HERMITE-HADAMARD TYPE INEQUALITIES FOR TWICE DIFFERENTIABLE CONVEX FUNCTIONS

SILVESTRU SEVER DRAGOMIR[1,2]

Abstract. In this paper, we show among others that, if $f,\ g,\ h:[a,b]\subset\mathbb{R}\to\mathbb{R}$ are twice differentiable functions on (a,b) with $g''(t)\le f''(t)\le h''(t)$ for almost every $t\in(a,b)$, then

$$\frac{1}{2}\left[g(x)+\frac{g(b)(b-x)+g(a)(x-a)}{b-a}\right]-\frac{1}{b-a}\int_a^b g(t)\,dt$$
$$\le\frac{1}{2}\left[f(x)+\frac{f(b)(b-x)+f(a)(x-a)}{b-a}\right]-\frac{1}{b-a}\int_a^b f(t)\,dt$$
$$\le\frac{1}{2}\left[h(x)+\frac{h(b)(b-x)+h(a)(x-a)}{b-a}\right]-\frac{1}{b-a}\int_a^b h(t)\,dt$$

for all $x\in(a,b)$. Applications for logarithm and power functions are also given.

1. Introduction

The following inequality holds for any convex function f defined on $\mathbb{R}$

$$f\left(\frac{a+b}{2}\right)\le\frac{1}{b-a}\int_a^b f(t)dt\le\frac{f(a)+f(b)}{2},\quad a,\ b\in\mathbb{R},\ a<b. \tag{1.1}$$

It was firstly discovered by Ch. Hermite in 1881 in the journal *Mathesis* (see [6]). But this result was nowhere mentioned in the mathematical literature and was not widely known as Hermite's result.

E. F. Beckenbach, a leading expert on the history and the theory of convex functions, wrote that this inequality was proven by J. Hadamard in 1893 [1]. In 1974, D. S. Mitrinović found Hermite's note in *Mathesis* [6]. Since (1.1) was known as Hadamard's inequality, the inequality is now commonly referred as the *Hermite-Hadamard inequality.* For a monograph devoted to this result see [5]. The recent survey paper [4] provides other related results.

Let $f:[a,b]\to\mathbb{R}$ be a convex function on $[a,b]$ and assume that $f'_+(a)$ and $f'_-(b)$ are finite. We recall the following improvement and reverse inequality for the first Hermite-Hadamard result that has been established in [2]

$$0\le\frac{1}{8}\left[f'_+\left(\frac{a+b}{2}\right)-f'_-\left(\frac{a+b}{2}\right)\right](b-a)$$
$$\le\frac{1}{b-a}\int_a^b f(t)\,dt-f\left(\frac{a+b}{2}\right)\le\frac{1}{8}(b-a)\left[f'_-(b)-f'_+(a)\right]. \tag{1.2}$$

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The following inequality that provides a reverse and improvement of the second Hermite-Hadamard result has been obtained in [3]

$$0 \le \frac{1}{8}\left[f'_+\left(\frac{a+b}{2}\right) - f'_-\left(\frac{a+b}{2}\right)\right](b-a) \le \frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(t)\,dt \le \frac{1}{8}(b-a)\left[f'_-(b) - f'_+(a)\right]. \tag{1.3}$$

The constant $\frac{1}{8}$ is best possible in both (1.2) and (1.3).

Motivated by the above results, in this paper, we show among others that, if $f,\ g,\ h : [a,b] \subset \mathbb{R} \to \mathbb{R}$ are twice differentiable functions on (a,b) with $g''(t) \le f''(t) \le h''(t)$ for almost every $t \in (a,b)$, then

$$\begin{aligned} &\frac{1}{2}\left[g(x) + \frac{g(b)(b-x) + g(a)(x-a)}{b-a}\right] - \frac{1}{b-a}\int_a^b g(t)\,dt \\ &\le \frac{1}{2}\left[f(x) + \frac{f(b)(b-x) + f(a)(x-a)}{b-a}\right] - \frac{1}{b-a}\int_a^b f(t)\,dt \\ &\le \frac{1}{2}\left[h(x) + \frac{h(b)(b-x) + h(a)(x-a)}{b-a}\right] - \frac{1}{b-a}\int_a^b h(t)\,dt \end{aligned}$$

for all $x \in (a,b)$. Applications for logarithm and power functions are also given.

2. Main Results

The following theorem is well known in the literature as Taylor's formula or Taylor's theorem with the integral remainder.

