ABOUT HERMITE-HADAMARD TYPE INEQUALITIES FOR $M_\varphi A$-S-CONVEX FUNCTIONS

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Abstract. The aim of this paper is to use the new concept of $M_\varphi A$-s-convexity, which is a generalization of several well-known concepts, of s-convexity, GA-s-convexity, harmonically s-convexity and (p,s)-convexity in order to give some new Hermite-Hadamard type inequalities for these kind of functions by using a new lemma. Then some examples are presented.

1. Introduction

The well-known inequality of Hermite-Hadamard was extended and generalized in many directions by many authors, like for example, [5, 4, 1, 12, 14, 3, 7, 15, 18, 8, 9, 10, 11, 17] and the references therein.

We start by recalling below the classical definition for the convex functions.

Definition 1. *A function $f : I \subset \mathbb{R} \to \mathbb{R}$ is said to be convex on an interval I if the inequality*

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) \tag{1}$$

holds for all $x, y \in I$ and $t \in [0,1]$. The function f is said to be concave on I if the inequality (1) takes place in reversed direction.

For other type of convexity, see also [16, 13].

Definition 2. *([18]) We consider I a real interval, $\varphi : I \to \mathbb{R}$ a continuous function and strictly monotonic function and $s \in (0,1]$.*

(a) A function $f : I \to \mathbb{R}$ is said to be $M_\varphi A$-s-convex in the first sense, if

$$f\left(\varphi^{-1}(t\varphi(x) + (1-t)\varphi(y))\right) \leq t^s f(x) + (1-t^s) f(y)$$

for all $x, y \in I$ and $t \in [0,1]$. If the above inequality is reversed then f is said to be $M_\varphi A$-s-concave in the first sense.

(b) A function $f : I \to \mathbb{R}$ is said to be $M_\varphi A$-s-convex in the second sense, if

$$f\left(\varphi^{-1}(t\varphi(x) + (1-t)\varphi(y))\right) \leq t^s f(x) + (1-t)^s f(y)$$

for all $x, y \in I$ and $t \in [0,1]$. If the above inequality is reversed then f is said to be $M_\varphi A$-s-concave in the second sense.

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Let f be a $M_{\varphi}A$-s-convex function. It is well-known that (see [18]) :

(i) If we take $\varphi : I \subset \mathbb{R} \to \mathbb{R}$, $\varphi(x) = x$, then $M_{\varphi}A$-convexity means GA-convexity.

(ii) If we consider $\varphi : I \subset (0, \infty) \to \mathbb{R}$, $\varphi(x) = lnx$, then $M_{\varphi}A$-convexity means convexity.

(iii) If we have $\varphi : I \subset (0, \infty) \to \mathbb{R}$, $\varphi(x) = x^{-1}$, then $M_{\varphi}A$-convexity means Harmonically-convexity.

(iv) If we have $\varphi : I \subset (0, \infty) \to \mathbb{R}$, $\varphi(x) = x^p$, $p \in \mathbb{R} - \{0\}$ then $M_{\varphi}A$-convexity means p-convexity.

The classical Hermite-Hadamard's inequality for convex functions is given below by,

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{3}$$

Moreover, if the function f is concave then the inequality (2) hold in reversed direction.

We need to mention the following results which will be used below.

Lemma 1. *(see [2]) Let $f : I^o \to \mathbb{R}, : I^o \subset [0, \infty)$ be a twice differentiable function on $: I^o$ where $a, b \in I,\ a < b$. If $f^{''} \in L[a, b]$, then the following equality holds:*

$$-f\left(\frac{a+b}{2}\right) + \frac{1}{b-a}\int_a^b f(x)dx =$$

$$= \frac{(b-a)^2}{16}[\int_0^1 t^2 f^{''}\left(t\frac{a+b}{2} + (1-t)a\right)dt + \int_0^1 (t-1)^2 f^{''}\left(tb + (1-t)\frac{a+b}{2}\right)dt].$$

Lemma 2. *(see [3]) Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a twice differentiable function on I^o, the interior of I, where $a, b \in I,\ a < b$. If $f^{''} \in L[a, b]$, then the following equality holds:*

$$-\frac{f\left(\frac{3a+b}{4}\right) + f\left(\frac{a+3b}{4}\right)}{2} + \frac{1}{b-a}\int_a^b f(x)dx =$$

$$= \frac{(b-a)^2}{128}[\int_0^1 t^2 f^{''}\left(t\frac{3a+b}{4} + (1-t)a\right)dt + \int_0^1 (t-1)^2 f^{''}\left(t\frac{a+b}{2} + (1-t)\frac{a+3b}{4}\right)dt +$$

$$+ \int_0^1 t^2 f^{''}\left(t\frac{a+3b}{4} + (1-t)\frac{a+b}{2}\right)dt + \int_0^1 (t-1)^2 f^{''}\left(tb + (1-t)\frac{a+3b}{4}\right)dt].$$

Lemma 3. *(see [17])*

Let f and $f^{'}$ be two differentiable functions on I°, where $f : I \to \mathbb{R}$ and $a, b \in I$ with $a < b$ and $\varphi : I \to \mathbb{R}$ be a continuous and strictly monotonic function such that $\varphi^{-1} : \varphi(I^{\circ}) \to I^{\circ}$ is continuously differentiable and $(\varphi^{-1})^{'}$ also continuously differentiable. If $f^{'}, f^{''} \in L[a, b]$ then we have

$$\frac{1}{\varphi(b)-\varphi(a)}\int_a^b f(x)\varphi^{'}(x)dx - f\left(\frac{a+b}{2}\right) + \frac{f^{'}}{\varphi^{'}}\left(\frac{a+b}{2}\right)\left(\varphi\left(\frac{a+b}{2}\right) - \frac{\varphi(a)+\varphi(b)}{2}\right) =$$

$$= \frac{1}{2\left(\varphi(b)-\varphi(a)\right)}\left[\left(\varphi\left(\frac{a+b}{2}\right)-\varphi(a)\right)^3 I_1 + \left(\varphi(b)-\varphi\left(\frac{a+b}{2}\right)\right)^3 I_2\right],$$

where

$$I_1 = \int_0^1 \{t^2 \left[(\varphi^{-1})'\left(t\varphi\left(\frac{a+b}{2}\right)+(1-t)\varphi(a)\right)\right]^2 f''\left(\varphi^{-1}\left(t\varphi\left(\frac{a+b}{2}\right)+(1-t)\varphi(a)\right)\right)+$$

$$+t^2(\varphi^{-1})''\left(t\varphi\left(\frac{a+b}{2}\right)+(1-t)\varphi(a)\right) f'\left(\varphi^{-1}\left(t\varphi\left(\frac{a+b}{2}\right)+(1-t)\varphi(a)\right)\right)\}dt,$$

and

$$I_2 = \int_0^1 \{(1-t)^2 \left[(\varphi^{-1})'\left(t\varphi(b)+(1-t)\varphi(\frac{a+b}{2})\right)\right]^2 f''\left(\varphi^{-1}\left(t\varphi(b)+(1-t)\varphi(\frac{a+b}{2})\right)\right)+$$

$$+(1-t)^2(\varphi^{-1})''\left(t\varphi(b)+(1-t)\varphi(\frac{a+b}{2})\right) f'\left(\varphi^{-1}\left(t\varphi(b)+(1-t)\varphi(\frac{a+b}{2})\right)\right)\}dt.$$

In this paper a new lemma is established and then the new concept of $M_\varphi A$-s-convexity in the second sense and respectively in the first sense is used, in order to establish several Hermite-Hadamard type inequalities for these functions. Some examples are then presented.

