DETERMINANT INEQUALITIES FOR POSITIVE DEFINITE MATRICES VIA ADDITIVE AND MULTIPLICATIVE YOUNG INEQUALITIES

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ABSTRACT. In this paper we prove among others that, if the positive definite matrices A, B of order n satisfy the condition

$$0 < mI_n \leq B - A \leq MI_n,$$

for some constants $0 < m < M$, where I_n is the identity matrix, then

$$\begin{aligned} 0 &\leq (1-t)\left[\det(A)\right]^{-1} + t\left[\det(A + mI_n)\right]^{-1} - \left[\det(A + mtI_n)\right]^{-1} \\ &\leq (1-t)\left[\det(A)\right]^{-1} + t\left[\det(B)\right]^{-1} - \left[\det((1-t)A + tB)\right]^{-1} \\ &\leq (1-t)\left[\det(A)\right]^{-1} + t\left[\det(A + MI_n)\right]^{-1} - \left[\det(A + MtI_n)\right]^{-1}, \end{aligned}$$

for all $t \in [0, 1]$.

1. Introduction

A real square matrix $A = (a_{ij})$, $i, j = 1, ..., n$ is *symmetric* provided $a_{ij} = a_{ji}$ for all $i, j = 1, ..., n$. A real symmetric matrix is said to be *positive definite* provided the quadratic form $Q(x) = \sum_{i,j=1}^{n} a_{ij}x_i x_j$ is positive for all $x = (x_1, ..., x_n) \in \mathbb{R}^n \setminus \{0\}$. It is well known that a necessary and sufficient condition for the symmetric matrix A to be positive definite, and we write $A > 0$, is that all determinants

$$\det(A_k) = \det(a_{ij}),\ i,\ j = 1, ..., k;\ k = 1, ..., n$$

are positive.

It is know that the following integral representation is valid, see [1, pp. 61-62] or [11, pp. 211-212]

$$J_n(A) := \int_{\mathbb{R}^n} \exp(-\langle Ax, x\rangle)dx := \int_{-\infty}^{\infty} ... \int_{-\infty}^{\infty} \exp(-\langle Ax, x\rangle)dx = \frac{\pi^{n/2}}{\left[\det(A)\right]^{1/2}}, \tag{1.1}$$

where A is a positive definite matrix of order n and $\langle \cdot, \cdot \rangle$ is the usual inner product on $\mathbb{R}^n$.

By utilizing the representation (1.1) and Hölder's integral inequality for multiple integrals one can prove the *logarithmic concavity* of the determinant that is due to Ky Fan ([1, p. 63] or [11, p. 212]), namely

$$\det((1-\lambda)A + \lambda B) \geq \left[\det(A)\right]^{1-\lambda}\left[\det(B)\right]^{\lambda} \tag{1.2}$$

for any positive definite matrices A, B and $\lambda \in [0, 1]$.

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By mathematical induction we can get a generalization of (1.2) which was obtained by L. Mirsky in [10], see also [11, p. 212]

$$\det\left(\sum_{j=1}^{m}\lambda_j A_j\right)\geq\prod_{j=1}^{m}\left[\det\left(A_j\right)\right]^{\lambda_j},\ m\geq 2, \tag{1.3}$$

where $\lambda_j>0$, $j=1,...,m$ with $\sum_{j=1}^{m}\lambda_j=1$ and $A_j>0$, $j=1,...,m$.

If we write (1.3) for $A_j=B_j^{-1}$ we get

$$\det\left(\sum_{j=1}^{m}\lambda_j B_j^{-1}\right)\geq\prod_{j=1}^{m}\left[\det\left(B_j^{-1}\right)\right]^{\lambda_j}=\left(\prod_{j=1}^{m}\left[\det\left(B_j\right)\right]^{\lambda_j}\right)^{-1},$$

which also gives

$$\prod_{j=1}^{m}\left[\det\left(A_j\right)\right]^{\lambda_j}\geq\det\left[\left(\sum_{j=1}^{m}\lambda_j A_j^{-1}\right)^{-1}\right], \tag{1.4}$$

where $\lambda_j>0$, $j=1,...,m$ with $\sum_{j=1}^{m}\lambda_j=1$ and $A_j>0$, $j=1,...,m$.