Lemma 1. *Let $I \subset \mathbb{R}$ be a closed interval, $a \in I$ and let n be a positive integer. If $g : I \longrightarrow \mathbb{C}$ is such that the n-derivative $g^{(n)}$ is absolutely continuous on I, then for each $x \in I$*

$$g(x) = T_n(g;a,x) + R_n(g;a,x), \tag{2.1}$$

where $T_n(g;c,y)$ is Taylor's polynomial, i.e.,

$$T_n(g;a,x) := \sum_{k=0}^{n} \frac{(x-a)^k}{k!} g^{(k)}(a). \tag{2.2}$$

Note that $g^{(0)} := g$ and $0! := 1$ and the remainder is given by

$$R_n(g;a,x) := \frac{1}{n!}\int_a^x (x-t)^n g^{(n+1)}(t)\,dt. \tag{2.3}$$

A simple proof of this result can be achieved by mathematical induction using the integration by parts formula in the Lebesgue integral.

For any integrable function h on an interval and any distinct numbers c, d in that interval, we have, by the change of variable $t = (1-s)c + sd$, $s \in [0,1]$ that

$$\int_c^d h(t)\,dt = (d-c)\int_0^1 h((1-s)c + sd)\,ds.$$

Therefore,

$$\begin{aligned}&\int_a^x g^{(n+1)}(t)(x-t)^n\,dt\\&=(x-a)\int_0^1 g^{(n+1)}((1-s)a+sx)(x-(1-s)a-sx)^n\,ds\\&=(x-a)^{n+1}\int_0^1 g^{(n+1)}((1-s)a+sx)(1-s)^n\,ds.\end{aligned}$$

The identity (2.1) can then be written as

$$\begin{aligned}g(x)&=\sum_{k=0}^{n}\frac{1}{k!}g^{(k)}(a)(x-a)^k\\&\quad+\frac{1}{n!}(x-a)^{n+1}\int_0^1 g^{(n+1)}((1-s)a+sx)(1-s)^n\,ds\end{aligned}\tag{2.4}$$

for all $x,\ a\in I$.

We have the following result concerning lower and upper bounds for the Jensen's gap:

Theorem 1. *Let $f,\ g:[a,b]\subset\mathbb{R}\to\mathbb{R}$ be twice differentiable functions on (a,b) with $g''(t)\le f''(t)$ for almost every $t\in(a,b)$, then we have the inequality*

$$\begin{aligned}&\frac{1}{b-a}\int_a^b g(t)\,dt-g(x)-g'(x)\left(\frac{a+b}{2}-x\right)\\&\le\frac{1}{b-a}\int_a^b f(t)\,dt-f(x)-f'(x)\left(\frac{a+b}{2}-x\right)\end{aligned}\tag{2.5}$$

for all $x\in(a,b)$.

In particular,

$$\frac{1}{b-a}\int_a^b g(t)\,dt-g\left(\frac{a+b}{2}\right)\le\frac{1}{b-a}\int_a^b f(t)\,dt-f\left(\frac{a+b}{2}\right).\tag{2.6}$$

Proof. We have from (2.4) for $n=2$ that

$$f(t)=f(x)+f'(x)(t-x)+(t-x)^2\int_0^1 f''((1-s)x+st)(1-s)\,ds\tag{2.7}$$

for all $t,\ x\in(a,b)$, where f is such that f' is absolutely continuos on $[a,b]$.

Since $g''(t)\le f''(t)$ for almost every $t\in(a,b)$, then

$$g''((1-s)x+st)\le f''((1-s)x+st)$$

for almost every $t\in(a,b)$ and all $s\in[0,1]$.

If we multiply by $1-s\ge0$ and integrate, then we get

$$\int_0^1 g''((1-s)x+st)(1-s)\,ds\le\int_0^1 f''((1-s)x+st)(1-s)\,ds,$$

which implies, by multiplying with $(t-x)^2\ge0$ that

$$\begin{aligned}&(t-x)^2\int_0^1 g''((1-s)x+st)(1-s)\,ds\\&\le(t-x)^2\int_0^1 f''((1-s)x+st)(1-s)\,ds,\end{aligned}\tag{2.8}$$

for all $t, x \in (a, b)$.

By utilizing the representation (2.7) we then get

$$g(t) - g(x) - g'(x)(t - x) \le f(t) - f(x) - f'(x)(t - x) \tag{2.9}$$

for all $t, x \in (a, b)$.