2. Several Hermite-Hadamard type inequalities for $M_\varphi A$-s-convex functions in the second sense and first sense

In this section is given a new lemma similar to [17] which will be used for establishing new inequalities that refine Hermite-Hadamard inequality for functions whose derivatives are $M_\varphi A$-s-convex functions in the second sense or in the first sense respectively.

Lemma 4. *Let f and $f^{'}$ be two differentiable functions on I°, where $f : I \to \mathbb{R}$ and $a, b \in I$ with $a < b$ and $\varphi : I \to \mathbb{R}$ be a continuous and strictly monotonic function such that $\varphi^{-1} : \varphi(I^{\circ}) \to I^{\circ}$ is continuously differentiable and $(\varphi^{-1})^{'}$ also continuously differentiable. If $f^{'}, f^{''} \in L[a, b]$ then we have*

$$\frac{1}{\varphi(b)-\varphi(a)}\int_a^b f(x)\varphi^{'}(x)dx - f\left(\frac{3a+b}{4}\right)\frac{\varphi\left(\frac{a+b}{2}\right)-\varphi(a)}{\varphi(b)-\varphi(a)} - f\left(\frac{a+3b}{4}\right)\frac{\varphi(b)-\varphi\left(\frac{a+b}{2}\right)}{\varphi(b)-\varphi(a)}+$$

$$+\frac{1}{2}\frac{\varphi\left(\frac{a+b}{2}\right)-\varphi(a)}{\varphi(b)-\varphi(a)}\frac{f^{'}}{\varphi^{'}}\left(\frac{3a+b}{4}\right)\left(2\varphi\left(\frac{3a+b}{4}\right)-\varphi(a)-\varphi\left(\frac{a+b}{2}\right)\right)+$$

$$+\frac{1}{2}\frac{\varphi(b)-\varphi\left(\frac{a+b}{2}\right)}{\varphi(b)-\varphi(a)}\frac{f^{'}}{\varphi^{'}}\left(\frac{a+3b}{4}\right)\left(2\varphi\left(\frac{a+3b}{4}\right)-\varphi\left(\frac{a+b}{2}\right)-\varphi(b)\right)=$$

$$= \frac{1}{2\left(\varphi(b)-\varphi(a)\right)}[\left(\varphi\left(\frac{3a+b}{4}\right)-\varphi(a)\right)^3 I_1 + \left(\varphi\left(\frac{a+b}{2}\right)-\varphi\left(\frac{3a+b}{4}\right)\right)^3 I_2+$$

$$+\left(\varphi\left(\frac{a+3b}{4}\right)-\varphi\left(\frac{a+b}{2}\right)\right)^3 I_3 + \left(\varphi(b)-\varphi\left(\frac{a+3b}{4}\right)\right)^3 I_4],$$

where

$$I_1 = \int_0^1 \{t^2 \left[(\varphi^{-1})'\left(t\varphi\left(\frac{3a+b}{4}\right)+(1-t)\varphi(a)\right)\right]^2 f''\left(\varphi^{-1}\left(t\varphi\left(\frac{3a+b}{4}\right)+(1-t)\varphi(a)\right)\right)+$$

$$+t^2(\varphi^{-1})''\left(t\varphi\left(\frac{3a+b}{4}\right)+(1-t)\varphi(a)\right)f'\left(\varphi^{-1}\left(t\varphi\left(\frac{3a+b}{4}\right)+(1-t)\varphi(a)\right)\right)\}dt,$$

$$I_2=\int_0^1\{(1-t)^2\left[(\varphi^{-1})'\left(t\varphi\left(\frac{a+b}{2}\right)+(1-t)\varphi(\frac{3a+b}{4})\right)\right]^2f''\left(\varphi^{-1}\left(t\varphi\left(\frac{a+b}{2}\right)+(1-t)\varphi(\frac{3a+b}{4})\right)\right)+$$
$$+(1-t)^2(\varphi^{-1})''\left(t\varphi\left(\frac{a+b}{2}\right)+(1-t)\varphi(\frac{3a+b}{4})\right)f'\left(\varphi^{-1}\left(t\varphi\left(\frac{a+b}{2}\right)+(1-t)\varphi(\frac{3a+b}{4})\right)\right)\}dt,$$

$$I_3=\int_0^1\{t^2\left[(\varphi^{-1})'\left(t\varphi\left(\frac{a+3b}{4}\right)+(1-t)\varphi(\frac{a+b}{2})\right)\right]^2f''\left(\varphi^{-1}\left(t\varphi\left(\frac{a+3b}{4}\right)+(1-t)\varphi(\frac{a+b}{2})\right)\right)+$$
$$+t^2(\varphi^{-1})''\left(t\varphi\left(\frac{a+3b}{4}\right)+(1-t)\varphi(\frac{a+b}{2})\right)f'\left(\varphi^{-1}\left(t\varphi\left(\frac{a+3b}{4}\right)+(1-t)\varphi(\frac{a+b}{2})\right)\right)\}dt$$

and

$$I_4=\int_0^1\{(1-t)^2\left[(\varphi^{-1})'\left(t\varphi(b)+(1-t)\varphi(\frac{a+3b}{4})\right)\right]^2f''\left(\varphi^{-1}\left(t\varphi(b)+(1-t)\varphi(\frac{a+3b}{4})\right)\right)+$$
$$+(1-t)^2(\varphi^{-1})''\left(t\varphi(b)+(1-t)\varphi(\frac{a+3b}{4})\right)f'\left(\varphi^{-1}\left(t\varphi(b)+(1-t)\varphi(\frac{a+3b}{4})\right)\right)\}dt.$$

