DETERMINANT INEQUALITIES FOR POSITIVE DEFINITE MATRICES VIA HEINZ MEANS

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Abstract. In this paper we prove among others that, if the positive definite matrices A, B of order n satisfy the condition

$$0 < mI_n \le B - A \le MI_n,$$

for some constants $0 < m < M$, where I_n is the identity matrix, then

$$\begin{aligned} 0 &\le \frac{1}{2}\left([\det (A+mI_n)]^{-1} + [\det (A)]^{-1}\right) \\ &- \int_0^1 [\det (A+mtI_n)]^{-1} dt \\ &\le \frac{1}{2}\left[[\det (A)]^{-1} + [\det (B)]^{-1}\right] - \int_0^1 [\det ((1-t)A + tB)]^{-1} dt \\ &\le \frac{1}{2}\left([\det (A+MI_n)]^{-1} + [\det (A)]^{-1}\right) \\ &- \int_0^1 [\det (A+MtI_n)]^{-1} dt. \end{aligned}$$

1. Introduction

We call Heinz means, the mean defined by

$$H_\nu (a,b) := \frac{1}{2}\left(a^{1-\nu}b^\nu + a^\nu b^{1-\nu}\right)$$

We call *Heron means*, the means defined by

$$F_\alpha (a,b) := (1-\alpha)\sqrt{ab} + \alpha \frac{a+b}{2},$$

where a, $b > 0$ and $\alpha \in [0,1]$.

In [2], Bhatia obtained the following interesting inequality between the Heinz and Heron means

$$H_\nu (a,b) \le F_{(2\nu-1)^2} (a,b) \tag{1.1}$$

where a, $b > 0$ and $\alpha \in [0,1]$.

This inequality can be written as

$$(0 \le)\, H_\nu (a,b) - \sqrt{ab} \le (2\nu - 1)^2 \left(\frac{a+b}{2} - \sqrt{ab}\right), \tag{1.2}$$

where a, $b > 0$ and $\alpha \in [0,1]$.

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Kittaneh and Manasrah [3], [4] provided a refinement and an additive reverse for Young inequality as follows:

$$r\left(\sqrt{a}-\sqrt{b}\right)^2 \le (1-\nu)\,a+\nu b-a^{1-\nu}b^{\nu} \le R\left(\sqrt{a}-\sqrt{b}\right)^2 \tag{1.3}$$

where $a,b>0$, $\nu\in[0,1]$, $r=\min\{1-\nu,\nu\}$ and $R=\max\{1-\nu,\nu\}$.

If we replace in (1.3) ν with $1-\nu$, add the obtained inequalities and divide by 2, then we get

$$r\left(\sqrt{a}-\sqrt{b}\right)^2 \le \frac{a+b}{2}-H_\nu(a,b) \le R\left(\sqrt{a}-\sqrt{b}\right)^2, \tag{1.4}$$

where $a,b>0$, $\nu\in[0,1]$.

A real square matrix $A=(a_{ij})$, $i,j=1,...,n$ is *symmetric* provided $a_{ij}=a_{ji}$ for all $i,j=1,...,n$. A real symmetric matrix is said to be *positive definite* provided the quadratic form $Q(x)=\sum_{i,j=1}^n a_{ij}x_ix_j$ is positive for all $x=(x_1,...,x_n)\in\mathbb{R}^n\setminus\{0\}$. It is well known that a necessary and sufficient condition for the symmetric matrix A to be positive definite, and we write $A>0$, is that all determinants

$$\det(A_k)=\det(a_{ij}),\ i,\ j=1,...,k;\ k=1,...,n$$

are positive.

It is know that the following integral representation is valid, see [1, pp. 61-62] or [11, pp. 211-212]

$$J_n(A):=\int_{\mathbb{R}^n}\exp(-\langle Ax,x\rangle)dx:=\int_{-\infty}^{\infty}...\int_{-\infty}^{\infty}\exp(-\langle Ax,x\rangle)dx = \frac{\pi^{n/2}}{[\det(A)]^{1/2}}, \tag{1.5}$$

where A is a positive definite matrix of order n and $\langle\cdot,\cdot\rangle$ is the usual inner product on $\mathbb{R}^n$.

By utilizing the representation (1.5) and Hölder's integral inequality for multiple integrals one can prove the *logarithmic concavity* of the determinant that is due to Ky Fan ([1, p. 63] or [11, p. 212]), namely

$$\det((1-\lambda)A+\lambda B)\ge[\det(A)]^{1-\lambda}[\det(B)]^{\lambda} \tag{1.6}$$

for any positive definite matrices A, B and $\lambda\in[0,1]$.

By mathematical induction we can get a generalization of (1.6) which was obtained by L. Mirsky in [10], see also [11, p. 212]

$$\det\left(\sum_{j=1}^m\lambda_jA_j\right)\ge\prod_{j=1}^m[\det(A_j)]^{\lambda_j},\ m\ge2, \tag{1.7}$$

where $\lambda_j>0$, $j=1,...,m$ with $\sum_{j=1}^m\lambda_j=1$ and $A_j>0$, $j=1,...,m$.

If we write (1.7) for $A_j=B_j^{-1}$ we get

$$\det\left(\sum_{j=1}^m\lambda_jB_j^{-1}\right)\ge\prod_{j=1}^m\left[\det\left(B_j^{-1}\right)\right]^{\lambda_j}=\left(\prod_{j=1}^m[\det(B_j)]^{\lambda_j}\right)^{-1},$$

which also gives

$$\prod_{j=1}^{m}[\det(A_j)]^{\lambda_j} \geq \det\left[\left(\sum_{j=1}^{m}\lambda_j A_j^{-1}\right)^{-1}\right], \tag{1.8}$$

where $\lambda_j > 0$, $j = 1, ..., m$ with $\sum_{j=1}^{m}\lambda_j = 1$ and $A_j > 0$, $j = 1, ..., m$.