Now, if we take the integral over $t \in [a, b]$, we get

$$\begin{aligned} &\int_a^b g(t)\,dt - g(x)(b - a) - g'(x)\left(\frac{1}{2}\left(b^2 - a^2\right) - x(b - a)\right) \\ &\le \int_a^b f(t)\,dt - f(x)(b - a) - f'(x)\left(\frac{1}{2}\left(b^2 - a^2\right) - x(b - a)\right) \end{aligned} \tag{2.10}$$

and by division with $b - a > 0$ we get the desired result (2.5). □

Corollary 1. *Let $f, g, h : [a, b] \subset \mathbb{R} \to \mathbb{R}$ be twice differentiable functions on (a, b) with $g''(t) \le f''(t) \le h''(t)$ for almost every $t \in (a, b)$, then we have the inequalities*

$$\begin{aligned} &\frac{1}{b-a}\int_a^b g(t)\,dt - g(x) - g'(x)\left(\frac{a+b}{2} - x\right) \\ &\le \frac{1}{b-a}\int_a^b f(t)\,dt - f(x) - f'(x)\left(\frac{a+b}{2} - x\right) \\ &\le \frac{1}{b-a}\int_a^b h(t)\,dt - h(x) - h'(x)\left(\frac{a+b}{2} - x\right) \end{aligned} \tag{2.11}$$

for all $x \in (a, b)$.

In particular,

$$\begin{aligned} \frac{1}{b-a}\int_a^b g(t)\,dt - g\left(\frac{a+b}{2}\right) &\le \frac{1}{b-a}\int_a^b f(t)\,dt - f\left(\frac{a+b}{2}\right) \\ &\le \frac{1}{b-a}\int_a^b h(t)\,dt - h\left(\frac{a+b}{2}\right). \end{aligned} \tag{2.12}$$

We also have:

Theorem 2. *Let $f, g : [a, b] \subset \mathbb{R} \to \mathbb{R}$ be twice differentiable functions on (a, b) with $g''(t) \le f''(t)$ for almost every $t \in (a, b)$, then we have the inequality*

$$\begin{aligned} &\frac{1}{2}\left[g(x) + \frac{g(b)(b - x) + g(a)(x - a)}{b - a}\right] - \frac{1}{b-a}\int_a^b g(t)\,dt \\ &\le \frac{1}{2}\left[f(x) + \frac{f(b)(b - x) + f(a)(x - a)}{b - a}\right] - \frac{1}{b-a}\int_a^b f(t)\,dt \end{aligned} \tag{2.13}$$

for all $x \in (a, b)$.

In particular,

$$\begin{aligned} &\frac{1}{2}\left[g\left(\frac{a+b}{2}\right) + \frac{g(b) + g(a)}{2}\right] - \frac{1}{b-a}\int_a^b g(t)\,dt \\ &\le \frac{1}{2}\left[f\left(\frac{a+b}{2}\right) + \frac{f(b) + f(a)}{2}\right] - \frac{1}{b-a}\int_a^b f(t)\,dt. \end{aligned} \tag{2.14}$$

Proof. If we replace t with x in (2.9) we have

$$g(x) - g(t) - g'(t)(x-t) \le f(x) - f(t) - f'(t)(x-t) \tag{2.15}$$

for all x, $t \in (a,b)$.

If we integrate (2.16) over t on $[a,b]$, then we get

$$\begin{aligned}&(b-a)\,g(x) - \int_a^b g(t)\,dt - \int_a^b g'(t)(x-t)\,dt \\ &\le (b-a)\,f(x) - \int_a^b f(t)\,dt - \int_a^b f'(t)(x-t)\,dt.\end{aligned} \tag{2.16}$$

Using integration by parts, we have

$$\begin{aligned}\int_a^b g'(t)(x-t)\,dt &= g(t)(x-t)\big|_a^b + \int_a^b g(t)\,dt \\ &= g(b)(x-b) - g(a)(x-a) + \int_a^b g(t)\,dt \\ &= -g(b)(b-x) - g(a)(x-a) + \int_a^b g(t)\,dt.\end{aligned}$$

Therefore

$$\begin{aligned}&(b-a)\,g(x) - \int_a^b g(t)\,dt - \int_a^b g'(t)(x-t)\,dt \\ &= (b-a)\,g(x) + g(b)(b-x) + g(a)(x-a) - 2\int_a^b g(t)\,dt\end{aligned}$$

for all $x \in (a,b)$.