Using the representation (1.1) one can also prove the result, see [11, p. 212],

$$\det\left(A\right)=\det\left(A_{1n}\right)\leq\det\left(A_{1k}\right)\det\left(A_{(k+1)n}\right),\ k=1,...,n; \tag{1.5}$$

where the determinant $\det\left(A_{rs}\right)$ is defined by

$$\det\left(A_{rs}\right)=\det\left(a_{ij}\right),i,\ j=r,...,s.$$

In particular,

$$\det\left(A\right)\leq a_{11}a_{22}...a_{nn}. \tag{1.6}$$

We recall also the Minkowski's type inequality,

$$\left[\det\left(A+B\right)\right]^{1/n}\geq\left[\det\left(A\right)\right]^{1/n}+\left[\det\left(B\right)\right]^{1/n} \tag{1.7}$$

for A, B positive definite matrices of order n. For other determinant inequalities see Chapter VIII of the classic book [11]. For some recent results see [5]-[9].

Motivated by the above results, in this paper we prove among others that, if the positive definite matrices A, B of order n satisfy the condition

$$0<mI_n\leq B-A\leq MI_n,$$

for some constants $0<m<M$, where I_n is the identity matrix, then

$$\begin{aligned}0&\leq(1-t)\left[\det\left(A\right)\right]^{-1}+t\left[\det\left(A+mI_n\right)\right]^{-1}-\left[\det\left(A+mtI_n\right)\right]^{-1}\\&\leq(1-t)\left[\det\left(A\right)\right]^{-1}+t\left[\det\left(B\right)\right]^{-1}-\left[\det\left((1-t)A+tB\right)\right]^{-1}\\&\leq(1-t)\left[\det\left(A\right)\right]^{-1}+t\left[\det\left(A+MI_n\right)\right]^{-1}-\left[\det\left(A+MtI_n\right)\right]^{-1},\end{aligned}$$

for all $t\in[0,1]$.

2. Additive Inequalities

We consider the function $f_t : [0,\infty) \to [0,\infty)$ defined for $t \in (0,1)$ by

$$f_t(u) = 1 - t + tu - u^t. \tag{2.1}$$

The following lemma holds.

Lemma 1. *For any $u \in [k,K] \subset [0,\infty)$ we have*

$$\max_{u\in[k,K]} f_t(u) = \Delta_t(k,K) := \begin{cases} f_t(k) & \text{if } K < 1, \\ \max\{f_t(k), f_t(K)\} & \text{if } k \le 1 \le K, \\ f_t(K) & \text{if } 1 < k \end{cases} \tag{2.2}$$

and

$$\min_{u\in[k,K]} f_t(u) = \delta_t(k,K) := \begin{cases} f_t(K) & \text{if } K < 1, \\ 0 & \text{if } k \le 1 \le K, \\ f_t(k) & \text{if } 1 < k. \end{cases} \tag{2.3}$$

Proof. The function f_t is differentiable and

$$f_t'(u) = t\left(1 - u^{t-1}\right) = t\frac{u^{1-t} - 1}{u^{1-t}},$$

which shows that the function f_t is decreasing on $[0,1]$ and increasing on $[1,\infty)$, $f_t(0) = 1 - t$, $f_t(1) = 0$ and the equation $f_t(u) = 1 - t$ for $u > 0$ has the unique solution $u_t = t^{\frac{1}{t-1}} > 1$.

Therefore, by considering the 3 possible situations for the location of the interval $[k,K]$ and the number 1 we get the desired bounds (2.2) and (2.3). □

Lemma 2. *Assume that a, $b > 0$ with $0 < k \le \frac{b}{a} \le K$, then*

$$0 \le \delta_t(k,K)\,a \le (1-t)\,a + tb - b^t a^{1-t} \le \Delta_t(k,K)\,a. \tag{2.4}$$

