

## DETERMINANT INEQUALITIES FOR POSITIVE DEFINITE MATRICES VIA BHATIA AND KITTANEH-MANASRAH RESULTS

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ABSTRACT. In this paper we prove among others that, if  $A$  and  $B$  are positive definite matrices, then

$$\begin{aligned} 0 &\leq \int_0^1 [\det((1-t)A + tB)]^{-1} dt - \left[ \det\left(\frac{A+B}{2}\right) \right]^{-1} \\ &\leq \frac{1}{3} \left[ \frac{1}{2} \left( [\det(A)]^{-1} + [\det(B)]^{-1} \right) - \left[ \det\left(\frac{A+B}{2}\right) \right]^{-1} \right] \\ &\leq \frac{1}{2} \left( [\det(A)]^{-1} + [\det(B)]^{-1} \right) - \int_0^1 [\det((1-t)A + tB)]^{-1} dt \\ &\leq \frac{4}{3} \left[ \frac{1}{2} \left( [\det(A)]^{-1} + [\det(B)]^{-1} \right) - \left[ \det\left(\frac{A+B}{2}\right) \right]^{-1} \right]. \end{aligned}$$

### 1. INTRODUCTION

We call Heinz means, the mean defined by

$$H_\nu(a, b) := \frac{1}{2} (a^{1-\nu} b^\nu + a^\nu b^{1-\nu})$$

We call *Heron means*, the means defined by

$$F_\alpha(a, b) := (1 - \alpha) \sqrt{ab} + \alpha \frac{a+b}{2},$$

where  $a, b > 0$  and  $\alpha \in [0, 1]$ .

In [2], Bhatia obtained the following interesting inequality between the Heinz and Heron means

$$(1.1) \quad H_\nu(a, b) \leq F_{(2\nu-1)^2}(a, b)$$

where  $a, b > 0$  and  $\alpha \in [0, 1]$ .

This inequality can be written as

$$(1.2) \quad (0 \leq) H_\nu(a, b) - \sqrt{ab} \leq (2\nu - 1)^2 \left( \frac{a+b}{2} - \sqrt{ab} \right),$$

where  $a, b > 0$  and  $\alpha \in [0, 1]$ .

Kittaneh and Manasrah [3], [4] provided a refinement and an additive reverse for Young inequality as follows:

$$(1.3) \quad r \left( \sqrt{a} - \sqrt{b} \right)^2 \leq (1 - \nu) a + \nu b - a^{1-\nu} b^\nu \leq R \left( \sqrt{a} - \sqrt{b} \right)^2$$

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where  $a, b > 0$ ,  $\nu \in [0, 1]$ ,  $r = \min\{1 - \nu, \nu\}$  and  $R = \max\{1 - \nu, \nu\}$ .

If we replace in (1.3)  $\nu$  with  $1 - \nu$ , add the obtained inequalities and divide by 2, then we get

$$(1.4) \quad r \left( \sqrt{a} - \sqrt{b} \right)^2 \leq \frac{a+b}{2} - H_\nu(a, b) \leq R \left( \sqrt{a} - \sqrt{b} \right)^2,$$

where  $a, b > 0$ ,  $\nu \in [0, 1]$ .

A real square matrix  $A = (a_{ij})$ ,  $i, j = 1, \dots, n$  is *symmetric* provided  $a_{ij} = a_{ji}$  for all  $i, j = 1, \dots, n$ . A real symmetric matrix is said to be *positive definite* provided the quadratic form  $Q(x) = \sum_{i,j=1}^n a_{ij}x_i x_j$  is positive for all  $x = (x_1, \dots, x_n) \in \mathbb{R}^n \setminus \{0\}$ . It is well known that a necessary and sufficient condition for the symmetric matrix  $A$  to be positive definite, and we write  $A > 0$ , is that all determinants

$$\det(A_k) = \det(a_{ij}), \quad i, j = 1, \dots, k; \quad k = 1, \dots, n$$

are positive.