Also,

$$\begin{aligned}&(b-a)\,f(x) - \int_a^b f(t)\,dt - \int_a^b f'(t)(x-t)\,dt \\ &= (b-a)\,f(x) + f(b)(b-x) + f(a)(x-a) - 2\int_a^b f(t)\,dt\end{aligned}$$

for all $x \in (a,b)$.

By utilizing (2.16) we get

$$\begin{aligned}&(b-a)\,g(x) + g(b)(b-x) + g(a)(x-a) - 2\int_a^b g(t)\,dt \\ &\le (b-a)\,f(x) + f(b)(b-x) + f(a)(x-a) - 2\int_a^b f(t)\,dt\end{aligned}$$

and by division with $2(b-a)$ we get (2.13). □

Corollary 2. *Let $f, g, h : [a,b] \subset \mathbb{R} \to \mathbb{R}$ be twice differentiable functions on (a,b) with $g''(t) \le f''(t) \le h''(t)$ for almost every $t \in (a,b)$, then we have the inequalities*

$$\begin{aligned}(2.17)\qquad & \frac{1}{2}\left[g(x) + \frac{g(b)(b-x) + g(a)(x-a)}{b-a}\right] - \frac{1}{b-a}\int_a^b g(t)\,dt \\ & \le \frac{1}{2}\left[f(x) + \frac{f(b)(b-x) + f(a)(x-a)}{b-a}\right] - \frac{1}{b-a}\int_a^b f(t)\,dt \\ & \le \frac{1}{2}\left[h(x) + \frac{h(b)(b-x) + h(a)(x-a)}{b-a}\right] - \frac{1}{b-a}\int_a^b h(t)\,dt\end{aligned}$$

for all $x \in (a,b)$.

In particular,

$$\begin{aligned}(2.18)\qquad & \frac{1}{2}\left[g\left(\frac{a+b}{2}\right) + \frac{g(b)+g(a)}{2}\right] - \frac{1}{b-a}\int_a^b g(t)\,dt \\ & \le \frac{1}{2}\left[f\left(\frac{a+b}{2}\right) + \frac{f(b)+f(a)}{2}\right] - \frac{1}{b-a}\int_a^b f(t)\,dt \\ & \le \frac{1}{2}\left[h\left(\frac{a+b}{2}\right) + \frac{h(b)+h(a)}{2}\right] - \frac{1}{b-a}\int_a^b h(t)\,dt.\end{aligned}$$

3. Examples for Means

We recall the following means:

a) The *arithmetic mean*

$$A(a,b) := \frac{a+b}{2}, \quad a,b > 0,$$

b) The *geometric mean*

$$G(a,b) := \sqrt{ab}; \quad a,b \ge 0,$$

c) The *harmonic mean*

$$H(a,b) := \frac{2}{\frac{1}{a} + \frac{1}{b}}; \quad a,b > 0,$$

d) The *identric mean*

$$I(a,b) := \begin{cases} \frac{1}{e}\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}} & \text{if } b \ne a \\ a & \text{if } b = a \end{cases}; \quad a,b > 0$$

e) The *logarithmic mean*

$$L(a,b) := \begin{cases} \frac{b-a}{\ln b - \ln a} & \text{if } b \ne a \\ a & \text{if } b = a \end{cases}; \quad a,b > 0$$

f) The *p-logarithmic mean*

$$L_p(a,b) := \begin{cases} \left(\dfrac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right)^{\frac{1}{p}} & \text{if} \quad b\neq a, \quad p\in\mathbb{R}\backslash\{-1,0\} \\ a & \text{if} \quad b=a \end{cases} ; \quad a,b>0.$$

It is well known that, if $L_{-1} := L$ and $L_0 := I$, then the function $\mathbb{R} \ni p \to L_p$ is monotonically strictly increasing. In particular, we have

$$H(a,b) \le G(a,b) \le L(a,b) \le I(a,b) \le A(a,b).$$

Consider the convex functions $g(x) = \frac{k}{p(p-1)}x^p$ and $h(x) = \frac{K}{p(p-1)}x^p$ with $p \in (-\infty,0)\cup(1,\infty)$, where $0 \le k < K$ and $x \in (a,b) \subset (0,\infty)$. Assume that f is twice differentiable on (a,b) and

$$0 \le kx^{p-2} \le f''(x) \le Kx^{p-2} \text{ for a.e. } x\in(a,b), \tag{3.1}$$

then by (2.12) we get

$$\begin{aligned} 0 &\le \frac{k}{p(p-1)}\left[\frac{1}{b-a}\int_a^b t^p dt - \left(\frac{a+b}{2}\right)^p\right] \\ &\le \frac{1}{b-a}\int_a^b f(t)\,dt - f\left(\frac{a+b}{2}\right) \\ &\le \frac{K}{p(p-1)}\left[\frac{1}{b-a}\int_a^b t^p dt - \left(\frac{a+b}{2}\right)^p\right]. \end{aligned} \tag{3.2}$$