Proof. If $u \in [k,K]$, then by Lemma 1 we have

$$\delta_t(k,K) \le 1 - t + tu - u^t \le \Delta_t(k,K). \tag{2.5}$$

If we take $u = \frac{b}{a}$ in (2.5), then we get

$$\delta_t(k,K) \le 1 - t + t\frac{b}{a} - \left(\frac{b}{a}\right)^t \le \Delta_t(k,K),$$

and by multiplying with a we obtain the desired result (2.4). □

Theorem 1. *Assume that the positive definite matrices A, B satisfy the condition*

$$0 < mI_n \le B - A \le MI_n, \tag{2.6}$$

for some constants $0 < m < M$, then

$$\begin{aligned} 0 &\le (1-t)\,[\det(A)]^{-1/2} + t\,[\det(A + mI_n)]^{-1/2} - [\det(A + mtI_n)]^{-1/2} \\ &\le (1-t)\,[\det(A)]^{-1/2} + t\,[\det(B)]^{-1/2} - [\det((1-t)\,A + tB)]^{-1/2} \\ &\le (1-t)\,[\det(A)]^{-1/2} + t\,[\det(A + MI_n)]^{-1/2} - [\det(A + MtI_n)]^{-1/2}, \end{aligned} \tag{2.7}$$

for all $t \in [0,1]$.

Also,

$$0 \leq \frac{[\det(A)]^{-1/2} + [\det(A + mI_n)]^{-1/2}}{2} - \frac{[\det(A + mtI_n)]^{-1/2} + [\det(A + m(1-t)I_n)]^{-1/2}}{2} \leq \frac{[\det(A)]^{-1/2} + [\det(B)]^{-1/2}}{2} - \frac{[\det((1-t)A + tB)]^{-1/2} + [\det(tA + (1-t)B)]^{-1/2}}{2} \leq \frac{[\det(A)]^{-1/2} + [\det(A + MI_n)]^{-1/2}}{2} - \frac{[\det(A + MtI_n)]^{-1/2} + [\det(A + M(1-t)I_n)]^{-1/2}}{2} \tag{2.8}$$

for all $t \in [0,1]$.

Proof. Let $a = \exp(-\langle Ax, x\rangle)$ and $b = \exp(-\langle Bx, x\rangle)$ for $x \in \mathbb{R}^n$. Then $\frac{b}{a} = \exp(-\langle (B-A)x, x\rangle)$ and since $0 < mI_n \leq B - A \leq MI_n$, hence

$$\exp\left(-M\|x\|^2\right) \leq \frac{b}{a} \leq \exp\left(-m\|x\|^2\right) < 1$$

If we apply the inequality (2.4) for $a = \exp(-\langle Ax, x\rangle)$, $b = \exp(-\langle Bx, x\rangle)$, $k = \exp\left(-M\|x\|^2\right)$ and $K = \exp\left(-m\|x\|^2\right) < 1$, then we get

$$\begin{aligned} 0 &\leq f_t\left(\exp\left(-m\|x\|^2\right)\right)\exp(-\langle Ax, x\rangle) \\ &\leq (1-t)\exp(-\langle Ax, x\rangle) + t\exp(-\langle Bx, x\rangle) \\ &\quad - \exp(-\langle ((1-t)A + tB)x, x\rangle) \\ &\leq f_t\left(\exp\left(-M\|x\|^2\right)\right)\exp(-\langle Ax, x\rangle), \end{aligned} \tag{2.9}$$

namely

$$\begin{aligned} 0 &\leq \left(1 - t + t\exp\left(-m\|x\|^2\right) - \exp\left(-mt\|x\|^2\right)\right)\exp(-\langle Ax, x\rangle) \\ &\leq (1-t)\exp(-\langle Ax, x\rangle) + t\exp(-\langle Bx, x\rangle) - \exp(-\langle ((1-t)A + tB)x, x\rangle) \\ &\leq \left(1 - t + t\exp\left(-M\|x\|^2\right) - \exp\left(-Mt\|x\|^2\right)\right)\exp(-\langle Ax, x\rangle). \end{aligned}$$

This inequality can be written as

$$\begin{aligned} 0 &\leq (1-t)\exp(-\langle Ax, x\rangle) + t\exp(-\langle (A + mI_n)x, x\rangle) \\ &\quad - \exp(-\langle (A + mtI_n)x, x\rangle) \\ &\leq (1-t)\exp(-\langle Ax, x\rangle) + t\exp(-\langle Bx, x\rangle) \\ &\quad - \exp(-\langle ((1-t)A + tB)x, x\rangle) \\ &\leq (1-t)\exp(-\langle Ax, x\rangle) + t\exp(-\langle (A + MI_n)x, x\rangle) \\ &\quad - \exp(-\langle (A + MtI_n)x, x\rangle), \end{aligned}$$

for $x \in \mathbb{R}^n$ and $t \in [0,1]$.