It is known that the following integral representation is valid, see [1, pp. 61-62] or [11, pp. 211-212]

$$(1.5) \quad \begin{aligned} J_n(A) &:= \int_{\mathbb{R}^n} \exp(-\langle Ax, x \rangle) dx := \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} \exp(-\langle Ax, x \rangle) dx \\ &= \frac{\pi^{n/2}}{[\det(A)]^{1/2}}, \end{aligned}$$

where  $A$  is a positive definite matrix of order  $n$  and  $\langle \cdot, \cdot \rangle$  is the usual inner product on  $\mathbb{R}^n$ .

By utilizing the representation (1.5) and Hölder's integral inequality for multiple integrals one can prove the *logarithmic concavity* of the determinant that is due to Ky Fan ([1, p. 63] or [11, p. 212]), namely

$$(1.6) \quad \det((1 - \lambda)A + \lambda B) \geq [\det(A)]^{1-\lambda} [\det(B)]^\lambda$$

for any positive definite matrices  $A, B$  and  $\lambda \in [0, 1]$ .

By mathematical induction we can get a generalization of (1.6) which was obtained by L. Mirsky in [10], see also [11, p. 212]

$$(1.7) \quad \det \left( \sum_{j=1}^m \lambda_j A_j \right) \geq \prod_{j=1}^m [\det(A_j)]^{\lambda_j}, \quad m \geq 2,$$

where  $\lambda_j > 0$ ,  $j = 1, \dots, m$  with  $\sum_{j=1}^m \lambda_j = 1$  and  $A_j > 0$ ,  $j = 1, \dots, m$ .

If we write (1.7) for  $A_j = B_j^{-1}$  we get

$$\det \left( \sum_{j=1}^m \lambda_j B_j^{-1} \right) \geq \prod_{j=1}^m [\det(B_j^{-1})]^{\lambda_j} = \left( \prod_{j=1}^m [\det(B_j)]^{\lambda_j} \right)^{-1},$$

which also gives

$$(1.8) \quad \prod_{j=1}^m [\det(A_j)]^{\lambda_j} \geq \det \left[ \left( \sum_{j=1}^m \lambda_j A_j^{-1} \right)^{-1} \right],$$

where  $\lambda_j > 0$ ,  $j = 1, \dots, m$  with  $\sum_{j=1}^m \lambda_j = 1$  and  $A_j > 0$ ,  $j = 1, \dots, m$ .

Using the representation (1.5) one can also prove the result, see [11, p. 212],

$$(1.9) \quad \det(A) = \det(A_{1n}) \leq \det(A_{1k}) \det(A_{(k+1)n}), \quad k = 1, \dots, n;$$

where the determinant  $\det(A_{rs})$  is defined by

$$\det(A_{rs}) = \det(a_{ij}), \quad i, j = r, \dots, s.$$

In particular,

$$(1.10) \quad \det(A) \leq a_{11}a_{22}\dots a_{nn}.$$

We recall also the Minkowski's type inequality,

$$(1.11) \quad [\det(A+B)]^{1/n} \geq [\det(A)]^{1/n} + [\det(B)]^{1/n}$$

for  $A, B$  positive definite matrices of order  $n$ . For other determinant inequalities see Chapter VIII of the classic book [11]. For some recent results see [5]-[9].