Proof. By integrating by parts two times $I_1,\ I_2,\ I_3$ and I_4 we get respectively,

$$I_1=\frac{t^2(f\circ\varphi^{-1})'\left(t\varphi\left(\frac{3a+b}{4}\right)+(1-t)\varphi(a)\right)}{\varphi\left(\frac{3a+b}{4}\right)-\varphi(a)}\Big|_0^1-2\int_0^1\frac{t(f\circ\varphi^{-1})'\left(t\varphi\left(\frac{3a+b}{4}\right)+(1-t)\varphi(a)\right)}{\varphi\left(\frac{3a+b}{4}\right)-\varphi(a)}dt=$$
$$=\frac{(f\circ\varphi^{-1})'\left(\varphi\left(\frac{3a+b}{4}\right)\right)}{\varphi\left(\frac{3a+b}{4}\right)-\varphi(a)}-2\frac{f\left(\frac{3a+b}{4}\right)}{\left(\varphi\left(\frac{3a+b}{4}\right)-\varphi(a)\right)^2}+2\int_0^1\frac{(f\circ\varphi^{-1})\left(t\varphi\left(\frac{3a+b}{4}\right)+(1-t)\varphi(a)\right)}{\left(\varphi\left(\frac{3a+b}{4}\right)-\varphi(a)\right)^2}dt,$$

$$I_2=\frac{(t-1)^2(f\circ\varphi^{-1})'\left(t\varphi\left(\frac{a+b}{2}\right)+(1-t)\varphi\left(\frac{3a+b}{4}\right)\right)}{\varphi\left(\frac{a+b}{2}\right)-\varphi\left(\frac{3a+b}{4}\right)}\Big|_0^1-2\int_0^1\frac{(t-1)(f\circ\varphi^{-1})'\left(t\varphi\left(\frac{a+b}{2}\right)+(1-t)\varphi\left(\frac{3a+b}{4}\right)\right)}{\varphi\left(\frac{a+b}{2}\right)-\varphi\left(\frac{3a+b}{4}\right)}dt=$$
$$=-\frac{(f\circ\varphi^{-1})'\left(\varphi\left(\frac{3a+b}{4}\right)\right)}{\varphi\left(\frac{a+b}{2}\right)-\varphi\left(\frac{3a+b}{4}\right)}-2\frac{f\left(\frac{3a+b}{4}\right)}{\left(\varphi\left(\frac{a+b}{2}\right)-\varphi\left(\frac{3a+b}{4}\right)\right)^2}+2\int_0^1\frac{(f\circ\varphi^{-1})\left(t\varphi\left(\frac{a+b}{2}\right)+(1-t)\varphi\left(\frac{3a+b}{4}\right)\right)}{\left(\varphi\left(\frac{a+b}{2}\right)-\varphi\left(\frac{3a+b}{4}\right)\right)^2}dt,$$

$$I_3=\frac{t^2(f\circ\varphi^{-1})'\left(t\varphi\left(\frac{a+3b}{4}\right)+(1-t)\varphi(\frac{a+b}{2})\right)}{\varphi\left(\frac{a+3b}{4}\right)-\varphi(\frac{a+b}{2})}\Big|_0^1-2\int_0^1\frac{t(f\circ\varphi^{-1})'\left(t\varphi\left(\frac{a+3b}{4}\right)+(1-t)\varphi(\frac{a+b}{2})\right)}{\varphi\left(\frac{a+3b}{4}\right)-\varphi(\frac{a+b}{2})}dt=$$
$$=\frac{(f\circ\varphi^{-1})'\left(\varphi\left(\frac{a+3b}{4}\right)\right)}{\varphi\left(\frac{a+3b}{4}\right)-\varphi(\frac{a+b}{2})}-2\frac{f\left(\frac{a+3b}{4}\right)}{\left(\varphi\left(\frac{a+3b}{4}\right)-\varphi(\frac{a+b}{2})\right)^2}+2\int_0^1\frac{(f\circ\varphi^{-1})\left(t\varphi\left(\frac{a+3b}{4}\right)+(1-t)\varphi(\frac{a+b}{2})\right)}{\left(\varphi\left(\frac{a+3b}{4}\right)-\varphi(\frac{a+b}{2})\right)^2}dt,$$

and

$$I_4=\frac{(1-t)^2(f\circ\varphi^{-1})'\left(t\varphi(b)+(1-t)\varphi(\frac{a+3b}{4})\right)}{\varphi(b)-\varphi(\frac{a+3b}{4})}\Big|_0^1-2\int_0^1\frac{(t-1)(f\circ\varphi^{-1})'\left(t\varphi(b)+(1-t)\varphi(\frac{a+3b}{4})\right)}{\varphi(b)-\varphi(\frac{a+3b}{4})}dt=$$
$$=-\frac{(f\circ\varphi^{-1})'\left(\varphi\left(\frac{a+3b}{4}\right)\right)}{\varphi(b)-\varphi(\frac{a+3b}{4})}-2\frac{f\left(\frac{a+3b}{4}\right)}{\left(\varphi(b)-\varphi(\frac{a+3b}{4})\right)^2}+2\int_0^1\frac{(f\circ\varphi^{-1})\left(t\varphi(b)+(1-t)\varphi(\frac{a+3b}{4})\right)}{\left(\varphi(b)-\varphi(\frac{a+3b}{4})\right)^2}dt.$$

Setting now, $x=\varphi^{-1}\left(t\varphi\left(\frac{3a+b}{4}\right)+(1-t)\varphi(a)\right)$ in I_1,
$x=\varphi^{-1}\left(t\varphi\left(\frac{a+b}{2}\right)+(1-t)\varphi\left(\frac{3a+b}{4}\right)\right)$ in I_2,
$x=\varphi^{-1}\left(t\varphi\left(\frac{a+3b}{4}\right)+(1-t)\varphi\left(\frac{a+b}{2}\right)\right)$ in I_3,
and $x=\varphi^{-1}\left(t\varphi(b)+(1-t)\varphi(\frac{a+3b}{4})\right)$ in I_4

we have respectively,

$dt = \frac{\varphi'(x)}{\varphi(\frac{3a+b}{4})-\varphi(a)}dx$, $dt = \frac{\varphi'(x)}{\varphi(\frac{a+b}{2})-\varphi(\frac{3a+b}{4})}dx$, $dt = \frac{\varphi'(x)}{\varphi(\frac{a+3b}{4})-\varphi(\frac{a+b}{2})}dx$, and $dt = \frac{\varphi'(x)}{\varphi(b)-\varphi(\frac{a+3b}{4})}dx$,

and then we obtain respectively,

$$I_1 = \frac{(f\circ\varphi^{-1})'\left(\varphi\left(\frac{3a+b}{4}\right)\right)}{\varphi\left(\frac{3a+b}{4}\right)-\varphi(a)} - 2\frac{f\left(\frac{3a+b}{4}\right)}{\left(\varphi\left(\frac{3a+b}{4}\right)-\varphi(a)\right)^2} + 2\int_a^{\frac{3a+b}{4}} \frac{f(x)\varphi'(x)}{(\varphi\left(\frac{3a+b}{4}\right)-\varphi(a))^3}dx,$$