If we take the integral over $x \in \mathbb{R}^n$, then we get

$$\begin{aligned}
0 &\leq (1-t)\int_{\mathbb{R}^n} \exp\left(-\langle Ax, x\rangle\right) dx + t\int_{\mathbb{R}^n} \exp\left(-\langle (A+mI_n)\,x, x\rangle\right) dx \\
&- \int_{\mathbb{R}^n} \exp\left(-\langle (A+mtI_n)\,x, x\rangle\right) dx \\
&\leq (1-t)\int_{\mathbb{R}^n} \exp\left(-\langle Ax, x\rangle\right) dx + t\int_{\mathbb{R}^n} \exp\left(-\langle Bx, x\rangle\right) dx \\
&- \int_{\mathbb{R}^n} \exp\left(-\langle ((1-t)\,A+tB)\,x, x\rangle\right) dx \\
&\leq (1-t)\int_{\mathbb{R}^n} \exp\left(-\langle Ax, x\rangle\right) dx + t\int_{\mathbb{R}^n} \exp\left(-\langle (A+MI_n)\,x, x\rangle\right) dx \\
&- \int_{\mathbb{R}^n} \exp\left(-\langle (A+MtI_n)\,x, x\rangle\right) dx,
\end{aligned}$$

for $t \in [0,1]$.

Using the representation (1.1) we get

$$\begin{aligned}
0 &\leq (1-t)\,J_n(A) + tJ_n(A+mI_n) - J_n(A+mtI_n) \\
&\leq (1-t)\,J_n(A) + tJ_n(B) - J_n((1-t)\,A+tB) \\
&\leq (1-t)\,J_n(A) + +tJ_n(A+MI_n) - J_n(A+MtI_n),
\end{aligned}$$

which, by the second equality in (1.1) gives (2.7).

If we replace t with $1-t$ in (2.7), then we have

$$\begin{aligned}
0 &\leq t\,[\det(A)]^{-1/2} + (1-t)\,[\det(A+mI_n)]^{-1/2} \\
&- [\det(A+m(1-t)\,I_n)]^{-1/2} \\
&\leq t\,[\det(A)]^{-1/2} + (1-t)\,[\det(B)]^{-1/2} \\
&- [\det(tA+(1-t)\,B)]^{-1/2} \\
&\leq t\,[\det(A)]^{-1/2} + (1-t)\,[\det(A+MI_n)]^{-1/2} \\
&- [\det(A+M(1-t)\,I_n)]^{-1/2},
\end{aligned} \tag{2.10}$$

for $t \in [0,1]$.

If we add (2.7) with (2.10) and divide by 2, then we get (2.8). □

Corollary 1. *With the assumptions of Theorem 1 we have*

$$\begin{aligned}
0 &\leq \frac{[\det(A)]^{-1/2} + [\det(A+mI_n)]^{-1/2}}{2} - \int_0^1 [\det(A+mtI_n)]^{-1/2}\,dt \\
&\leq \frac{[\det(A)]^{-1/2} + [\det(B)]^{-1/2}}{2} - \int_0^1 [\det((1-t)\,A+tB)]^{-1/2}\,dt \\
&\leq \frac{[\det(A)]^{-1/2} + [\det(A+MI_n)]^{-1/2}}{2} - \int_0^1 [\det(A+MtI_n)]^{-1/2}\,dt.
\end{aligned} \tag{2.11}$$

The proof follows by taking the integral over $t \in [0,1]$ in (2.7).

If we take the square in the representation (1.1), then we get

$$\left(\int_{\mathbb{R}^n} \exp(-\langle x, Ax\rangle)dx\right)^2 = \frac{\pi^n}{\det(A)}.$$

Since

$$\left(\int_{\mathbb{R}^n} \exp(-\langle Ax, x\rangle)dx\right)^2 = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \exp(-\langle Ax, x\rangle)\exp(-\langle Ay, y\rangle)dxdy$$
$$= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \exp(-\langle Ax, x\rangle - \langle Ay, y\rangle))dxdy,$$

hence

$$K_n(A) := \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \exp(-\langle Ax, x\rangle - \langle Ay, y\rangle))dxdy = \frac{\pi^n}{\det(A)} \tag{2.12}$$

for A a positive definite matrix of order n and $\langle\cdot,\cdot\rangle$ is the usual inner product on $\mathbb{R}^n$.