Motivated by the above results, in this paper we prove among others that, if  $A$  and  $B$  are positive definite matrices, then

$$\begin{aligned} 0 &\leq \int_0^1 [\det((1-t)A + tB)]^{-1} dt - \left[ \det\left(\frac{A+B}{2}\right) \right]^{-1} \\ &\leq \frac{1}{3} \left[ \frac{1}{2} \left( [\det(A)]^{-1} + [\det(B)]^{-1} \right) - \left[ \det\left(\frac{A+B}{2}\right) \right]^{-1} \right] \\ &\leq \frac{1}{2} \left( [\det(A)]^{-1} + [\det(B)]^{-1} \right) - \int_0^1 [\det((1-t)A + tB)]^{-1} dt \\ &\leq \frac{4}{3} \left[ \frac{1}{2} \left( [\det(A)]^{-1} + [\det(B)]^{-1} \right) - \left[ \det\left(\frac{A+B}{2}\right) \right]^{-1} \right]. \end{aligned}$$

## 2. MAIN RESULTS

Our first main result is as follows:

**Theorem 1.** *For any positive definite matrices  $A, B$  and  $t \in [0, 1]$ ,*

$$(2.1) \quad \begin{aligned} 0 &\leq \frac{1}{2} \left[ [\det((1-t)A + tB)]^{-1/2} + [\det(tA + (1-t)B)]^{-1/2} \right] \\ &\quad - \left[ \det\left(\frac{A+B}{2}\right) \right]^{-1/2} \\ &\leq (2t-1)^2 \left[ \frac{1}{2} \left( [\det(A)]^{-1/2} + [\det(B)]^{-1/2} \right) - \left[ \det\left(\frac{A+B}{2}\right) \right]^{-1/2} \right] \end{aligned}$$

and

$$(2.2) \quad \begin{aligned} 0 &\leq r \left[ \frac{1}{2} \left( [\det(A)]^{-1/2} + [\det(B)]^{-1/2} \right) - \left[ \det\left(\frac{A+B}{2}\right) \right]^{-1/2} \right] \\ &\leq \frac{1}{2} \left( [\det(A)]^{-1/2} + [\det(B)]^{-1/2} \right) \\ &\quad - \frac{1}{2} \left[ [\det((1-t)A + tB)]^{-1/2} + [\det(tA + (1-t)B)]^{-1/2} \right] \\ &\leq R \left[ \frac{1}{2} \left( [\det(A)]^{-1/2} + [\det(B)]^{-1/2} \right) - \left[ \det\left(\frac{A+B}{2}\right) \right]^{-1/2} \right], \end{aligned}$$

where  $r = \min\{1 - t, t\}$  and  $R = \max\{1 - t, t\}$ .

*Proof.* From (1.2) we get for  $a = \exp(-\langle Ax, x \rangle)$ ,  $b = \exp(-\langle Bx, x \rangle)$  with  $x \in \mathbb{R}^n$  that

$$(2.3) \quad 0 \leq \frac{1}{2} (\exp(-\langle ((1-t)A + tB)x, x \rangle) + \exp(-\langle (tA + (1-t)B)x, x \rangle)) \\ - \exp\left(-\left\langle \frac{A+B}{2}x, x \right\rangle\right) \\ \leq (2t-1)^2 \\ \times \left(\frac{1}{2} (\exp(-\langle Ax, x \rangle) + \exp(-\langle Bx, x \rangle)) - \exp\left(-\left\langle \frac{A+B}{2}x, x \right\rangle\right)\right).$$

By taking the integral on  $\mathbb{R}^n$  in (2.3), then we get

$$0 \leq \frac{1}{2} \left( \int_{\mathbb{R}^n} \exp(-\langle ((1-t)A + tB)x, x \rangle) dx \\ + \int_{\mathbb{R}^n} \exp(-\langle (tA + (1-t)B)x, x \rangle) dx \right) - \int_{\mathbb{R}^n} \exp\left(-\left\langle \frac{A+B}{2}x, x \right\rangle\right) dx \\ \leq (2t-1)^2 \left( \frac{1}{2} \left( \int_{\mathbb{R}^n} \exp(-\langle Ax, x \rangle) dx + \int_{\mathbb{R}^n} \exp(-\langle Bx, x \rangle) dx \right) \right. \\ \left. - \exp\left(-\left\langle \frac{A+B}{2}x, x \right\rangle\right) \right)$$

and by (1.5) we obtain

$$0 \leq \frac{1}{2} [J_n((1-t)A + tB) + J_n(tA + (1-t)B)] - J_n\left(\frac{A+B}{2}\right) \\ \leq (2t-1)^2 \left[ \frac{1}{2} (J_n(A) + J_n(B)) - J_n\left(\frac{A+B}{2}\right) \right]$$

which by (1.5), we derive (2.1).

