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ESTIMATES OF JENSEN'S GAP FOR TWICE DIFFERENTIABLE CONVEX FUNCTIONS

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Abstract. Let $(\Omega, \mathcal{A}, \mu)$ be a measurable space consisting of a set Ω, a σ-algebra $\mathcal{A}$ of parts of Ω and a countably additive and positive measure μ on $\mathcal{A}$ with values in $\mathbb{R} \cup \{\infty\}$. Assume, for simplicity, that $\int_\Omega d\mu = 1$. In this paper we show among others that, if Φ, Ψ, $\Gamma : [m, M] \subset \mathbb{R} \to \mathbb{R}$ are twice differentiable function on (m, M) with $\Psi''(x) \le \Phi''(x) \le \Gamma''(x)$ for almost every $x \in (m, M)$ and $f : \Omega \to [m, M]$ so that f, f^2, $\Phi \circ f$, $\Phi' \circ f$, $\Psi \circ f$, $\Psi' \circ f$, $\Gamma \circ f$, $\Gamma' \circ f \in L(\Omega, \mu)$, then we have the inequalities

$$\int_\Omega \Psi \circ f d\mu - \Psi\left(\int_\Omega f d\mu\right) \le \int_\Omega \Phi \circ f d\mu - \Phi\left(\int_\Omega f d\mu\right) \le \int_\Omega \Gamma \circ f d\mu - \Gamma\left(\int_\Omega f d\mu\right).$$

Applications for exponential, logarithm and power functions are also given.

1. Introduction

Let $(\Omega, \mathcal{A}, \mu)$ be a measurable space consisting of a set Ω, a σ-algebra $\mathcal{A}$ of parts of Ω and a countably additive and positive measure μ on $\mathcal{A}$ with values in $\mathbb{R} \cup \{\infty\}$. Assume, for simplicity, that $\int_\Omega d\mu(t) = 1$. Consider the *Lebesgue space*

$$L(\Omega, \mu) := \{f : \Omega \to \mathbb{R},\ f \text{ is } \mu\text{-measurable and } \int_\Omega |f(t)|\, d\mu(t) < \infty\}.$$

For simplicity of notation we write everywhere in the sequel $\int_\Omega w d\mu$ instead of $\int_\Omega w(t)\, d\mu(t)$.

In order to provide a reverse of the celebrated Jensen's integral inequality for convex functions, the author obtained in [6] and [9] the following result:

Theorem 1. *Let $\Phi : I \to \mathbb{R}$ be a continuous convex function on the interval of real numbers I and $m, M \in \mathbb{R}$, $m < M$ with $[m, M] \subset \mathring{I}$, $\mathring{I}$ the interior of I. If $f : \Omega \to [m, M]$ is so that $\Phi \circ f$, f, $\Phi' \circ f$, $(\Phi' \circ f) f \in L(\Omega, \mu)$, then we have the*

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RGMIA

inequality:

$$0 \leq \int_\Omega \Phi \circ f d\mu - \Phi\left(\int_\Omega f d\mu\right) \leq \int_\Omega (\Phi' \circ f) f d\mu - \int_\Omega \Phi' \circ f d\mu \int_\Omega f d\mu \tag{1.1}$$
$$\leq \frac{1}{2}\left[\Phi'_-(M) - \Phi'_+(m)\right] \int_\Omega \left| f - \int_\Omega f d\mu \right| d\mu$$
$$\leq \frac{1}{2}\left[\Phi'_-(M) - \Phi'_+(m)\right] \left[\int_\Omega f^2 d\mu - \left(\int_\Omega f d\mu\right)^2\right]^{\frac{1}{2}}$$
$$\leq \frac{1}{4}\left[\Phi'_-(M) - \Phi'_+(m)\right](M-m).$$

Remark 1. *We notice that the inequality between the first and the second term in (1.1) in the discrete case was proved in 1994 by Dragomir & Ionescu, see* [12].