$$I_2 = -\frac{(f\circ\varphi^{-1})'\left(\varphi\left(\frac{3a+b}{4}\right)\right)}{\varphi\left(\frac{a+b}{2}\right)-\varphi\left(\frac{3a+b}{4}\right)} - 2\frac{f\left(\frac{3a+b}{4}\right)}{\left(\varphi\left(\frac{a+b}{2}\right)-\varphi\left(\frac{3a+b}{4}\right)\right)^2} + 2\int_{\frac{3a+b}{4}}^{\frac{a+b}{2}} \frac{f(x)\varphi'(x)}{(\varphi\left(\frac{a+b}{2}\right)-\varphi\left(\frac{3a+b}{4}\right))^3}dx,$$

$$I_3 = -\frac{(f\circ\varphi^{-1})'\left(\varphi\left(\frac{a+3b}{4}\right)\right)}{\varphi\left(\frac{a+3b}{4}\right)-\varphi(\frac{a+b}{2})} - 2\frac{f\left(\frac{a+3b}{4}\right)}{\left(\varphi\left(\frac{a+3b}{4}\right)-\varphi(\frac{a+b}{2})\right)^2} + 2\int_{\frac{a+b}{2}}^{\frac{a+3b}{4}} \frac{f(x)\varphi'(x)}{\left(\varphi\left(\frac{a+3b}{4}\right)-\varphi(\frac{a+b}{2})\right)^3}dx,$$

$$I_2 = -\frac{(f\circ\varphi^{-1})'\left(\varphi\left(\frac{a+3b}{4}\right)\right)}{\varphi(b)-\varphi(\frac{a+3b}{4})} - 2\frac{f\left(\frac{a+3b}{4}\right)}{\left(\varphi(b)-\varphi(\frac{a+3b}{4})\right)^2} + 2\int_{\frac{a+3b}{4}}^{b} \frac{f(x)\varphi'(x)}{\left(\varphi(b)-\varphi(\frac{a+3b}{4})\right)^3}dx.$$

Now multiplying I_1 by $\left(\varphi\left(\frac{3a+b}{4}\right)-\varphi(a)\right)^3$, I_2 by $\left(\varphi(\frac{a+b}{2})-\varphi\left(\frac{3a+b}{4}\right)\right)^3$, I_3 by $\left(\varphi(\frac{a+3b}{4})-\varphi\left(\frac{a+b}{2}\right)\right)^3$ and I_4 by $\left(\varphi(b)-\varphi\left(\frac{a+3b}{4}\right)\right)^3$, and adding then the obtained expressions we get,

$$\left(\varphi\left(\frac{3a+b}{4}\right)-\varphi(a)\right)^3 I_1 + \left(\varphi(\frac{a+b}{2})-\varphi\left(\frac{3a+b}{4}\right)\right)^3 I_2 +$$

$$+\left(\varphi(\frac{a+3b}{4})-\varphi\left(\frac{a+b}{2}\right)\right)^3 I_3 + \left(\varphi(b)-\varphi\left(\frac{a+3b}{4}\right)\right)^3 I_4 =$$

$$= 2\int_a^b f(x)\varphi'(x)dx - 2f\left(\frac{3a+b}{4}\right)(\varphi\left(\frac{a+b}{2}\right)-\varphi(a)) - 2f\left(\frac{a+3b}{4}\right)(\varphi(b)-\varphi\left(\frac{a+b}{2}\right))+$$

$$+\frac{f'}{\varphi'}\left(\frac{3a+b}{4}\right)\left(2\varphi\left(\frac{3a+b}{4}\right)-\varphi(a)-\varphi\left(\frac{a+b}{2}\right)\right)(\varphi\left(\frac{a+b}{2}\right)-\varphi(a))+$$

$$+\frac{f'}{\varphi'}\left(\frac{3a+b}{4}\right)\left(2\varphi\left(\frac{3a+b}{4}\right)-\varphi(a)-\varphi\left(\frac{a+b}{2}\right)\right)(\varphi(b)-\varphi\left(\frac{a+b}{2}\right))$$

and from here the desired inequality. We used above also the properties of the derivatives of inverse of the function φ.

□

Theorem 1. *Let $f : I \subset \mathbb{R}_+ \to \mathbb{R}$ be differentiable and f' differentiable on I°, and $a, b \in I^\circ$ with $a < b$, $\varphi : I \to \mathbb{R}$ be a continuous and strictly monotonic function so that $\varphi^{-1} : \varphi(I^\circ) \to I^\circ$ is continuously differentiable, $(\varphi^{-1})'$ continuously differentiable and $f', f'' \in L[a, b]$.*

If $|f'|$ and $|f''|$ are $M_\varphi A$-s-convex function in the second sense on $[a, b]$ and $s \in (0, 1]$, then

$$|\frac{1}{\varphi(b)-\varphi(a)}\int_a^b f(x)\varphi'(x)dx - f\left(\frac{3a+b}{4}\right)\frac{\varphi\left(\frac{a+b}{2}\right)-\varphi(a)}{\varphi(b)-\varphi(a)} - f\left(\frac{a+3b}{4}\right)\frac{\varphi(b)-\varphi\left(\frac{a+b}{2}\right)}{\varphi(b)-\varphi(a)}+$$

$$+\frac{1}{2}\frac{\varphi\left(\frac{a+b}{2}\right)-\varphi(a)}{\varphi(b)-\varphi(a)}\frac{f^{'}}{\varphi^{'}}\left(\frac{3a+b}{4}\right)\left(2\varphi\left(\frac{3a+b}{4}\right)-\varphi(a)-\varphi\left(\frac{a+b}{2}\right)\right)+$$
$$+\frac{1}{2}\frac{\varphi(b)-\varphi\left(\frac{a+b}{2}\right)}{\varphi(b)-\varphi(a)}\frac{f^{'}}{\varphi^{'}}\left(\frac{a+3b}{4}\right)\left(2\varphi\left(\frac{a+3b}{4}\right)-\varphi\left(\frac{a+b}{2}\right)-\varphi(b)\right)|\leq$$