From (1.4) we have

$$r \left( \frac{1}{2} (\exp(-\langle Ax, x \rangle) + \exp(-\langle Bx, x \rangle)) - \exp\left(-\left\langle \frac{A+B}{2}x, x \right\rangle\right) \right) \\ \leq \frac{1}{2} (\exp(-\langle Ax, x \rangle) + \exp(-\langle Bx, x \rangle)) \\ - \frac{1}{2} (\exp(-\langle ((1-t)A + tB)x, x \rangle) + \exp(-\langle (tA + (1-t)B)x, x \rangle)) \\ \leq R \left( \frac{1}{2} (\exp(-\langle Ax, x \rangle) + \exp(-\langle Bx, x \rangle)) - \exp\left(-\left\langle \frac{A+B}{2}x, x \right\rangle\right) \right),$$

for  $x \in \mathbb{R}^n$ .

By taking the integral on  $\mathbb{R}^n$  we deduce the desired result (2.2). □

**Corollary 1.** For any positive definite matrices  $A, B$  we have

$$(2.4) \quad 0 \leq \int_0^1 [\det((1-t)A + tB)]^{-1/2} dt - \left[ \det\left(\frac{A+B}{2}\right) \right]^{-1/2} \\ \leq \frac{1}{3} \left[ \frac{1}{2} \left( [\det(A)]^{-1/2} + [\det(B)]^{-1/2} \right) - \left[ \det\left(\frac{A+B}{2}\right) \right]^{-1/2} \right]$$

and

$$\begin{aligned}
 (2.5) \quad & \frac{1}{3} \left[ \frac{1}{2} \left( [\det(A)]^{-1/2} + [\det(B)]^{-1/2} \right) - \left[ \det \left( \frac{A+B}{2} \right) \right]^{-1/2} \right] \\
 & \leq \frac{1}{2} \left( [\det(A)]^{-1/2} + [\det(B)]^{-1/2} \right) - \int_0^1 [\det((1-t)A + tB)]^{-1/2} dt \\
 & \leq \frac{4}{3} \left[ \frac{1}{2} \left( [\det(A)]^{-1/2} + [\det(B)]^{-1/2} \right) - \left[ \det \left( \frac{A+B}{2} \right) \right]^{-1/2} \right].
 \end{aligned}$$

*Proof.* Taking the integral over  $t$  in (2.1), we get

$$\begin{aligned}
 (2.6) \quad 0 & \leq \frac{1}{2} \left[ \int_0^1 [\det((1-t)A + tB)]^{-1/2} dt + \int_0^1 [\det(tA + (1-t)B)]^{-1/2} dt \right] \\
 & \quad - \left[ \det \left( \frac{A+B}{2} \right) \right]^{-1/2} \\
 & \leq \left[ \frac{1}{2} \left( [\det(A)]^{-1/2} + [\det(B)]^{-1/2} \right) - \left[ \det \left( \frac{A+B}{2} \right) \right]^{-1/2} \right] \\
 & \quad \times \int_0^1 (2t-1)^2 dt.
 \end{aligned}$$

Observe that

$$\int_0^1 [\det((1-t)A + tB)]^{-1/2} dt = \int_0^1 [\det(tA + (1-t)B)]^{-1/2} dt$$

and

$$\int_0^1 (2t-1)^2 dt = \frac{1}{3},$$

then by (2.6) we obtain (2.4).