We have:

Theorem 2. *Assume that the positive definite matrices A, B satisfy the condition (2.6) for some constants $0 < m < M$, then*

$$\begin{aligned} 0 &\le (1-t)[\det(A)]^{-1} + t[\det(A+mI_n)]^{-1} - [\det(A+mtI_n)]^{-1} \\ &\le (1-t)[\det(A)]^{-1} + t[\det(B)]^{-1} - [\det((1-t)A+tB)]^{-1} \\ &\le (1-t)[\det(A)]^{-1} + t[\det(A+MI_n)]^{-1} - [\det(A+MtI_n)]^{-1}, \end{aligned} \tag{2.13}$$

for all $t \in [0,1]$.

Also,

$$\begin{aligned} 0 &\le \frac{[\det(A)]^{-1} + [\det(A+mI_n)]^{-1}}{2} \\ &\quad - \frac{[\det(A+mtI_n)]^{-1} + [\det(A+m(1-t)I_n)]^{-1}}{2} \\ &\le \frac{[\det(A)]^{-1} + [\det(B)]^{-1}}{2} \\ &\quad - \frac{[\det((1-t)A+tB)]^{-1} + [\det(tA+(1-t)B)]^{-1}}{2} \\ &\le \frac{[\det(A)]^{-1} + [\det(A+MI_n)]^{-1}}{2} \\ &\quad - \frac{[\det(A+MtI_n)]^{-1} + [\det(A+M(1-t)I_n)]^{-1}}{2} \end{aligned} \tag{2.14}$$

for all $t \in [0,1]$.

Proof. Let $a = \exp(-\langle Ax, x\rangle - \langle Ay, y\rangle)$ and $b = \exp(-\langle Bx, x\rangle - \langle By, y\rangle)$ for x, $y \in \mathbb{R}^n$. Then

$$\frac{b}{a} = \exp(-\langle (B-A)x, x\rangle - \langle (B-A)y, y\rangle)$$

and since $0 < mI_n \le B - A \le MI_n$, hence

$$\exp\left(-M\left(\|x\|^2 + \|y\|^2\right)\right) \le \frac{b}{a} \le \exp\left(-m\left(\|x\|^2 + \|y\|^2\right)\right) < 1$$

If we apply the inequality (2.4) for

$$a = \exp(-\langle Ax, x\rangle - \langle Ay, y\rangle),\ b = \exp(-\langle Bx, x\rangle - \langle By, y\rangle),$$

$k=\exp\left(-M\left(\|x\|^2+\|y\|^2\right)\right)$ and $K=\exp\left(-m\left(\|x\|^2+\|y\|^2\right)\right)<1$, then we get

$$\begin{aligned}
0 &\le f_t\left(\exp\left(-m\left(\|x\|^2+\|y\|^2\right)\right)\right)\exp\left(-\langle Ax,x\rangle-\langle Ay,y\rangle\right)\\
&\le (1-t)\exp\left(-\langle Ax,x\rangle-\langle Ay,y\rangle\right)+t\exp\left(-\langle Bx,x\rangle-\langle By,y\rangle\right)\\
&-\exp\left(-\langle((1-t)A+tB)x,x\rangle-\langle((1-t)A+tB)y,y\rangle\right)\\
&\le f_t\left(\exp\left(-M\left(\|x\|^2+\|y\|^2\right)\right)\right)\exp\left(-\langle Ax,x\rangle-\langle Ay,y\rangle\right),
\end{aligned}$$