$$\leq\frac{1}{2|\varphi(b)-\varphi(a)|}\{|\varphi\left(\frac{3a+b}{4}\right)-\varphi(a)|^3[|f^{''}\left(\frac{3a+b}{4}\right)|A_{\varphi,s+2,0}(a,b,4,0)+$$
$$+|f^{''}(a)|A_{\varphi,2,s}(a,b,4,0)+|f^{'}\left(\frac{3a+b}{4}\right)|B_{\varphi,s+2,0}(a,b,4,0)+|f^{'}(a)|B_{\varphi,2,s}(a,b,4,0)]+$$
$$+|\varphi\left(\frac{a+b}{2}\right)-\varphi(\frac{3a+b}{4})|^3[|f^{''}\left(\frac{a+b}{2}\right)|A_{\varphi,s,2}(a,b,3,1)+|f^{''}(\frac{3a+b}{4})|A_{\varphi,0,s+2}(a,b,3,1)+$$
$$+|f^{'}\left(\frac{a+b}{2}\right)|B_{\varphi,s,2}(a,b,3,1)+|f^{'}(\frac{3a+b}{4})|B_{\varphi,0,s+2}(a,b,3,1)]+$$
$$+|\varphi\left(\frac{a+3b}{4}\right)-\varphi(\frac{a+b}{2})|^3[|f^{''}\left(\frac{a+3b}{4}\right)|A_{\varphi,s+2,0}(a,b,2,2)+|f^{''}\left(\frac{a+b}{2}\right)|A_{\varphi,2,s}(a,b,2,2)+$$
$$+|f^{'}\left(\frac{a+3b}{4}\right)|B_{\varphi,s+2,0}(a,b,2,2)+|f^{'}(\frac{a+b}{2})|B_{\varphi,2,s}(a,b,2,2)]+$$
$$+|\varphi(b)-\varphi\left(\frac{3a+b}{4}\right)|^3[|f^{''}(b)|A_{\varphi,s,2}(a,b,1,3)+|f^{''}\left(\frac{a+3b}{4}\right)|A_{\varphi,0,s+2}(a,b,1,3)+$$
$$|f^{'}(b)|B_{\varphi,s,2}(a,b,1,3)+|f^{'}\left(\frac{a+3b}{4}\right)|B_{\varphi,0,s+2}(a,b,1,3)]\},$$

where $A_{\varphi,n,m}(a,b,l,k)=\int_0^1 t^n(1-t)^m[(\varphi^{-1})^{'}(t\varphi\left(\frac{(l-1)a+(k+1)b}{4}\right)+(1-t)\varphi(\frac{la+kb}{4}))]^2dt,$
$B_{\varphi,n,m}(a,b,l,k)=\int_0^1 t^n(1-t)^m|(\varphi^{-1})^{''}(t\varphi\left(\frac{(l-1)a+(k+1)b}{4}\right)+(1-t)\varphi(\frac{la+kb}{4}))|dt.$

Proof. By using previous lemma and that $|f^{'}|,|f^{''}|$ are $M_\varphi A$-s-convex function in the second sense on $[a,b]$ we get

$$|\frac{\left(\varphi\left(\frac{3a+b}{4}\right)-\varphi(a)\right)^3I_1+\left(\varphi(\frac{a+b}{2})-\varphi\left(\frac{3a+b}{4}\right)\right)^3I_2+\left(\varphi(\frac{a+3b}{4})-\varphi\left(\frac{a+b}{2}\right)\right)^3I_3+\left(\varphi(b)-\varphi\left(\frac{a+3b}{4}\right)\right)^3I_4}{2(\varphi(b)-\varphi(a))}|\leq$$

$$\leq\frac{1}{2|\varphi(b)-\varphi(a)|}\{|\varphi\left(\frac{3a+b}{4}\right)-\varphi(a)|^3[\int_0^1 t^2\left((\varphi^{-1})^{'}(t\varphi(\frac{3a+b}{4})+(1-t)\varphi(a))\right)^2\cdot$$
$$\cdot\left(t^s|f^{''}(\frac{3a+b}{4})|+(1-t)^s|f^{''}(a)|\right)dt+\int_0^1 t^2|(\varphi^{-1})^{''}(t\varphi(\frac{3a+b}{4})+(1-t)\varphi(a))|\cdot$$
$$\cdot\left(t^s|f^{'}(\frac{3a+b}{4})|+(1-t)^s|f^{'}(a)|\right)dt]+$$
$$+|\varphi\left(\frac{a+b}{2}\right)-\varphi(\frac{3a+b}{4})|^3[\int_0^1(t-1)^2\left((\varphi^{-1})^{'}(t\varphi(\frac{a+b}{2})+(1-t)\varphi(\frac{3a+b}{4}))\right)^2\cdot$$
$$\cdot\left(t^s|f^{''}(\frac{a+b}{2})|+(1-t)^s|f^{''}(\frac{3a+b}{4})|\right)dt+\int_0^1(t-1)^2|(\varphi^{-1})^{''}(t\varphi(\frac{a+b}{2})+(1-t)\varphi(\frac{3a+b}{4}))|\cdot$$
$$\cdot\left(t^s|f^{'}(\frac{a+b}{2})|+(1-t)^s|f^{'}(\frac{3a+b}{4})|\right)dt]+$$
$$+|\varphi\left(\frac{a+3b}{4}\right)-\varphi(\frac{a+b}{2})|^3[\int_0^1 t^2\left((\varphi^{-1})^{'}(t\varphi(\frac{a+3b}{4})+(1-t)\varphi(\frac{a+b}{2}))\right)^2\cdot$$

$$\cdot\left(t^s|f''(\frac{a+3b}{4})|+(1-t)^s|f''(\frac{a+b}{2})|\right)dt+\int_0^1 t^2|(\varphi^{-1})''(t\varphi(\frac{a+3b}{4})+(1-t)\varphi(\frac{a+b}{2}))|\cdot$$
$$\cdot\left(t^s|f'(\frac{a+3b}{4})|+(1-t)^s|f'(\frac{a+b}{2})|\right)dt]+$$
$$+|\varphi(b)-\varphi(\frac{a+3b}{4})|^3[\int_0^1(t-1)^2\left((\varphi^{-1})'(t\varphi(b)+(1-t)\varphi(\frac{a+3b}{4}))\right)^2\cdot$$
$$\cdot\left(t^s|f''(b)|+(1-t)^s|f''(\frac{a+3b}{4})|\right)dt+\int_0^1(t-1)^2|(\varphi^{-1})''(t\varphi(b)+(1-t)\varphi(\frac{a+3b}{4}))|\cdot$$
$$\cdot\left(t^s|f'(b)|+(1-t)^s|f'(\frac{a+3b}{4})|\right)dt]\}.$$

From here by calculus and by using previous notations we get the desired inequality. □

Theorem 2. *Let $f: I\subset\mathbb{R}_+\to\mathbb{R}$ be differentiable and f' differentiable on I°, and $a,b\in I^\circ$ with $a<b$, $\varphi: I\to\mathbb{R}$ be a continuous and strictly monotonic function so that $\varphi^{-1}:\varphi(I^\circ)\to I^\circ$ is continuously differentiable, $(\varphi^{-1})'$ continuously differentiable and $f',f''\in L[a,b]$.*