For a real function $g : [m, M] \to \mathbb{R}$ and two distinct points α, $\beta \in [m, M]$ we recall that the *divided difference* of g in these points is defined by

$$[\alpha, \beta; g] := \frac{g(\beta) - g(\alpha)}{\beta - \alpha}.$$

Upper and lower bounds for the Jensen's gap were also obtained in [10]:

Theorem 2. *Let $\Phi : I \to \mathbb{R}$ be a continuous convex function on the interval of real numbers I and m, $M \in \mathbb{R}$, $m < M$ with $[m, M] \subset \mathring{I}$. If $f : \Omega \to [m, M]$, is μ-measurable and such that f, $\Phi \circ f \in L(\Omega, \mu)$, then by assuming that $\int_\Omega f d\mu \neq m$, M, we have*

$$\left|\int_\Omega \left|\Phi(f) - \Phi\left(\int_\Omega f d\mu\right)\right| \operatorname{sgn}\left(f - \int_\Omega f d\mu\right) d\mu\right| \tag{1.2}$$
$$\leq \int_\Omega (\Phi \circ f) d\mu - \Phi\left(\int_\Omega f d\mu\right)$$
$$\leq \frac{1}{2}\left(\left[\int_\Omega f d\mu, M; \Phi\right] - \left[m, \int_\Omega f d\mu; \Phi\right]\right) \int_\Omega \left| f - \int_\Omega f d\mu\right| d\mu$$
$$\leq \frac{1}{2}\left(\left[\int_\Omega f d\mu, M; \Phi\right] - \left[m, \int_\Omega f d\mu; \Phi\right]\right) \left[\int_\Omega f^2 d\mu - \left(\int_\Omega f d\mu\right)^2\right]^{\frac{1}{2}}$$
$$\leq \frac{1}{4}\left(\left[\int_\Omega f d\mu, M; \Phi\right] - \left[m, \int_\Omega f d\mu; \Phi\right]\right)(M-m).$$

The constant $\frac{1}{2}$ in the second inequality from (1.2) is best possible.

For other recent reverses of Jensen inequality and applications to divergence measures see [8], [9], [10] and the survey paper [11]. More related results may be found in [1]-[4], [7], [11] and [11]-[14].

Motivated by the above results, In this paper we show among others that, if Φ, Ψ, $\Gamma : [m, M] \subset \mathbb{R} \to \mathbb{R}$ are twice differentiable function on (m, M) with $\Psi''(x) \leq \Phi''(x) \leq \Gamma''(x)$ for almost every $x \in (m, M)$ and $f : \Omega \to [m, M]$ so that f, f^2,

RGMIA

$\Phi \circ f,\ \Phi' \circ f,\ \Psi \circ f,\ \Psi' \circ f,\ \Gamma \circ f,\ \Gamma' \circ f \in L(\Omega, \mu)$, then we have the inequalities

$$\int_\Omega \Psi \circ f d\mu - \Psi\left(\int_\Omega f d\mu\right) \le \int_\Omega \Phi \circ f d\mu - \Phi\left(\int_\Omega f d\mu\right)$$
$$\le \int_\Omega \Gamma \circ f d\mu - \Gamma\left(\int_\Omega f d\mu\right).$$

Applications for exponential, logarithm and power functions are also given.

2. Main Results

The following theorem is well known in the literature as Taylor's formula or Taylor's theorem with the integral remainder.

Lemma 1. *Let $I \subset \mathbb{R}$ be a closed interval, $a \in I$ and let n be a positive integer. If $g : I \longrightarrow \mathbb{C}$ is such that the n-derivative $g^{(n)}$ is absolutely continuous on I, then for each $x \in I$*

$$g(x) = T_n(g; a, x) + R_n(g; a, x), \tag{2.1}$$

where $T_n(g; c, y)$ is Taylor's polynomial, i.e.,

$$T_n(g; a, x) := \sum_{k=0}^{n} \frac{(x-a)^k}{k!} g^{(k)}(a). \tag{2.2}$$

Note that $g^{(0)} := g$ and $0! := 1$ and the remainder is given by

$$R_n(g; a, x) := \frac{1}{n!} \int_a^x (x-t)^n g^{(n+1)}(t)\, dt. \tag{2.3}$$

A simple proof of this theorem can be achieved by mathematical induction using the integration by parts formula in the Lebesgue integral.

For any integrable function h on an interval and any distinct numbers c, d in that interval, we have, by the change of variable $t = (1-s)c + sd$, $s \in [0,1]$ that

$$\int_c^d h(t)\, dt = (d-c) \int_0^1 h((1-s)c + sd)\, ds.$$

Therefore,

$$\int_a^x g^{(n+1)}(t)(x-t)^n\, dt$$
$$= (x-a) \int_0^1 g^{(n+1)}((1-s)a + sx)(x - (1-s)a - sx)^n\, ds$$
$$= (x-a)^{n+1} \int_0^1 g^{(n+1)}((1-s)a + sx)(1-s)^n\, ds.$$

The identity (2.1) can then be written as

$$g(x) = \sum_{k=0}^{n} \frac{1}{k!} g^{(k)}(a)(x-a)^k + \frac{1}{n!}(x-a)^{n+1} \int_0^1 g^{(n+1)}((1-s)a + sx)(1-s)^n\, ds \tag{2.4}$$

for all x, $a \in I$.