namely

$$\begin{aligned}
0 &\le (1-t)\exp\left(-\langle Ax,x\rangle-\langle Ay,y\rangle\right)\\
&+t\exp\left(-\langle(A+mI_n)x,x\rangle-\langle(A+mI_n)y,y\rangle\right)\\
&-\exp\left(-\langle(A+mtI_n)x,x\rangle-\langle(A+mtI_n)y,y\rangle\right)\\
&\le (1-t)\exp\left(-\langle Ax,x\rangle-\langle Ay,y\rangle\right)+t\exp\left(-\langle Bx,x\rangle-\langle By,y\rangle\right)\\
&-\exp\left(-\langle((1-t)A+tB)x,x\rangle-\langle((1-t)A+tB)y,xy\rangle\right)\\
&\le (1-t)\exp\left(-\langle Ax,x\rangle-\langle Ay,y\rangle\right)\\
&+t\exp\left(-\langle(A+MI_n)x,x\rangle-\langle(A+MI_n)y,y\rangle\right)\\
&-\exp\left(-\langle(A+MtI_n)x,x\rangle-\langle(A+MtI_n)y,y\rangle\right),
\end{aligned}$$

for $x,\ y\in\mathbb{R}^n$ and $t\in[0,1]$.

If we take the double integral over $x,y\in\mathbb{R}^n$, then we get

$$\begin{aligned}
0 &\le (1-t)\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\exp\left(-\langle Ax,x\rangle-\langle Ay,y\rangle\right)dxdy\\
&+t\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\exp\left(-\langle(A+mI_n)x,x\rangle-\langle(A+mI_n)y,y\rangle\right)dxdy\\
&-\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\exp\left(-\langle(A+mtI_n)x,x\rangle-\langle(A+mtI_n)y,y\rangle\right)dxdy\\
&\le (1-t)\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\exp\left(-\langle Ax,x\rangle-\langle Ay,y\rangle\right)dxdy\\
&+t\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\exp\left(-\langle Bx,x\rangle-\langle By,y\rangle\right)dxdy\\
&-\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\exp\left(-\langle((1-t)A+tB)x,x\rangle-\langle((1-t)A+tB)y,xy\rangle\right)dxdy\\
&\le (1-t)\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\exp\left(-\langle Ax,x\rangle-\langle Ay,y\rangle\right)dxdy\\
&+t\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\exp\left(-\langle(A+MI_n)x,x\rangle-\langle(A+MI_n)y,y\rangle\right)dxdy\\
&-\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\exp\left(-\langle(A+MtI_n)x,x\rangle-\langle(A+MtI_n)y,y\rangle\right)dxdy,
\end{aligned}$$

and by making use of the representation (2.12). □

Corollary 2. *With the assumptions of Theorem 2 we have*

$$0 \le \frac{[\det(A)]^{-1} + [\det(A+mI_n)]^{-1}}{2} - \int_0^1 [\det(A+mtI_n)]^{-1} dt \tag{2.15}$$
$$\le \frac{[\det(A)]^{-1} + [\det(B)]^{-1}}{2} - \int_0^1 [\det((1-t)A+tB)]^{-1} dt$$
$$\le \frac{[\det(A)]^{-1} + [\det(A+MI_n)]^{-1}}{2} - \int_0^1 [\det(A+MtI_n)]^{-1} dt.$$

The proof follows by taking the integral over $t \in [0,1]$ in (2.13).

3. Multiplicative Inequalities

We consider the function $g_t : (0,\infty) \to (0,\infty)$ defined for $t \in (0,1)$ by

$$1 \le g_t(u) = \frac{1-t+tu}{u^t} = (1-t)u^{-t} + tu^{1-t}. \tag{3.1}$$

For $[k,K] \subset (0,\infty)$ define the quantities

$$\Gamma_t(k,K) := \begin{cases} g_t(k) \text{ if } K<1, \\ \max\{g_t(k), g_t(K)\} \text{ if } k \le 1 \le K, \\ g_t(K) \text{ if } 1<k \end{cases} \tag{3.2}$$

$$= \begin{cases} (1-t)k^{-t} + tk^{1-t} \text{ if } K<1, \\ \max\{(1-t)k^{-t} + tk^{1-t}, (1-t)K^{-t} + tK^{1-t}\} \\ \text{if } k \le 1 \le K, \\ (1-t)K^{-t} + tK^{1-t} \text{ if } 1<k \end{cases}$$

and

$$\gamma_t(k,K) := \begin{cases} g_t(K) \text{ if } K<1, \\ 1 \text{ if } k \le 1 \le K, \\ g_t(k) \text{ if } 1<k. \end{cases} \tag{3.3}$$

$$= \begin{cases} (1-t)K^{-t} + tK^{1-t} \text{ if } K<1, \\ 1 \text{ if } k \le 1 \le K, \\ (1-t)k^{-t} + tk^{1-t} \text{ if } 1<k. \end{cases}$$

The following lemma holds.