If $|f'|$ and $|f''|$ are $M_\varphi A$-s-convex function in the first sense on $[a,b]$ and $s\in(0,1]$, then

$$|\frac{1}{\varphi(b)-\varphi(a)}\int_a^b f(x)\varphi'(x)dx-f\left(\frac{3a+b}{4}\right)\frac{\varphi\left(\frac{a+b}{2}\right)-\varphi(a)}{\varphi(b)-\varphi(a)}-f\left(\frac{a+3b}{4}\right)\frac{\varphi(b)-\varphi\left(\frac{a+b}{2}\right)}{\varphi(b)-\varphi(a)}+$$
$$+\frac{1}{2}\frac{\varphi\left(\frac{a+b}{2}\right)-\varphi(a)}{\varphi(b)-\varphi(a)}\frac{f'}{\varphi'}\left(\frac{3a+b}{4}\right)\left(2\varphi\left(\frac{3a+b}{4}\right)-\varphi(a)-\varphi\left(\frac{a+b}{2}\right)\right)+$$
$$+\frac{1}{2}\frac{\varphi(b)-\varphi\left(\frac{a+b}{2}\right)}{\varphi(b)-\varphi(a)}\frac{f'}{\varphi'}\left(\frac{a+3b}{4}\right)\left(2\varphi\left(\frac{a+3b}{4}\right)-\varphi\left(\frac{a+b}{2}\right)-\varphi(b)\right)|\le$$
$$\le\frac{1}{2|\varphi(b)-\varphi(a)|}\{|\varphi\left(\frac{3a+b}{4}\right)-\varphi(a)|^3[|f''\left(\frac{3a+b}{4}\right)|A_{\varphi,s+2,0}(a,b,4,0)+$$
$$+|f''(a)|A'_{\varphi,2,s}(a,b,4,0)+|f'\left(\frac{3a+b}{4}\right)|B_{\varphi,s+2,0}(a,b,4,0)+|f'(a)|B'_{\varphi,2,s}(a,b,4,0)]+$$
$$+|\varphi\left(\frac{a+b}{2}\right)-\varphi(\frac{3a+b}{4})|^3[|f''\left(\frac{a+b}{2}\right)|A_{\varphi,s,2}(a,b,3,1)+|f''(\frac{3a+b}{4})|C'_{\varphi,2,s}(a,b,3,1)+$$
$$+|f'\left(\frac{a+b}{2}\right)|B_{\varphi,s,2}(a,b,3,1)+|f'(\frac{3a+b}{4})|D'_{\varphi,2,s}(a,b,3,1)]+$$
$$+|\varphi\left(\frac{a+3b}{4}\right)-\varphi(\frac{a+b}{2})|^3[|f''\left(\frac{a+3b}{4}\right)|A_{\varphi,s+2,0}(a,b,2,2)+|f''\left(\frac{a+b}{2}\right)|A'_{\varphi,2,s}(a,b,2,2)+$$
$$+|f'\left(\frac{a+3b}{4}\right)|B_{\varphi,s+2,0}(a,b,2,2)+|f'(\frac{a+b}{2})|B'_{\varphi,2,s}(a,b,2,2)]+$$
$$+|\varphi(b)-\varphi\left(\frac{3a+b}{4}\right)|^3[|f''(b)|A_{\varphi,s,2}(a,b,1,3)+|f''\left(\frac{a+3b}{4}\right)|C'_{\varphi,2,s}(a,b,1,3)+$$
$$+|f'(b)|B_{\varphi,s,2}(a,b,1,3)+|f'\left(\frac{a+3b}{4}\right)|D'_{\varphi,2,s}(a,b,1,3)]\},$$

where $A^{'}_{\varphi,n,m}(a,b,l,k)=\int_0^1 t^n(1-t^m)[(\varphi^{-1})^{'}(t\varphi\left(\frac{(l-1)a+(k+1)b}{4}\right)+(1-t)\varphi(\frac{la+kb}{4}))]^2dt,$

$B^{'}_{\varphi,n,m}(a,b,l,k)=\int_0^1 t^n(1-t^m)|(\varphi^{-1})^{''}(t\varphi\left(\frac{(l-1)a+(k+1)b}{4}\right)+(1-t)\varphi(\frac{la+kb}{4}))|dt,$

$C^{'}_{\varphi,n,m}(a,b,l,k)=\int_0^1 (t-1)^n(1-t^m)[(\varphi^{-1})^{'}(t\varphi\left(\frac{(l-1)a+(k+1)b}{4}\right)+(1-t)\varphi(\frac{la+kb}{4}))]^2dt,$

$D^{'}_{\varphi,n,m}(a,b,l,k)=\int_0^1 (t-1)^n(1-t^m)|(\varphi^{-1})^{''}(t\varphi\left(\frac{(l-1)a+(k+1)b}{4}\right)+(1-t)\varphi(\frac{la+kb}{4}))|dt,$

and $A_{\varphi,n,m}(a,b,l,k)$, $B_{\varphi,n,m}(a,b,l,k)$ *are like in previous theorem.*

Proof. We use the definition of $M_\varphi A$-s-convexity in the first sense on $[a,b]$ like in previous theorem and the demonstration will be similar. □

Theorem 3. *Let* $f: I\subset\mathbb{R}_+\to\mathbb{R}$ *be differentiable and* $f^{'}$ *differentiable on* I°, *and* $a,b\in I^\circ$ *with* $a<b$, $\varphi: I\to\mathbb{R}$ *be a continuous and strictly monotonic function so that* $\varphi^{-1}:\varphi(I^\circ)\to I^\circ$ *is continuously differentiable,* $(\varphi^{-1})^{'}$ *continuously differentiable and* $f^{'}, f^{''}\in L[a,b]$ *and* $q>1$, $\frac{1}{p}+\frac{1}{q}=1$.

If $|f^{'}|^q$ *and* $|f^{''}|^q$ *are* $M_\varphi A$*-s-convexity in the second sense on* $[a,b]$ *and* $s\in(0,1]$ *then we have,*

$$|\frac{1}{\varphi(b)-\varphi(a)}\int_a^b f(x)\varphi^{'}(x)dx-f\left(\frac{3a+b}{4}\right)\frac{\varphi\left(\frac{a+b}{2}\right)-\varphi(a)}{\varphi(b)-\varphi(a)}-f\left(\frac{a+3b}{4}\right)\frac{\varphi(b)-\varphi\left(\frac{a+b}{2}\right)}{\varphi(b)-\varphi(a)}+$$

$$+\frac{1}{2}\frac{\varphi\left(\frac{a+b}{2}\right)-\varphi(a)}{\varphi(b)-\varphi(a)}\frac{f^{'}}{\varphi^{'}}\left(\frac{3a+b}{4}\right)\left(2\varphi\left(\frac{3a+b}{4}\right)-\varphi(a)-\varphi\left(\frac{a+b}{2}\right)\right)+$$

$$+\frac{1}{2}\frac{\varphi(b)-\varphi\left(\frac{a+b}{2}\right)}{\varphi(b)-\varphi(a)}\frac{f^{'}}{\varphi^{'}}\left(\frac{a+3b}{4}\right)\left(2\varphi\left(\frac{a+3b}{4}\right)-\varphi\left(\frac{a+b}{2}\right)-\varphi(b)\right)|\le$$