From (2.2) we get

$$\begin{aligned}
 (2.7) \quad 0 & \leq \left[ \frac{1}{2} \left( [\det(A)]^{-1/2} + [\det(B)]^{-1/2} \right) - \left[ \det \left( \frac{A+B}{2} \right) \right]^{-1/2} \right] \\
 & \quad \times \int_0^1 \min\{1-t, t\} dt \\
 & \leq \frac{1}{2} \left( [\det(A)]^{-1/2} + [\det(B)]^{-1/2} \right) \\
 & \quad - \frac{1}{2} \left[ \int_0^1 [\det((1-t)A + tB)]^{-1/2} dt + \int_0^1 [\det(tA + (1-t)B)]^{-1/2} dt \right] \\
 & \leq \left[ \frac{1}{2} \left( [\det(A)]^{-1/2} + [\det(B)]^{-1/2} \right) - \left[ \det \left( \frac{A+B}{2} \right) \right]^{-1/2} \right] \\
 & \quad \times \int_0^1 \max\{1-t, t\} dt.
 \end{aligned}$$

Since

$$\int_0^1 \min\{1-t, t\} dt = \frac{1}{3} \quad \text{and} \quad \int_0^1 \max\{1-t, t\} dt = \frac{3}{4},$$

hence by (2.7) we get (2.5). □

If we take the square in the representation (1.5), then we get

$$\left( \int_{\mathbb{R}^n} \exp(-\langle x, Ax \rangle) dx \right)^2 = \frac{\pi^n}{\det(A)}.$$

Since

$$\begin{aligned} \left( \int_{\mathbb{R}^n} \exp(-\langle Ax, x \rangle) dx \right)^2 &= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \exp(-\langle Ax, x \rangle) \exp(-\langle Ay, y \rangle) dx dy \\ &= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \exp(-\langle Ax, x \rangle - \langle Ay, y \rangle) dx dy, \end{aligned}$$

hence

$$(2.8) \quad K_n(A) := \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \exp(-\langle Ax, x \rangle - \langle Ay, y \rangle) dx dy = \frac{\pi^n}{\det(A)}$$

for  $A$  a positive definite matrix of order  $n$  and  $\langle \cdot, \cdot \rangle$  is the usual inner product on  $\mathbb{R}^n$ .

**Theorem 2.** For any positive definite matrices  $A, B$  and  $t \in [0, 1]$ ,

$$(2.9) \quad \begin{aligned} 0 &\leq \frac{1}{2} \left[ [\det((1-t)A + tB)]^{-1} + [\det(tA + (1-t)B)]^{-1} \right] \\ &\quad - \left[ \det\left(\frac{A+B}{2}\right) \right]^{-1} \\ &\leq (2t-1)^2 \left[ \frac{1}{2} \left( [\det(A)]^{-1} + [\det(B)]^{-1} \right) - \left[ \det\left(\frac{A+B}{2}\right) \right]^{-1} \right] \end{aligned}$$

and

$$(2.10) \quad \begin{aligned} 0 &\leq r \left[ \frac{1}{2} \left( [\det(A)]^{-1} + [\det(B)]^{-1} \right) - \left[ \det\left(\frac{A+B}{2}\right) \right]^{-1} \right] \\ &\leq \frac{1}{2} \left( [\det(A)]^{-1} + [\det(B)]^{-1} \right) \\ &\quad - \frac{1}{2} \left[ [\det((1-t)A + tB)]^{-1} + [\det(tA + (1-t)B)]^{-1} \right] \\ &\leq R \left[ \frac{1}{2} \left( [\det(A)]^{-1} + [\det(B)]^{-1} \right) - \left[ \det\left(\frac{A+B}{2}\right) \right]^{-1} \right], \end{aligned}$$

where  $r = \min\{1-t, t\}$  and  $R = \max\{1-t, t\}$ .

