \quad (132)$$

$$\left[\left\| \left\| \left(D^{2\alpha}_{x_0+;e^t} A_1 \right) \circ \log \right\|_p \right\|_{\infty, [e^{x_0}, e^b]} \int_{x_0}^b \|A_2(x)\|_q (e^x - e^{x_0})^{2\alpha} dx \right] +$$

$$\left[\left\|\left\|\left(D^{2\alpha}_{x_0-;e^t}A_2\right)\circ\log\right\|_q\right\|_{\infty,[e^a,e^{x_0}]}\int_a^{x_0}\|A_1(x)\|_p\left(e^{x_0}-e^x\right)^{2\alpha}dx\right]+$$

$$\left.\left[\left\|\left\|\left(D^{2\alpha}_{x_0+;e^t}A_2\right)\circ\log\right\|_q\right\|_{\infty,[e^{x_0},e^b]}\int_{x_0}^{b}\|A_1(x)\|_p\left(e^x-e^{x_0}\right)^{2\alpha}dx\right]\right\}.$$

We continue with

Corollary 40 *(to Corollary 31) All as in Corollary 31, with $g(t)=t$. Then*

1) for $\gamma,\delta>1:\frac{1}{\gamma}+\frac{1}{\delta}=1$; $\frac{1}{2\delta}<\alpha<1$, we have

$$\|\Omega(A_1,A_2)(x_0)\|_1\leq\frac{1}{\Gamma(2\alpha)\left(\gamma(2\alpha-1)+1\right)^{\frac{1}{\gamma}}}$$

$$\left\{\left[\left\|\left\|D^{2\alpha}_{x_0-}A_1\right\|_p\right\|_{\delta,[a,x_0]}\int_a^{x_0}\|A_2(x)\|_q(x_0-x)^{2\alpha-\frac{1}{\delta}}dx\right]+\right. \tag{133}$$

$$\left[\left\|\left\|D^{2\alpha}_{*x_0}A_1\right\|_p\right\|_{\delta,[x_0,b]}\int_{x_0}^{b}\|A_2(x)\|_q(x-x_0)^{2\alpha-\frac{1}{\delta}}dx\right]+$$

$$\left[\left\|\left\|D^{2\alpha}_{x_0-}A_2\right\|_q\right\|_{\delta,[a,x_0]}\int_a^{x_0}\|A_1(x)\|_p(x_0-x)^{2\alpha-\frac{1}{\delta}}dx\right]+$$

$$\left.\left[\left\|\left\|D^{2\alpha}_{*x_0}A_2\right\|_q\right\|_{\delta,[x_0,b]}\int_{x_0}^{b}\|A_1(x)\|_p(x-x_0)^{2\alpha-\frac{1}{\delta}}dx\right]\right\},$$

2) if $\frac{1}{2}\leq\alpha<1$, we have

$$\|\Omega(A_1,A_2)(x_0)\|_1\leq\frac{1}{\Gamma(2\alpha)}$$

$$\left\{\left[\left\|\left\|D^{2\alpha}_{x_0-}A_1\right\|_p\right\|_{L_1([a,x_0])}\int_a^{x_0}\|A_2(x)\|_q(x_0-x)^{2\alpha-1}dx\right]+\right. \tag{134}$$

$$\left[\left\|\left\|D^{2\alpha}_{*x_0}A_1\right\|_p\right\|_{L_1([x_0,b])}\int_{x_0}^{b}\|A_2(x)\|_q(x-x_0)^{2\alpha-1}dx\right]+$$

$$\left[\left\|\left\|D^{2\alpha}_{x_0-}A_2\right\|_q\right\|_{L_1([a,x_0])}\int_a^{x_0}\|A_1(x)\|_p(x_0-x)^{2\alpha-1}dx\right]+$$

$$\left.\left[\left\|\left\|D^{2\alpha}_{*x_0}A_2\right\|_q\right\|_{L_1([x_0,b])}\int_{x_0}^{b}\|A_1(x)\|_p(x-x_0)^{2\alpha-1}dx\right]\right\},$$

and

3)