$$\le\frac{1}{2}\frac{|\varphi\left(\frac{3a+b}{4}\right)-\varphi(a)|^3}{|\varphi(b)-\varphi(a)|}\{(A^{''}_{\varphi,2p,0}(a,b,4,0))^{\frac{1}{p}}\left(\frac{|f^{''}\left(\frac{3a+b}{4}\right)|^q+|f^{''}(a)|^q}{s+1}\right)^{\frac{1}{q}}+$$

$$+(B^{''}_{\varphi,2p,0}(a,b,4,0))^{\frac{1}{p}}\left(\frac{|f^{'}\left(\frac{3a+b}{4}\right)|^q+|f^{'}(a)|^q}{s+1}\right)^{\frac{1}{q}}\}+$$

$$+\frac{1}{2}\frac{\varphi\left(\frac{a+b}{2}\right)-\varphi\left(\frac{3a+b}{4}\right)|^3}{|\varphi(b)-\varphi(a)|}\{(A^{''}_{\varphi,0,2p}(a,b,1,3))^{\frac{1}{p}}\left(\frac{|f^{''}\left(\frac{a+b}{2}\right)|^q+|f^{''}\left(\frac{3a+b}{4}\right)|^q}{s+1}\right)^{\frac{1}{q}}+$$

$$+(B^{''}_{\varphi,0,2p}(a,b,1,3))^{\frac{1}{p}}\left(\frac{|f^{'}\left(\frac{a+b}{2}\right)|^q+|f^{''}\left(\frac{3a+b}{4}\right)|^q}{s+1}\right)^{\frac{1}{q}}\},$$

$$+\frac{1}{2}\frac{|\varphi\left(\frac{a+3b}{4}\right)-\varphi(\frac{a+b}{2})|^3}{|\varphi(b)-\varphi(a)|}\{(A^{''}_{\varphi,2p,0}(a,b,2,2))^{\frac{1}{p}}\left(\frac{|f^{''}\left(\frac{a+3b}{4}\right)|^q+|f^{''}(\frac{a+b}{2})|^q}{s+1}\right)^{\frac{1}{q}}+$$

$$+(B^{''}_{\varphi,2p,0}(a,b,2,2))^{\frac{1}{p}}\left(\frac{|f^{'}\left(\frac{a+3b}{4}\right)|^q+|f^{'}(\frac{a+b}{2})|^q}{s+1}\right)^{\frac{1}{q}}\}+$$

$$+\frac{1}{2}\frac{\varphi(b)-\varphi\left(\frac{a+3b}{4}\right)|^3}{|\varphi(b)-\varphi(a)|}\{(A^{''}_{\varphi,0,2p}(a,b,3,1))^{\frac{1}{p}}\left(\frac{|f^{''}(b)|^q+|f^{''}\left(\frac{a+3b}{4}\right)|^q}{s+1}\right)^{\frac{1}{q}}+$$

$$+(B''_{\varphi,0,2p}(a,b,3,1))^{\frac{1}{p}}\left(\frac{|f'(b)|^q+|f''\left(\frac{a+3b}{4}\right)|^q}{s+1}\right)^{\frac{1}{q}}\},$$

where $A''_{\varphi,n,m}(a,b,l,k)=\int_0^1 t^n(1-t)^m[(\varphi^{-1})'\left(t\varphi\left(\frac{(l-1)a+(k+1)b}{4}\right)+(1-t)\varphi(\frac{la+kb}{4})\right)]^{2p}dt,$

$B''_{\varphi,n,m}(a,b,l,k)=\int_0^1 t^n(1-t)^m|(\varphi^{-1})''\left(t\varphi\left(\frac{(l-1)a+(k+1)b}{4}\right)+(1-t)\varphi(\frac{la+kb}{4}\right)|^p dt.$

Proof. Using Lemma 4 and Holder's inequality we have succesively

$$|\frac{1}{\varphi(b)-\varphi(a)}\int_a^b f(x)\varphi'(x)dx-f\left(\frac{3a+b}{4}\right)\frac{\varphi\left(\frac{a+b}{2}\right)-\varphi(a)}{\varphi(b)-\varphi(a)}-f\left(\frac{a+3b}{4}\right)\frac{\varphi(b)-\varphi\left(\frac{a+b}{2}\right)}{\varphi(b)-\varphi(a)}+$$

$$+\frac{1}{2}\frac{\varphi\left(\frac{a+b}{2}\right)-\varphi(a)}{\varphi(b)-\varphi(a)}\frac{f'}{\varphi'}\left(\frac{3a+b}{4}\right)\left(2\varphi\left(\frac{3a+b}{4}\right)-\varphi(a)-\varphi\left(\frac{a+b}{2}\right)\right)+$$

$$+\frac{1}{2}\frac{\varphi(b)-\varphi\left(\frac{a+b}{2}\right)}{\varphi(b)-\varphi(a)}\frac{f'}{\varphi'}\left(\frac{a+3b}{4}\right)\left(2\varphi\left(\frac{a+3b}{4}\right)-\varphi\left(\frac{a+b}{2}\right)-\varphi(b)\right)|\leq$$

$$\leq\frac{1}{2}\frac{|\varphi\left(\frac{3a+b}{4}\right)-\varphi(a)|^3}{|\varphi(b)-\varphi(a)|}\{\left(\int_0^1 t^{2p}[(\varphi^{-1})'\left(t\varphi\left(\frac{3a+b}{4}\right)+(1-t)\varphi(a)\right)]^{2p}dt\right)^{\frac{1}{p}}\cdot$$

$$\cdot\left(\int_0^1|f''(\varphi^{-1})\left(t\varphi\left(\frac{3a+b}{4}\right)+(1-t)\varphi(a)\right)|^q dt\right)^{\frac{1}{q}}+$$

$$+\left(\int_0^1 t^{2p}|(\varphi^{-1})''\left(t\varphi\left(\frac{3a+b}{4}\right)+(1-t)\varphi(a)\right)|^p dt\right)^{\frac{1}{p}}\cdot$$

$$\cdot\left(\int_0^1|f'(\varphi^{-1})\left(t\varphi\left(\frac{3a+b}{4}\right)+(1-t)\varphi(a)\right)|^q dt\right)^{\frac{1}{q}}\}+$$

$$+\frac{1}{2}\frac{|\varphi(\frac{a+b}{2})-\varphi\left(\frac{3a+b}{4}\right)|^3}{|\varphi(b)-\varphi(a)|}\{\left(\int_0^1(1-t)^{2p}[(\varphi^{-1})'\left(t\varphi(\frac{a+b}{2})+(1-t)\varphi\left(\frac{3a+b}{4}\right)\right)]^{2p}dt\right)^{\frac{1}{p}}\cdot$$

$$\cdot\left(\int_0^1|f''(\varphi^{-1})\left(t\varphi(\frac{a+b}{2})+(1-t)\varphi\left(\frac{3a+b}{4}\right)\right)|^q dt\right)^{\frac{1}{q}}+$$