We have the following result concerning lower and upper bounds for the Jensen's gap:

RGMIA

Theorem 3. *Let* $\Phi, \Psi : [m, M] \subset \mathbb{R} \to \mathbb{R}$ *be twice differentiable function on* (m, M) *with* $\Psi''(x) \le \Phi''(x)$ *for almost every* $x \in (m, M)$ *and* $f : \Omega \to [m, M]$ *so that* f, f^2, $\Phi \circ f$, $\Phi' \circ f$, $\Psi \circ f$, $\Psi' \circ f \in L(\Omega, \mu)$. *Then we have the inequality:*

$$\int_\Omega \Psi \circ f d\mu - \Psi\left(\int_\Omega f d\mu\right) \le \int_\Omega \Phi \circ f d\mu - \Phi\left(\int_\Omega f d\mu\right). \tag{2.5}$$

Proof. We have from (2.4) for $n = 2$ that

$$\Phi(x) = \Phi(c) + \Phi'(c)(x - c) + (x - c)^2 \int_0^1 \Phi''((1 - s)c + sx)(1 - s) ds \tag{2.6}$$

for all x, $c \in (m, M)$, where Φ is such that Φ' is absolutely continuos on $[m, M]$.

Since $\Psi''(x) \le \Phi''(x)$ for almost every $x \in (m, M)$, then

$$\Psi''((1 - s)c + sx) \le \Phi''((1 - s)c + sx)$$

for almost every $x \in (m, M)$ and all $s \in [0, 1]$.

If we multiply by $1 - s \ge 0$ and integrate, then we get

$$\int_0^1 \Psi''((1 - s)c + sx)(1 - s) ds \le \int_0^1 \Phi''((1 - s)c + sx)(1 - s) ds,$$

which implies, by multiplying with $(x - c)^2 \ge 0$ that

$$\begin{aligned} &(x - c)^2 \int_0^1 \Psi''((1 - s)c + sx)(1 - s) ds \\ &\le (x - c)^2 \int_0^1 \Phi''((1 - s)c + sx)(1 - s) ds, \end{aligned} \tag{2.7}$$

for all x, $c \in (m, M)$.

By utilising the representation (2.6) we then get

$$\Psi(x) - \Psi(c) - \Psi'(c)(x - c) \le \Phi(x) - \Phi(c) - \Phi'(c)(x - c) \tag{2.8}$$

for all x, $c \in (m, M)$.

Let $t \in \Omega$ and take $x = f(t)$ and $c = \int_\Omega f d\mu$ to get

$$\begin{aligned} &\Psi(f(t)) - \Psi\left(\int_\Omega f d\mu\right) - \Psi'\left(\int_\Omega f d\mu\right)\left(f(t) - \int_\Omega f d\mu\right) \\ &\le \Phi(f(t)) - \Phi\left(\int_\Omega f d\mu\right) - \Phi'\left(\int_\Omega f d\mu\right)\left(f(t) - \int_\Omega f d\mu\right) \end{aligned}$$

for $t \in \Omega$.

By taking the integral over $d\mu(t)$ we get

$$\begin{aligned} &\int_\Omega \Psi(f(t)) d\mu(t) - \Psi\left(\int_\Omega f d\mu\right) \int_\Omega d\mu(t) \\ &- \Psi'\left(\int_\Omega f d\mu\right)\left(\int_\Omega f(t) d\mu(t) - \int_\Omega f d\mu\right) \\ &\le \int_\Omega \Phi(f(t)) d\mu(t) - \Phi\left(\int_\Omega f d\mu\right) \int_\Omega d\mu(t) \\ &- \Phi'\left(\int_\Omega f d\mu\right)\left(\int_\Omega f(t) d\mu(t) - \int_\Omega f d\mu\right), \end{aligned}$$

which gives (2.5). □

RGMIA

Corollary 1. *Let $\Phi,\ \Psi,\ \Gamma : [m,M] \subset \mathbb{R} \to \mathbb{R}$ be twice differentiable function on (m,M) with $\Psi''(x) \le \Phi''(x) \le \Gamma''(x)$ for almost every $x \in (m,M)$ and $f : \Omega \to [m,M]$ so that $f,\ f^2,\ \Phi\circ f,\ \Phi'\circ f,\ \Psi\circ f,\ \Psi'\circ f,\ \Gamma\circ f,\ \Gamma'\circ f \in L(\Omega,\mu)$. Then we have the inequalities*

$$\int_\Omega \Psi\circ f d\mu - \Psi\left(\int_\Omega f d\mu\right) \le \int_\Omega \Phi\circ f d\mu - \Phi\left(\int_\Omega f d\mu\right) \le \int_\Omega \Gamma\circ f d\mu - \Gamma\left(\int_\Omega f d\mu\right). \tag{2.9}$$