$$\|\Omega(A_1,A_2)(x_0)\|_1\leq\frac{1}{\Gamma(2\alpha+1)}$$

$$\left\{\left[\left\|\left\|D_{x_0-}^{2\alpha}A_1\right\|_p\right\|_{\infty,[a,x_0]}\int_a^{x_0}\left\|A_2(x)\right\|_q(x_0-x)^{2\alpha}dx\right]+\right. \tag{135}$$

$$\left[\left\|\left\|D_{*x_0}^{2\alpha}A_1\right\|_p\right\|_{\infty,[x_0,b]}\int_{x_0}^{b}\left\|A_2(x)\right\|_q(x-x_0)^{2\alpha}dx\right]+$$

$$\left[\left\|\left\|D_{x_0-}^{2\alpha}A_2\right\|_q\right\|_{\infty,[a,x_0]}\int_a^{x_0}\left\|A_1(x)\right\|_p(x_0-x)^{2\alpha}dx\right]+$$

$$\left.\left[\left\|\left\|D_{*x_0}^{2\alpha}A_2\right\|_q\right\|_{\infty,[x_0,b]}\int_{x_0}^{b}\left\|A_1(x)\right\|_p(x-x_0)^{2\alpha}dx\right]\right\}.$$

Next come sequential fractional Grüss inequalities:

Corollary 41 *(to Theorem 38) All as in Theorem 38, with $g(t)=e^t$. Then*
1) if $\gamma,\delta>1:\frac{1}{\gamma}+\frac{1}{\delta}=1$, $\frac{1}{2\delta}<\alpha<1$, we have

$$\left\|\Delta(A_1,A_2)\right\|_1\leq\frac{W_7(A_1,A_2)\left(e^b-e^a\right)^{2\alpha-\frac{1}{\delta}}(b-a)}{\Gamma(2\alpha)\left(\gamma(2\alpha-1)+1\right)^{\frac{1}{\gamma}}}$$

$$\left[\int_a^b\left(\left\|A_1(x)\right\|_p+\left\|A_2(x)\right\|_q\right)dx\right], \tag{136}$$

2) if $\frac{1}{2}\leq\alpha<1$, we have

$$\left\|\Delta(A_1,A_2)\right\|_1\leq\frac{W_8(A_1,A_2)\left(e^b-e^a\right)^{2\alpha-1}(b-a)}{\Gamma(2\alpha)}$$

$$\left[\int_a^b\left(\left\|A_1(x)\right\|_p+\left\|A_2(x)\right\|_q\right)dx\right], \tag{137}$$

and
3)

$$\left\|\Delta(A_1,A_2)\right\|_1\leq\frac{W_9(A_1,A_2)\left(e^b-e^a\right)^{2\alpha}(b-a)}{\Gamma(2\alpha+1)}$$

$$\left[\int_a^b\left(\left\|A_1(x)\right\|_p+\left\|A_2(x)\right\|_q\right)dx\right], \tag{138}$$

We finish with

Corollary 42 *(to Theorem 38) All as in Theorem 38, with $g(t)=t$. Then*
1) if $\gamma,\delta>1:\frac{1}{\gamma}+\frac{1}{\delta}=1$, $\frac{1}{2\delta}<\alpha<1$, we have

$$\left\|\Delta(A_1,A_2)\right\|_1\leq\frac{W_7(A_1,A_2)(b-a)^{2\alpha+\frac{1}{\gamma}}}{\Gamma(2\alpha)\left(\gamma(2\alpha-1)+1\right)^{\frac{1}{\gamma}}}$$

$$\left[\int_a^b \left(\|A_1(x)\|_p + \|A_2(x)\|_q\right) dx\right], \tag{139}$$

2) if $\frac{1}{2} \leq \alpha < 1$*, we have*

$$\|\Delta(A_1, A_2)\|_1 \leq \frac{W_8(A_1, A_2)(b-a)^{2\alpha}}{\Gamma(2\alpha)}$$

$$\left[\int_a^b \left(\|A_1(x)\|_p + \|A_2(x)\|_q\right) dx\right], \tag{140}$$

and

3)

$$\|\Delta(A_1, A_2)\|_1 \leq \frac{W_9(A_1, A_2)(b-a)^{2\alpha+1}}{\Gamma(2\alpha+1)}$$

$$\left[\int_a^b \left(\|A_1(x)\|_p + \|A_2(x)\|_q\right) dx\right]. \tag{141}$$

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