$$+\left(\int_0^1(1-t)^{2p}|(\varphi^{-1})''\left(t\varphi(\frac{a+b}{2})+(1-t)\varphi\left(\frac{3a+b}{4}\right)\right)|^p dt\right)^{\frac{1}{p}}\cdot$$

$$\cdot\left(\int_0^1|f'(\varphi^{-1})\left(t\varphi(\frac{a+b}{2})+(1-t)\varphi\left(\frac{3a+b}{4}\right)\right)|^q dt\right)^{\frac{1}{q}}\}+$$

$$+\frac{1}{2}\frac{|\varphi\left(\frac{a+3b}{4}\right)-\varphi(\frac{a+b}{2})|^3}{|\varphi(b)-\varphi(a)|}\{\left(\int_0^1 t^{2p}[(\varphi^{-1})'\left(t\varphi\left(\frac{a+3b}{4}\right)+(1-t)\varphi(\frac{a+b}{2})\right)]^{2p}dt\right)^{\frac{1}{p}}\cdot$$

$$\cdot\left(\int_0^1|f''(\varphi^{-1})\left(t\varphi\left(\frac{a+3b}{4}\right)+(1-t)\varphi(\frac{a+b}{2})\right)|^q dt\right)^{\frac{1}{q}}+$$

$$+\left(\int_0^1 t^{2p}|(\varphi^{-1})''\left(t\varphi\left(\frac{a+3b}{4}\right)+(1-t)\varphi(\frac{a+b}{2})\right)|^p dt\right)^{\frac{1}{p}}\cdot$$

$$\cdot\left(\int_0^1|f'(\varphi^{-1})\left(t\varphi\left(\frac{a+3b}{4}\right)+(1-t)\varphi(\frac{a+b}{2})\right)|^q dt\right)^{\frac{1}{q}}\}+$$

$$+\frac{1}{2}\frac{|\varphi(b)-\varphi\left(\frac{a+3b}{4}\right)|^{3}}{|\varphi(b)-\varphi(a)|}\{\left(\int_{0}^{1}(t-1)^{2p}[(\varphi^{-1})^{'}\left(t\varphi\left(b\right)+(1-t)\varphi(\frac{a+3b}{4})\right)]^{2p}dt\right)^{\frac{1}{p}}\cdot$$

$$\cdot\left(\int_{0}^{1}|f^{''}\left(\varphi^{-1}\right)\left(t\varphi\left(b\right)+(1-t)\varphi(\frac{a+3b}{4})\right)|^{q}dt\right)^{\frac{1}{q}}+$$

$$+\left(\int_{0}^{1}(t-1)^{2p}|(\varphi^{-1})^{''}\left(t\varphi\left(b\right)+(1-t)\varphi(\frac{a+3b}{4})\right)|^{p}dt\right)^{\frac{1}{p}}\cdot$$

$$\cdot\left(\int_{0}^{1}|f^{'}\left(\varphi^{-1}\right)\left(t\varphi\left(b\right)+(1-t)\varphi(\frac{a+3b}{4})\right)|^{q}dt\right)^{\frac{1}{q}}\},$$

and from here using that $|f^{'}|^{q}$ and $|f^{''}|^{q}$ are $M_{\varphi}A$-s-convexity in the second sense on $[a,b]$ we get the desired inequality.

□

References

[1] Alomari, M., Darus, M., Kirmaci, U. S., Some inequalities of Hermite-Hadamard type for s-convex functions, *Acta Mathematica Scientia*, (2011) 31 B(4), 1643-1652.

[2] Ciurdariu, L., A note concerning several Hermite-Hadamard inequalities for different types of convex functions, *Int. J. of Math. Anal.*, **6(33)** (2012), 1623-1639.

[3] Ciurdariu, L., On some Hermite-Hadamard type inequalities for functions whose power of absolute value of derivatives are $(\alpha, m)-$ convex, *Int. J. of Math. Anal.*, **6(48)** (2012), 2361-2383.

[4] Dragomir, S. S., Pearce, C. E. M., Selected topic on Hermite-Hadamard inequalities and applications, *Melbourne and Adelaide* December, (2001).

[5] Dragomir, S. S., Fitzpatrick, S., The Hadamard's inequality for s-convex functionsin the second sense, *Demonstratio Math.* , **32 (4)** (1999), 687-696.

[6] Hadamard, J., Etude sur le proprietes des fonctions entieres en particulier d' une fonction consideree par Riemann, J. Math. Pures Appl., 58, 171-215 (1893).

[7] Iscan, Imdat, Kunt, M., Yazici, N., Gozutok, Tuncay, K., New general integral inequalities for Lipschitzian functions via Riemann-Liouville fractional integrals and applications, *Joirnal of Inequalities and Special Functions*, **7** 4, (2016), 1-12.

[8] Iscan I., Hermite-Hadamard type inequalities for harmonically convex functions, Hacettepe Journal of Mathematics and Statistics, 43(6), (2014), 935-942.

[9] Iscan I., Some new general integral inequalities for h-convex and h-concave functions, Adv. Pure Appl. Math. 5(1), (2014), 21-29.

[10] Iscan I., Hermite-Hadamard type inequalities for GA-s-convex functions, Le Matematiche, Vol. LXIX (2014)- Fasc. II, pp. 129-146.

[11] Khan M. Adil, Khurshid Y., Dragomir S. S., and Ullah R., Inequalities of Hermite-Hadamard type with applications, Punjab Univ. J. Math. 50(3), (2018) 1-12.

[12] Kirmaci, U. S., Inequalities for differentiable mappings and applications to special means of real numbers and to midpoint formula, *Appl. Math. Comput.*, **147 (1)** (2014), 137-146.

[13] Mihesan, V. G., A generalization of the convexity, Seminar of Functional Equations, *Approx. and Convex,* Cluj-Napoca, Romania (1993).

[14] Set, E., New inequalities of Ostrowski type for mappings whose derivatives are s-convex in the second via fractional integrals, *Comput. Math. Appl.* **(2010)** Art ID:531976, 7 pages.

[15] Tekin Toplu , Imdat Iscan and Kadakal Mahir, Hyperbolic type convexity and some new inequalities, *Honan Mathematical J.* **42**, (2020) No. 2, pp. 301-318.

[16] Toader, Gh., On a generalization of the convexity, *Mathematica*, 30 (53) (1988), 83-87.

[17] Tirtirau L., Several Hermite-Hadamard type inequalities for $M_{\varphi}A$-s-convex functions, *Int. J. of Math. Anal.*, **15(7)** (2021), 321-330.

[18] Turhan S., Kunt M., Iscan I., On Hermite-Hadamard type inequalities with respect to the generalization of some types of s-convexity, Konuralp Journal of Mathematics, 8(1) (2020) 165-174.

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