Example 1. *Consider the convex functions $\Psi(x) = \frac{k}{p(p-1)}x^p$ and $\Gamma(x) = \frac{K}{p(p-1)}x^p$ with $p \in (-\infty,0)\cup(1,\infty)$, where $0 \le k < K$ and $x \in (m,M) \subset (0,\infty)$. Assume that Φ is twice differentiable on (m,M) and*

$$0 \le kx^{p-2} \le \Phi''(x) \le Kx^{p-2} \text{ for a.e. } x \in (m,M),$$

then by (2.9) we get

$$\begin{aligned} 0 &\le \frac{k}{p(p-1)}\left[\int_\Omega f^p d\mu - \left(\int_\Omega f d\mu\right)^p\right] \\ &\le \int_\Omega \Phi\circ f d\mu - \Phi\left(\int_\Omega f d\mu\right) \\ &\le \frac{K}{p(p-1)}\left[\int_\Omega f^p d\mu - \left(\int_\Omega f d\mu\right)^p\right]. \end{aligned} \tag{2.10}$$

In particular, if

$$0 \le k \le \Phi''(x) \le K \text{ for a.e. } x \in (m,M),$$

then by (2.9) we get

$$\begin{aligned} 0 &\le \frac{k}{2}\left[\int_\Omega f^2 d\mu - \left(\int_\Omega f d\mu\right)^2\right] \le \int_\Omega \Phi\circ f d\mu - \Phi\left(\int_\Omega f d\mu\right) \\ &\le \frac{K}{2}\left[\int_\Omega f^2 d\mu - \left(\int_\Omega f d\mu\right)^2\right]. \end{aligned} \tag{2.11}$$

If

$$0 \le kx^{-3} \le \Phi''(x) \le Kx^{-3} \text{ for a.e. } x \in (m,M),$$

then by (2.9) we get

$$\begin{aligned} 0 &\le \frac{k}{p(p-1)}\left[\int_\Omega f^{-1} d\mu - \left(\int_\Omega f d\mu\right)^{-1}\right] \\ &\le \int_\Omega \Phi\circ f d\mu - \Phi\left(\int_\Omega f d\mu\right) \\ &\le \frac{K}{p(p-1)}\left[\int_\Omega f^{-1} d\mu - \left(\int_\Omega f d\mu\right)^{-1}\right]. \end{aligned} \tag{2.12}$$

Example 2. *Consider the convex functions $\Psi(x) = -k\ln x$ and $\Gamma(x) = -K\ln x$, where $0 < k < K$ and $x \in (m,M) \subset (0,\infty)$. Assume that Φ is twice differentiable on (m,M) and*

$$0 \le kx^{-2} \le \Phi''(x) \le Kx^{-2} \text{ for a.e. } x \in (m,M),$$

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then by (2.9) we get

$$0\le k\left[\ln\left(\int_\Omega f d\mu\right)-\int_\Omega \ln(f)\,d\mu\right]\le\int_\Omega \Phi\circ f d\mu-\Phi\left(\int_\Omega f d\mu\right)\le K\left[\ln\left(\int_\Omega f d\mu\right)-\int_\Omega \ln(f)\,d\mu\right]. \tag{2.13}$$

Example 3. *Consider the convex functions* $\Psi(x)=\frac{k}{\alpha^2}\exp(\alpha x)$ *and* $\Gamma(x)=\frac{K}{\alpha^2}\exp(\alpha x)$ *with* $\alpha\neq 0$, *where* $0\le k<K$ *and* $x\in(m,M)\subset\mathbb{R}$. *Assume that* Φ *is twice differentiable on* (m,M) *and*

$$0\le k\exp(\alpha x)\le\Phi''(x)\le K\exp(\alpha x)\ \text{ for a.e. } x\in(m,M),$$

then by (2.9) we get

$$\begin{aligned}0&\le\frac{k}{\alpha^2}\left[\int_\Omega \exp(\alpha f)\,d\mu-\exp\left(\alpha\int_\Omega f d\mu\right)\right]\\&\le\int_\Omega \Phi\circ f d\mu-\Phi\left(\int_\Omega f d\mu\right)\\&\le\frac{K}{\alpha^2}\left[\int_\Omega \exp(\alpha f)\,d\mu-\exp\left(\alpha\int_\Omega f d\mu\right)\right].\end{aligned} \tag{2.14}$$