If $p \neq -1$, then

$$\frac{1}{b-a}\int_a^b t^p dt = \frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)} = [L_p(a,b)]^p$$

and by (3.2) we get

$$\begin{aligned} 0 &\le \frac{k}{p(p-1)}\left([L_p(a,b)]^p - [A(a,b)]^p\right) \le \frac{1}{b-a}\int_a^b f(t)\,dt - f\left(\frac{a+b}{2}\right) \\ &\le \frac{K}{p(p-1)}\left([L_p(a,b)]^p - [A(a,b)]^p\right). \end{aligned} \tag{3.3}$$

For $p=2$ we obtain from (3.3) that

$$0 \le \frac{k}{24}(b-a)^2 \le \frac{1}{b-a}\int_a^b f(t)\,dt - f\left(\frac{a+b}{2}\right) \le \frac{K}{24}(b-a)^2, \tag{3.4}$$

provided that

$$0 \le k \le f''(x) \le K \text{ for a.e. } x\in(a,b). \tag{3.5}$$

For $p=-1$ we obtain from (3.2) that

$$\begin{aligned} 0 &\le \frac{k}{2}\left([L(a,b)]^{-1} - [A(a,b)]^{-1}\right) \le \frac{1}{b-a}\int_a^b f(t)\,dt - f\left(\frac{a+b}{2}\right) \\ &\le \frac{K}{2}\left([L(a,b)]^{-1} - [A(a,b)]^{-1}\right), \end{aligned} \tag{3.6}$$

provided that

$$0 \le kx^{-3} \le f''(x) \le Kx^{-3} \text{ for a.e. } x\in(a,b). \tag{3.7}$$

If f satisfies the condition (3.1), then by (2.18) we get

$$0 \leq \frac{k}{p(p-1)}\left\{\frac{1}{2}\left[\left(\frac{a+b}{2}\right)^{p}+\frac{b^{p}+a^{p}}{2}\right]-\frac{1}{b-a}\int_{a}^{b}t^{p}dt\right\} \tag{3.8}$$
$$\leq \frac{1}{2}\left[f\left(\frac{a+b}{2}\right)+\frac{f(b)+f(a)}{2}\right]-\frac{1}{b-a}\int_{a}^{b}f(t)\,dt$$
$$\leq \frac{K}{p(p-1)}\left\{\frac{1}{2}\left[\left(\frac{a+b}{2}\right)^{p}+\frac{b^{p}+a^{p}}{2}\right]-\frac{1}{b-a}\int_{a}^{b}t^{p}dt\right\}.$$

If $p \neq -1$, then

$$0 \leq \frac{k}{p(p-1)}\left\{\frac{1}{2}\left([A(a,b)]^{p}+A(a^{p},b^{p})\right)-[L_{p}(a,b)]^{p}\right\} \tag{3.9}$$
$$\leq \frac{1}{2}\left[f\left(\frac{a+b}{2}\right)+\frac{f(b)+f(a)}{2}\right]-\frac{1}{b-a}\int_{a}^{b}f(t)\,dt$$
$$\leq \frac{K}{p(p-1)}\left\{\frac{1}{2}\left([A(a,b)]^{p}+A(a^{p},b^{p})\right)-[L_{p}(a,b)]^{p}\right\}.$$

For $p=2$ we get

$$0 \leq \frac{k}{48}(b-a)^{2} \tag{3.10}$$
$$\leq \frac{1}{2}\left[f\left(\frac{a+b}{2}\right)+\frac{f(b)+f(a)}{2}\right]-\frac{1}{b-a}\int_{a}^{b}f(t)\,dt$$
$$\leq \frac{K}{48}(b-a)^{2},$$

provided that f satisfies the condition (3.5).