*Proof.* From (1.2) we get for  $a = \exp(-\langle Ax, x \rangle - \langle Ay, y \rangle)$ ,  $b = \exp(-\langle Bx, x \rangle - \langle By, y \rangle)$  with  $x, y \in \mathbb{R}^n$  that

$$\begin{aligned}
 (2.11) \quad 0 &\leq \frac{1}{2} [\exp(-\langle ((1-t)A + tB)x, x \rangle - \langle ((1-t)A + tB)y, y \rangle) \\
 &\quad + \exp(-\langle (tA + (1-t)B)x, x \rangle - \langle (tA + (1-t)B)y, y \rangle)] \\
 &\quad - \exp\left(-\left\langle \frac{A+B}{2}x, x \right\rangle - \left\langle \frac{A+B}{2}y, y \right\rangle\right) \\
 &\leq (2t-1)^2 \\
 &\quad \times \left[ \frac{1}{2} (\exp(-\langle Ax, x \rangle - \langle Ay, y \rangle) + \exp(-\langle Bx, x \rangle - \langle By, y \rangle)) \right. \\
 &\quad \left. - \exp\left(-\left\langle \frac{A+B}{2}x, x \right\rangle - \left\langle \frac{A+B}{2}y, y \right\rangle\right) \right].
 \end{aligned}$$

If we take the double integral on  $\mathbb{R}^n \times \mathbb{R}^n$ , then we get

$$\begin{aligned}
 (2.12) \quad &\frac{1}{2} \left[ \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \exp(-\langle ((1-t)A + tB)x, x \rangle - \langle ((1-t)A + tB)y, y \rangle) dx dy \right. \\
 &\quad \left. + \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \exp(-\langle (tA + (1-t)B)x, x \rangle - \langle (tA + (1-t)B)y, y \rangle) dx dy \right] \\
 &\quad - \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \exp\left(-\left\langle \frac{A+B}{2}x, x \right\rangle - \left\langle \frac{A+B}{2}y, y \right\rangle\right) dx dy \\
 &\leq (2t-1)^2 \\
 &\quad \times \left[ \frac{1}{2} \left( \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \exp(-\langle Ax, x \rangle - \langle Ay, y \rangle) dx dy \right. \right. \\
 &\quad \left. \left. + \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \exp(-\langle Bx, x \rangle - \langle By, y \rangle) dx dy \right) \right. \\
 &\quad \left. - \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \exp\left(-\left\langle \frac{A+B}{2}x, x \right\rangle - \left\langle \frac{A+B}{2}y, y \right\rangle\right) dx dy \right].
 \end{aligned}$$

By utilising the representation (2.8) we get

$$\begin{aligned}
 0 &\leq \frac{1}{2} [K_n((1-t)A + tB) + K_n(tA + (1-t)B)] - K_n\left(\frac{A+B}{2}\right) \\
 &\leq (2t-1)^2 \left( \frac{1}{2} [K_n(A) + K_n(B)] - K_n\left(\frac{A+B}{2}\right) \right),
 \end{aligned}$$

which is equivalent to (2.9).

The inequality (2.10) follows in a similar way from (1.4). □

**Corollary 2.** *For any positive definite matrices  $A, B$  we have*

$$\begin{aligned}
 (2.13) \quad 0 &\leq \int_0^1 [\det((1-t)A + tB)]^{-1} dt - \left[ \det\left(\frac{A+B}{2}\right) \right]^{-1} \\
 &\leq \frac{1}{3} \left[ \frac{1}{2} \left( [\det(A)]^{-1} + [\det(B)]^{-1} \right) - \left[ \det\left(\frac{A+B}{2}\right) \right]^{-1} \right] \\
 &\leq \frac{1}{2} \left( [\det(A)]^{-1} + [\det(B)]^{-1} \right) - \int_0^1 [\det((1-t)A + tB)]^{-1} dt \\
 &\leq \frac{4}{3} \left[ \frac{1}{2} \left( [\det(A)]^{-1} + [\det(B)]^{-1} \right) - \left[ \det\left(\frac{A+B}{2}\right) \right]^{-1} \right].
 \end{aligned}$$