Consider the convex functions $\Psi(x)=\frac{1}{\alpha^2}\exp(\alpha x)$, $\Phi(x)=\frac{1}{\beta^2}\exp(\beta x)$ *and* $\Gamma(x)=\frac{1}{\gamma^2}\exp(\gamma x)$ *with* $0<\alpha\le\beta\le\gamma$ *then,*

$$\Psi''(x)\le\Phi''(x)\le\Gamma''(x)\ \text{ for } x\ge 0$$

and by (2.9) for $f\ge 0$ *we get*

$$\begin{aligned}0&\le\frac{1}{\alpha^2}\left[\int_\Omega \exp(\alpha f)\,d\mu-\exp\left(\alpha\int_\Omega f d\mu\right)\right]\\&\le\frac{1}{\beta^2}\left[\int_\Omega \exp(\beta f)\,d\mu-\exp\left(\beta\int_\Omega f d\mu\right)\right]\\&\le\frac{1}{\gamma^2}\left[\int_\Omega \exp(\gamma f)\,d\mu-\exp\left(\gamma\int_\Omega f d\mu\right)\right].\end{aligned} \tag{2.15}$$

We have:

Theorem 4. *Let* $\Phi,\Psi:[m,M]\subset\mathbb{R}\to\mathbb{R}$ *be twice differentiable function on* (m,M) *with* $\Psi''(x)\le\Phi''(x)$ *for almost every* $x\in(m,M)$ *and* $f:\Omega\to[m,M]$ *so that* f, f^2, $\Phi\circ f$, $\Phi'\circ f$, $\Psi\circ f$, $\Psi'\circ f\in L(\Omega,\mu)$. *Then we have the inequality:*

$$\begin{aligned}&\int_\Omega \Psi\circ f d\mu-\Psi\left(\frac{f(m)+f(M)}{2}\right)\\&-\Psi'\left(\frac{f(m)+f(M)}{2}\right)\left(\int_\Omega f d\mu-\frac{f(m)+f(M)}{2}\right)\\&\le\int_\Omega \Phi\circ f d\mu-\Phi\left(\frac{f(m)+f(M)}{2}\right)\\&-\Phi'\left(\frac{f(m)+f(M)}{2}\right)\left(\int_\Omega f d\mu-\frac{f(m)+f(M)}{2}\right)\end{aligned} \tag{2.16}$$

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Proof. From (2.8) we get by taking $c=\frac{f(m)+f(M)}{2}$ and $x=f(t)$, $t\in\Omega$, that

$$\begin{aligned}\Psi(f(t))&-\Psi\left(\frac{f(m)+f(M)}{2}\right)\\&-\Psi'\left(\frac{f(m)+f(M)}{2}\right)\left(f(t)-\frac{f(m)+f(M)}{2}\right)\\&\leq\Phi(f(t))-\Phi\left(\frac{f(m)+f(M)}{2}\right)\\&-\Phi'\left(\frac{f(m)+f(M)}{2}\right)\left(f(t)-\frac{f(m)+f(M)}{2}\right)\end{aligned}\tag{2.17}$$

for $t\in\Omega$.

If we take the integral in (2.17), then we get

$$\begin{aligned}\int_\Omega&\Psi(f(t))\,d\mu(t)-\Psi\left(\frac{f(m)+f(M)}{2}\right)\int_\Omega d\mu(t)\\&-\Psi'\left(\frac{f(m)+f(M)}{2}\right)\left(\int_\Omega f(t)\,d\mu(t)-\frac{f(m)+f(M)}{2}\int_\Omega d\mu(t)\right)\\&\leq\int_\Omega\Phi(f(t))\,d\mu(t)-\Phi\left(\frac{f(m)+f(M)}{2}\right)\int_\Omega d\mu(t)\\&-\Phi'\left(\frac{f(m)+f(M)}{2}\right)\left(\int_\Omega f(t)\,d\mu(t)-\frac{f(m)+f(M)}{2}\int_\Omega d\mu(t)\right)\end{aligned}\tag{2.18}$$