For $p=-1$ we obtain from (3.2) that

$$0 \leq \frac{k}{2}\left\{\frac{1}{2}\left[[A(a,b)]^{-1}+[H(a,b)]^{-1}\right]-[L(a,b)]^{-1}\right\} \tag{3.11}$$
$$\leq \frac{1}{2}\left[f\left(\frac{a+b}{2}\right)+\frac{f(b)+f(a)}{2}\right]-\frac{1}{b-a}\int_{a}^{b}f(t)\,dt$$
$$\leq \frac{K}{2}\left\{\frac{1}{2}\left[[A(a,b)]^{-1}+[H(a,b)]^{-1}\right]-[L(a,b)]^{-1}\right\},$$

provided that f satisfies the condition (3.5).

Consider the convex functions $g(x)=-k\ln x$ and $h(x)=-K\ln x$, where $0<k<K$ and $x\in(m,M)\subset(0,\infty)$. Assume that f is twice differentiable on (m,M) and

$$0 \leq kx^{-2} \leq f''(x) \leq Kx^{-2} \text{ for a.e. } x\in(m,M),$$

then by (2.12) we get

$$
\begin{aligned}
0 &\leq k\left[\ln\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b \ln t dt\right] \\
&\leq \frac{1}{b-a}\int_a^b f(t)\,dt - f\left(\frac{a+b}{2}\right) \\
&\leq K\left[\ln\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b \ln t dt\right]. \qquad (3.12)
\end{aligned}
$$

Observe that

$$
\frac{1}{b-a}\int_a^b \ln t dt = \frac{b\ln b - b - a\ln a + a}{b-a} = \ln I(a,b)
$$

and by (3.12) we get

$$
0 \leq \ln\left(\frac{A(a,b)}{I(a,b)}\right)^k \leq \frac{1}{b-a}\int_a^b f(t)\,dt - f\left(\frac{a+b}{2}\right) \leq \ln\left(\frac{A(a,b)}{I(a,b)}\right)^K,
$$

which is equivalent to

$$
\begin{aligned}
1 &\leq \left(\frac{A(a,b)}{I(a,b)}\right)^k \leq \exp\left[\frac{1}{b-a}\int_a^b f(t)\,dt - f\left(\frac{a+b}{2}\right)\right] \\
&\leq \left(\frac{A(a,b)}{I(a,b)}\right)^K. \qquad (3.13)
\end{aligned}
$$

Further, if we use (2.18), then we get

$$
\begin{aligned}
0 &\leq k\left\{\frac{1}{b-a}\int_a^b \ln t dt - \frac{1}{2}\left[\ln\left(\frac{a+b}{2}\right)+\frac{\ln(b)+\ln(a)}{2}\right]\right\} \\
&\leq \frac{1}{2}\left[f\left(\frac{a+b}{2}\right)+\frac{f(b)+f(a)}{2}\right]-\frac{1}{b-a}\int_a^b f(t)\,dt \\
&\leq K\left\{\frac{1}{b-a}\int_a^b \ln t dt - \frac{1}{2}\left[\ln\left(\frac{a+b}{2}\right)+\frac{\ln(b)+\ln(a)}{2}\right]\right\},
\end{aligned}
$$

namely

$$
\begin{aligned}
0 &\leq k\left\{\ln I(a,b) - \ln\left(\sqrt{A(a,b)\,G(a,b)}\right)\right\} \\
&\leq \frac{1}{2}\left[f\left(\frac{a+b}{2}\right)+\frac{f(b)+f(a)}{2}\right]-\frac{1}{b-a}\int_a^b f(t)\,dt \\
&\leq K\left\{\ln I(a,b) - \ln\left(\sqrt{A(a,b)\,G(a,b)}\right)\right\},
\end{aligned}
$$

which is equivalent to

$$\begin{aligned}
1 &\leq \left(\frac{I(a,b)}{\sqrt{A(a,b)\,G(a,b)}}\right)^{k} \\
&\leq \exp\left\{\frac{1}{2}\left[f\left(\frac{a+b}{2}\right)+\frac{f(b)+f(a)}{2}\right]-\frac{1}{b-a}\int_a^b f(t)\,dt\right\} \\
&\leq \left(\frac{I(a,b)}{\sqrt{A(a,b)\,G(a,b)}}\right)^{K}.
\end{aligned}$$

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[1]Mathematics, College of Engineering & Science, Victoria University, PO Box 14428, Melbourne City, MC 8001, Australia.

E-mail address: sever.dragomir@vu.edu.au

URL: http://rgmia.org/dragomir

[2]DST-NRF Centre of Excellence in the Mathematical, and Statistical Sciences, School of Computer Science, & Applied Mathematics, University of the Witwatersrand,, Johannesburg, South Africa.