Lemma 3. *For any $u \in [k,K] \subset (0,\infty)$ we have*

$$\max_{u\in[k,K]} g_t(u) = \Gamma_t(k,K)$$

and

$$\min_{u\in[k,K]} g_t(u) = \gamma_t(k,K).$$

Proof. The function g_t is differentiable and

$$g_t'(u) = (1-t)\, t u^{-t-1}(u-1),$$

which shows that the function g_t is decreasing on $(0,1)$ and increasing on $[1,\infty)$. We have $g_t(1)=1$, $\lim_{u\to 0+} g_t(u) = +\infty$, $\lim_{u\to\infty} g_t(u) = +\infty$ and $g_t\left(\frac{1}{u}\right) = g_{1-t}(u)$ for any $u>0$ and $t\in(0,1)$.

Therefore, by considering the 3 possible situations for the location of the interval $[k,K]$ and the number 1 we get the desired bounds (2.4) and (2.5). □

Lemma 4. *Assume that* a, $b>0$ *with* $0<k\le \frac{b}{a}\le K$*, then*

$$\gamma_t(k,K)\, a^{1-t}b^t \le (1-t)\,a + tb \le \Gamma_t(k,K)\, a^{1-t}b^t. \tag{3.4}$$

Proof. From Lemma 3 we have

$$\gamma_t(k,K) \le \frac{1-t+t\frac{b}{a}}{\left(\frac{b}{a}\right)^t} \le \Gamma_t(k,K),$$

namely

$$\gamma_t(k,K)\left(\frac{b}{a}\right)^t \le 1-t+t\frac{b}{a} \le \Gamma_t(k,K)\left(\frac{b}{a}\right)^t.$$

If we multiply these inequalities by a, then we get (3.4). □

Theorem 3. *Assume that the positive definite matrices* A, B *satisfy the condition*

$$0 < mI_n \le B-A,$$

for some constant $0<m$*, then*

$$\begin{aligned}&(1-t)\left[\det\left((1-t)A+tB-tmI_n\right)\right]^{-1/2}\\ &+t\left[\det\left((1-t)A+tB+(1-t)mI_n\right)\right]^{-1/2}\\ &\le (1-t)\left[\det(A)\right]^{-1/2}+t\left[\det(B)\right]^{-1/2}\end{aligned} \tag{3.5}$$

for all $t\in[0,1]$.

In particular, for $t=1/2$,

$$\begin{aligned}&\left[\det\left(\frac{A+B}{2}-\frac{m}{2}I_n\right)\right]^{-1/2}+\left[\det\left(\frac{A+B}{2}+\frac{m}{2}I_n\right)\right]^{-1/2}\\ &\le \left[\det(A)\right]^{-1/2}+\left[\det(B)\right]^{-1/2}.\end{aligned} \tag{3.6}$$

Proof. If $0<k\le\frac{b}{a}\le K<1$, then by (3.4) we get

$$g_t(K)\, a^{1-t}b^t \le (1-t)\,a+tb$$

namely

$$\left[(1-t)K^{-t}+tK^{1-t}\right] a^{1-t}b^t \le (1-t)\,a+tb \tag{3.7}$$

If we apply the inequality (3.7) for $a=\exp(-\langle Ax,x\rangle)$, $b=\exp(-\langle Bx,x\rangle)$ and $K=\exp\left(-m\|x\|^2\right)<1$, then we get

$$\begin{aligned}&\left[(1-t)\exp\left(tm\|x\|^2\right)+t\exp\left(-m(1-t)\|x\|^2\right)\right]\\&\times\exp(-\langle(1-t)Ax,x\rangle-\langle tBx,x\rangle)\\&\le(1-t)\exp(-\langle Ax,x\rangle)+t\exp(-\langle Bx,x\rangle)\end{aligned}$$