and since $\int_\Omega d\mu(t)=1$, hence by (2.18) we get (2.16). □

Corollary 2. *Let Φ, Ψ, $\Gamma:[m,M]\subset\mathbb{R}\to\mathbb{R}$ be twice differentiable function on (m,M) with $\Psi''(x)\leq\Phi''(x)\leq\Gamma''(x)$ for almost every $x\in(m,M)$ and $f:\Omega\to[m,M]$ so that f, f^2, $\Phi\circ f$, $\Phi'\circ f$, $\Psi\circ f$, $\Psi'\circ f$, $\Gamma\circ f$, $\Gamma'\circ f\in L(\Omega,\mu)$. Then we have the inequalities*

$$\begin{aligned}\int_\Omega&\Psi\circ f d\mu-\Psi\left(\frac{f(m)+f(M)}{2}\right)\\&-\Psi'\left(\frac{f(m)+f(M)}{2}\right)\left(\int_\Omega fd\mu-\frac{f(m)+f(M)}{2}\right)\\&\leq\int_\Omega\Phi\circ fd\mu-\Phi\left(\frac{f(m)+f(M)}{2}\right)\\&-\Phi'\left(\frac{f(m)+f(M)}{2}\right)\left(\int_\Omega fd\mu-\frac{f(m)+f(M)}{2}\right)\\&\leq\int_\Omega\Gamma\circ fd\mu-\Gamma\left(\frac{f(m)+f(M)}{2}\right)\\&-\Gamma'\left(\frac{f(m)+f(M)}{2}\right)\left(\int_\Omega fd\mu-\frac{f(m)+f(M)}{2}\right).\end{aligned}\tag{2.19}$$

Example 4. *Consider the convex functions $\Psi(x)=\frac{k}{p(p-1)}x^p$ and $\Gamma(x)=\frac{K}{p(p-1)}x^p$ with $p\in(-\infty,0)\cup(1,\infty)$, where $0\leq k<K$ and $x\in(m,M)\subset(0,\infty)$. Assume that Φ is twice differentiable on (m,M) and*

$$0\leq kx^{p-2}\leq\Phi''(x)\leq Kx^{p-2}\text{ for a.e. }x\in(m,M),$$

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then by (2.19)

$$0\le k\left[\int_{\Omega}f^{p}d\mu-p\left(\frac{f(m)+f(M)}{2}\right)^{p-1}\int_{\Omega}fd\mu+(p-1)\left(\frac{f(m)+f(M)}{2}\right)^{p}\right]\le\int_{\Omega}\Phi\circ fd\mu-\Phi\left(\frac{f(m)+f(M)}{2}\right)-\Phi'\left(\frac{f(m)+f(M)}{2}\right)\left(\int_{\Omega}fd\mu-\frac{f(m)+f(M)}{2}\right)\le K\left[\int_{\Omega}f^{p}d\mu-p\left(\frac{f(m)+f(M)}{2}\right)^{p-1}\int_{\Omega}fd\mu+(p-1)\left(\frac{f(m)+f(M)}{2}\right)^{p}\right]. \tag{2.20}$$

If

$$0\le k\le\Phi''(x)\le K \text{ for a.e. } x\in(m,M),$$

then

$$0\le k\left[\int_{\Omega}f^{2}d\mu-2\left(\frac{f(m)+f(M)}{2}\right)\int_{\Omega}fd\mu+\left(\frac{f(m)+f(M)}{2}\right)^{2}\right]\le\int_{\Omega}\Phi\circ fd\mu-\Phi\left(\frac{f(m)+f(M)}{2}\right)-\Phi'\left(\frac{f(m)+f(M)}{2}\right)\left(\int_{\Omega}fd\mu-\frac{f(m)+f(M)}{2}\right)\le K\left[\int_{\Omega}f^{2}d\mu-2\left(\frac{f(m)+f(M)}{2}\right)\int_{\Omega}fd\mu+\left(\frac{f(m)+f(M)}{2}\right)^{2}\right]. \tag{2.21}$$

Example 5. *Consider the convex functions* $\Psi(x)=-k\ln x$ *and* $\Gamma(x)=-K\ln x$, *where* $0<k<K$ *and* $x\in(m,M)\subset(0,\infty)$. *Assume that* Φ *is twice differentiable on* (m,M) *and*

$$0\le kx^{-2}\le\Phi''(x)\le Kx^{-2} \text{ for a.e. } x\in(m,M),$$

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then by (2.19) we get

$$\begin{aligned}
&k\left[\ln\left(\frac{f(m)+f(M)}{2}\right)-\int_\Omega \ln f d\mu+\right.\\
&\left.-\left(\frac{f(m)+f(M)}{2}\right)^{-1}\int_\Omega f d\mu+1\right]\\
&\leq \int_\Omega \Phi\circ f d\mu-\Phi\left(\frac{f(m)+f(M)}{2}\right)\\
&-\Phi'\left(\frac{f(m)+f(M)}{2}\right)\left(\int_\Omega f d\mu-\frac{f(m)+f(M)}{2}\right)\\
&\leq K\left[\ln\left(\frac{f(m)+f(M)}{2}\right)-\int_\Omega \ln f d\mu+\right.\\
&\left.-\left(\frac{f(m)+f(M)}{2}\right)^{-1}\int_\Omega f d\mu+1\right].
\end{aligned} \tag{2.22}$$