### 3. THE CASE OF HERMITIAN MATRICES

A complex square matrix  $H = (h_{ij})$ ,  $i, j = 1, \dots, n$  is said to be Hermitian provided  $h_{ij} = \overline{h_{ji}}$  for all  $i, j = 1, \dots, n$ . A Hermitian matrix is said to be positive definite if the Hermitian form  $P(z) = \sum_{i,j=1}^n a_{ij} z_i \overline{z_j}$  is positive for all  $z = (z_1, \dots, z_n) \in \mathbb{C}^n \setminus \{0\}$ .

It is known that, see for instance [11, p. 215], for a positive definite Hermitian matrix  $H$ , we have

$$(3.1) \quad K_n(H) := \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \exp(-\langle \overline{z}, Hz \rangle) dx dy = \frac{\pi^n}{\det(H)},$$

where  $z = x + iy$  and  $dx$  and  $dy$  denote integration over real  $n$ -dimensional space  $\mathbb{R}^n$ . Here the inner product  $\langle x, y \rangle$  is understood in the real sense, i.e.  $\langle x, y \rangle = \sum_{k=1}^n x_k y_k$ .

On making use of a similar argument to the one in Theorem 2 for the representation  $K_n(\cdot)$  we can state the following result as well:

**Theorem 3.** *For any positive definite Hermitian matrices  $H, K$  and  $t \in [0, 1]$ ,*

$$\begin{aligned}
 (3.2) \quad 0 &\leq \frac{1}{2} \left[ [\det((1-t)H + tK)]^{-1} + [\det(tH + (1-t)K)]^{-1} \right] \\
 &\quad - \left[ \det\left(\frac{H+K}{2}\right) \right]^{-1} \\
 &\leq (2t-1)^2 \left[ \frac{1}{2} \left( [\det(H)]^{-1} + [\det(K)]^{-1} \right) - \left[ \det\left(\frac{H+K}{2}\right) \right]^{-1} \right]
 \end{aligned}$$

and

$$\begin{aligned}
 (3.3) \quad 0 &\leq r \left[ \frac{1}{2} \left( [\det(H)]^{-1} + [\det(K)]^{-1} \right) - \left[ \det\left(\frac{H+K}{2}\right) \right]^{-1} \right] \\
 &\leq \frac{1}{2} \left( [\det(H)]^{-1} + [\det(K)]^{-1} \right) \\
 &\quad - \frac{1}{2} \left[ [\det((1-t)H + tK)]^{-1} + [\det(tH + (1-t)K)]^{-1} \right] \\
 &\leq R \left[ \frac{1}{2} \left( [\det(H)]^{-1} + [\det(K)]^{-1} \right) - \left[ \det\left(\frac{H+K}{2}\right) \right]^{-1} \right],
 \end{aligned}$$

where  $r = \min\{1-t, t\}$  and  $R = \max\{1-t, t\}$ .



Also

$$\begin{aligned}
 (3.4) \quad 0 &\leq \int_0^1 [\det((1-t)H + tK)]^{-1} dt - \left[ \det\left(\frac{H+K}{2}\right) \right]^{-1} \\
 &\leq \frac{1}{3} \left[ \frac{1}{2} \left( [\det(H)]^{-1} + [\det(K)]^{-1} \right) - \left[ \det\left(\frac{H+K}{2}\right) \right]^{-1} \right] \\
 &\leq \frac{1}{2} \left( [\det(H)]^{-1} + [\det(K)]^{-1} \right) - \int_0^1 [\det((1-t)H + tK)]^{-1} dt \\
 &\leq \frac{4}{3} \left[ \frac{1}{2} \left( [\det(H)]^{-1} + [\det(K)]^{-1} \right) - \left[ \det\left(\frac{H+K}{2}\right) \right]^{-1} \right].
 \end{aligned}$$

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