This is equivalent to

$$\begin{aligned}&(1-t)\exp\left(tm\|x\|^2-\langle(1-t)Ax,x\rangle-\langle tBx,x\rangle\right)\\&+t\exp\left(-m(1-t)\|x\|^2-\langle(1-t)Ax,x\rangle-\langle tBx,x\rangle\right)\\&\le(1-t)\exp(-\langle Ax,x\rangle)+t\exp(-\langle Bx,x\rangle)\end{aligned}$$

namely

$$\begin{aligned}&(1-t)\exp(-\langle((1-t)A+tB-tmI_n)x,x\rangle)\\&+t\exp(-\langle((1-t)A+tB+(1-t)mI_n)x,x\rangle)\\&\le(1-t)\exp(-\langle Ax,x\rangle)+t\exp(-\langle Bx,x\rangle)\end{aligned}$$

for all $x\in\mathbb{R}^n$ and $t\in[0,1]$.

Observe that

$$(1-t)A+tB-tmI_n=A+t(B-A)-tmI_n\ge A+tmI_n-tmI_n=A>0$$

and

$$(1-t)A+tB+(1-t)mI_n>0.$$

By taking the integral on $\mathbb{R}^n$, we get

$$\begin{aligned}&(1-t)\int_{\mathbb{R}^n}\exp(-\langle((1-t)A+tB-tmI_n)x,x\rangle)\,dx\\&+t\int_{\mathbb{R}^n}\exp(-\langle((1-t)A+tB+(1-t)mI_n)x,x\rangle)\,dx\\&\le(1-t)\int_{\mathbb{R}^n}\exp(-\langle Ax,x\rangle)\,dx+t\int_{\mathbb{R}^n}\exp(-\langle Bx,x\rangle)\,dx,\end{aligned}$$

namely, by (1.1)

$$\begin{aligned}&(1-t)J_n((1-t)A+tB-tmI_n)+tJ_n((1-t)A+tB+(1-t)mI_n)\\&\le(1-t)J_n(A)+tJ_n(B),\end{aligned}$$

which gives (3.5). □

By utilizing a similar argument to the one in the proof of Theorem 2 we can finally state:

Theorem 4. *Assume that the positive definite matrices A, B satisfy the condition*

$$0<mI_n\le B-A,$$

for some constant $0 < m$, *then*

$$(1-t)\left[\det\left((1-t)A + tB - tmI_n\right)\right]^{-1} + t\left[\det\left((1-t)A + tB + (1-t)mI_n\right)\right]^{-1} \le (1-t)\left[\det(A)\right]^{-1} + t\left[\det(B)\right]^{-1} \tag{3.8}$$

for all $t \in [0,1]$.

In particular, for $t = 1/2$,

$$\left[\det\left(\frac{A+B}{2} - \frac{m}{2}I_n\right)\right]^{-1} + \left[\det\left(\frac{A+B}{2} + \frac{m}{2}I_n\right)\right]^{-1} \le \left[\det(A)\right]^{-1} + \left[\det(B)\right]^{-1}. \tag{3.9}$$

A complex square matrix $H = (h_{ij})$, $i, j = 1, ..., n$ is said to be Hermitian provided $h_{ij} = \overline{h_{ji}}$ for all i, $j = 1, ..., n$. A Hermitian matrix is said to be positive definite if the Hermitian form $P(z) = \sum_{i,j=1}^{n} a_{ij} z_i \overline{z_j}$ is positive for all $z = (z_1, ..., z_n) \in \mathbb{C}^n \setminus \{0\}$.

It is known that, see for instance [11, p. 215], for a positive definite Hermitian matrix H, we have

$$K_n(H) := \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \exp\left(-\langle \overline{z}, Hz \rangle\right) dxdy = \frac{\pi^n}{\det(H)}, \tag{3.10}$$

where $z = x + iy$ and dx and dy denote integration over real n-dimensional space $\mathbb{R}^n$. Here the inner product $\langle x, y \rangle$ is understood in the real sense, i.e. $\langle x, y \rangle = \sum_{k=1}^{n} x_k y_k$.

On making use of a similar argument to the one in Theorem 2 and Theorem 4 for the representation $K_n(\cdot)$ we can state the same inequalities for positive definite Hermitian matrices H and K.

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