Example 6. *Consider the convex functions* $\Psi(x)=\frac{k}{\alpha^2}\exp(\alpha x)$ *and* $\Gamma(x)=\frac{K}{\alpha^2}\exp(\alpha x)$ *with* $\alpha\neq 0$, *where* $0\leq k<K$ *and* $x\in(m,M)\subset\mathbb{R}$. *Assume that* Φ *is twice differentiable on* (m,M) *and*

$$0\leq k\exp(\alpha x)\leq \Phi''(x)\leq K\exp(\alpha x) \text{ for a.e. } x\in(m,M),$$

then by (2.9) we get

$$\begin{aligned}
&\frac{k}{\alpha^2}\left[\int_\Omega \exp(\alpha f)d\mu-\exp\left(\alpha\frac{f(m)+f(M)}{2}\right)\right.\\
&\left.-\alpha\exp\left(\alpha\frac{f(m)+f(M)}{2}\right)\left(\int_\Omega f d\mu-\frac{f(m)+f(M)}{2}\right)\right]\\
&\leq \int_\Omega \Phi\circ f d\mu-\Phi\left(\frac{f(m)+f(M)}{2}\right)\\
&-\Phi'\left(\frac{f(m)+f(M)}{2}\right)\left(\int_\Omega f d\mu-\frac{f(m)+f(M)}{2}\right)\\
&\leq \frac{K}{\alpha^2}\left[\int_\Omega \exp(\alpha f)d\mu-\exp\left(\alpha\frac{f(m)+f(M)}{2}\right)\right.\\
&\left.-\alpha\exp\left(\alpha\frac{f(m)+f(M)}{2}\right)\left(\int_\Omega f d\mu-\frac{f(m)+f(M)}{2}\right)\right].
\end{aligned} \tag{2.23}$$

3. Discrete Inequalities

Let Φ, Ψ, $\Gamma:[m,M]\subset\mathbb{R}\to\mathbb{R}$ be twice differentiable function on (m,M) with $\Psi''(x)\leq\Phi''(x)\leq\Gamma''(x)$ for almost every $x\in(m,M)$ and $x_i\in(m,M)$, $p_i\geq 0$ for $i\in\{1,...,m\}$ with $\sum_{i=1}^n p_i=1$. Then we have the inequalities

$$\begin{aligned}
\sum_{i=1}^n p_i\Psi(x_i)-\Psi\left(\sum_{i=1}^n p_ix_i\right)&\leq\sum_{i=1}^n p_i\Phi(x_i)-\Phi\left(\sum_{i=1}^n p_ix_i\right)\\
&\leq\sum_{i=1}^n p_i\Gamma(x_i)-\Gamma\left(\sum_{i=1}^n p_ix_i\right).
\end{aligned} \tag{3.1}$$

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Consider the convex functions $\Psi(x)=\frac{k}{p(p-1)}x^{p}$ and $\Gamma(x)=\frac{K}{p(p-1)}x^{p}$ with $p\in(-\infty,0)\cup(1,\infty)$, where $0\le k<K$ and $x\in(m,M)\subset(0,\infty)$. Assume that Φ is twice differentiable on (m,M) and

$$0\le kx^{p-2}\le\Phi''(x)\le Kx^{p-2}\text{ for a.e. }x\in(m,M),$$

then by (2.9) we get

$$\begin{aligned}0&\le\frac{k}{p(p-1)}\left[\sum_{i=1}^{n}p_ix_i^{p}-\left(\sum_{i=1}^{n}p_ix_i\right)^{p}\right]\\&\le\sum_{i=1}^{n}p_i\Phi(x_i)-\Phi\left(\sum_{i=1}^{n}p_ix_i\right)\\&\le\frac{K}{p(p-1)}\left[\sum_{i=1}^{n}p_ix_i^{p}-\left(\sum_{i=1}^{n}p_ix_i\right)^{p}\right].\end{aligned}\tag{3.2}$$

In particular, if

$$0\le k\le\Phi''(x)\le K\text{ for a.e. }x\in(m,M),$$

then by (3.2) we get

$$\begin{aligned}0&\le\frac{k}{2}\left[\sum_{i=1}^{n}p_ix_i^{2}-\left(\sum_{i=1}^{n}p_ix_i\right)^{2}\right]\\&\le\sum_{i=1}^{n}p_i\Phi(x_i)-\Phi\left(\sum_{i=1}^{n}p_ix_i\right)\\&\le\frac{K}{2}\left[\sum_{i=1}^{n}p_ix_i^{2}-\left(\sum_{i=1}^{n}p_ix_i\right)^{2}\right].\end{aligned}\tag{3.3}$$

If

$$0\le kx^{-3}\le\Phi''(x)\le Kx^{-3}\text{ for a.e. }x\in(m,M),$$

then by (3.2) we get

$$\begin{aligned}0&\le\frac{k}{p(p-1)}\left[\sum_{i=1}^{n}p_ix_i^{-1}-\left(\sum_{i=1}^{n}p_ix_i\right)^{-1}\right]\\&\le\sum_{i=1}^{n}p_i\Phi(x_i)-\Phi\left(\sum_{i=1}^{n}p_ix_i\right)\\&\le\frac{K}{p(p-1)}\left[\sum_{i=1}^{n}p_ix_i^{-1}-\left(\sum_{i=1}^{n}p_ix_i\right)^{-1}\right].\end{aligned}\tag{3.4}$$

Assume that Φ is twice differentiable on (m,M) and

$$0\le kx^{-2}\le\Phi''(x)\le Kx^{-2}\text{ for a.e. }x\in(m,M),$$

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then by (2.9) we get

$$0 \le k\left[\ln\left(\sum_{i=1}^{n} p_i x_i\right) - \ln\left(\prod_{i=1}^{n} x_i^{p_i}\right)\right] \le \sum_{i=1}^{n} p_i \Phi(x_i) - \Phi\left(\sum_{i=1}^{n} p_i x_i\right) \le K\left[\ln\left(\sum_{i=1}^{n} p_i x_i\right) - \ln\left(\prod_{i=1}^{n} x_i^{p_i}\right)\right], \tag{3.5}$$

which is equivalent to

$$1 \le \left(\frac{\sum_{i=1}^{n} p_i x_i}{\prod_{i=1}^{n} x_i^{p_i}}\right)^k \le \frac{\exp\left(\sum_{i=1}^{n} p_i \Phi(x_i)\right)}{\exp \Phi\left(\sum_{i=1}^{n} p_i x_i\right)} \le \left(\frac{\sum_{i=1}^{n} p_i x_i}{\prod_{i=1}^{n} x_i^{p_i}}\right)^K. \tag{3.6}$$

Assume that Φ is twice differentiable on (m, M) and

$$0 \le k \exp(\alpha x) \le \Phi''(x) \le K \exp(\alpha x) \text{ for a.e. } x \in (m, M),$$

then by (3.2) we get

$$\begin{aligned} 0 &\le \frac{k}{\alpha^2}\left[\sum_{i=1}^{n} p_i \exp(\alpha x_i) - \exp\left(\alpha \sum_{i=1}^{n} p_i x_i\right)\right] \\ &\le \sum_{i=1}^{n} p_i \Phi(x_i) - \Phi\left(\sum_{i=1}^{n} p_i x_i\right) \\ &\le \frac{K}{\alpha^2}\left[\sum_{i=1}^{n} p_i \exp(\alpha x_i) - \exp\left(\alpha \sum_{i=1}^{n} p_i x_i\right)\right]. \end{aligned} \tag{3.7}$$

If $0 < \alpha \le \beta \le \gamma$ then by (2.15) for $x_i \ge 0$ we get

$$\begin{aligned} 0 &\le \frac{1}{\alpha^2}\left[\sum_{i=1}^{n} p_i \exp(\alpha x_i) - \exp\left(\alpha \sum_{i=1}^{n} p_i x_i\right)\right] \\ &\le \frac{1}{\beta^2}\left[\sum_{i=1}^{n} p_i \exp(\beta x_i) - \exp\left(\beta \sum_{i=1}^{n} p_i x_i\right)\right] \\ &\le \frac{1}{\gamma^2}\left[\sum_{i=1}^{n} p_i \exp(\gamma x_i) - \exp\left(\gamma \sum_{i=1}^{n} p_i x_i\right)\right]. \end{aligned} \tag{3.8}$$

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