Using the representation (1.5) one can also prove the result, see [11, p. 212],

$$\det(A) = \det(A_{1n}) \leq \det(A_{1k})\det\left(A_{(k+1)n}\right),\ k = 1, ..., n; \tag{1.9}$$

where the determinant $\det(A_{rs})$ is defined by

$$\det(A_{rs}) = \det(a_{ij}), i,\ j = r, ..., s.$$

In particular,

$$\det(A) \leq a_{11}a_{22}...a_{nn}. \tag{1.10}$$

We recall also the Minkowski's type inequality,

$$[\det(A+B)]^{1/n} \geq [\det(A)]^{1/n} + [\det(B)]^{1/n} \tag{1.11}$$

for A, B positive definite matrices of order n. For other determinant inequalities see Chapter VIII of the classic book [11]. For some recent results see [5]-[9].

Motivated by the above results, in this paper we prove among others that, if the positive definite matrices A, B of order n satisfy the condition

$$0 < mI_n \leq B - A \leq MI_n,$$

for some constants $0 < m < M$, where I_n is the identity matrix, then

$$\begin{aligned}0 &\leq \frac{1}{2}\left([\det(A+mI_n)]^{-1} + [\det(A)]^{-1}\right)\\ &\quad - \int_0^1 [\det(A+mtI_n)]^{-1}dt\\ &\leq \frac{1}{2}\left[[\det(A)]^{-1} + [\det(B)]^{-1}\right] - \int_0^1 [\det((1-t)A+tB)]^{-1}dt\\ &\leq \frac{1}{2}\left([\det(A+MI_n)]^{-1} + [\det(A)]^{-1}\right)\\ &\quad - \int_0^1 [\det(A+MtI_n)]^{-1}dt.\end{aligned}$$

2. Main Results

We need the following simple analytic inequality:

Lemma 1. *Assume that* a, $b > 0$ *with*

$$0 < k \leq \frac{b}{a} \leq K. \tag{2.1}$$

Then

$$\frac{1}{2}\left(k^t + k^{1-t}\right)a \leq H_t(a,b) \leq \frac{1}{2}\left(K^t + K^{1-t}\right)a, \tag{2.2}$$

for all $t \in [0,1]$.

Proof. If we consider the function $f_t : [0, \infty) \to \mathbb{R}$ for $t \in [0,1]$ defined by

$$f_t(u) = \frac{1}{2}\left(u^t + u^{1-t}\right),$$

then

$$f_t'(u) = \frac{1}{2}\left(tu^{t-1} + (1-t)u^{-t}\right),$$

which is positive for $u \in (0, \infty)$.

Therefore f_t is increasing on $(0, \infty)$ and

$$f_t(k) = \min_{u \in [k,K]} f_t(u) \le f_t(u) \le \max_{u \in [k,K]} f_t(u) = f_t(K)$$

for any $u \in [k, K]$.

Now, if $0 \le k \le \frac{b}{a} \le K$, then

$$\frac{1}{2}\left(k^t + k^{1-t}\right) \le \frac{1}{2}\left(\left(\frac{b}{a}\right)^t + \left(\frac{b}{a}\right)^{1-t}\right) \le \frac{1}{2}\left(K^t + K^{1-t}\right),$$

namely

$$\frac{1}{2}\left(k^t + k^{1-t}\right)a \le \frac{1}{2}\left(a^{1-t}b^t + a^t b^{1-t}\right) \le \frac{1}{2}\left(K^t + K^{1-t}\right)a,$$

and the inequality (2.2) is proved. □

Theorem 1. *Assume that the positive definite matrices A, B satisfy the condition*

$$0 < mI_n \le B - A \le MI_n, \tag{2.3}$$

for some constants $0 < m < M$, then

$$\begin{aligned} &\frac{1}{2}\left(\left[\det\left(A + MtI_n\right)\right]^{-1/2} + \left[\det\left(A + M(1-t)I_n\right)\right]^{-1/2}\right) \\ &\le \frac{1}{2}\left(\left[\det\left((1-t)A + tB\right)\right]^{-1/2} + \left[\det\left(tA + (1-t)B\right)\right]^{-1/2}\right) \\ &\le \frac{1}{2}\left(\left[\det\left(A + mtI_n\right)\right]^{-1/2} + \left[\det\left(A + m(1-t)I_n\right)\right]^{-1/2}\right). \end{aligned} \tag{2.4}$$

Also,

$$\begin{aligned} \int_0^1 \left[\det\left(A + MtI_n\right)\right]^{-1/2} dt &\le \int_0^1 \left[\det\left((1-t)A + tB\right)\right]^{-1/2} dt \\ &\le \int_0^1 \left[\det\left(A + mtI_n\right)\right]^{-1/2} dt. \end{aligned} \tag{2.5}$$

Proof. Let $a = \exp(-\langle Ax, x\rangle)$ and $b = \exp(-\langle Bx, x\rangle)$ for $x \in \mathbb{R}^n$. Then $\frac{b}{a} = \exp(-\langle (B - A)x, x\rangle)$ and since $0 < mI_n \le B - A \le MI_n$, hence

$$\exp\left(-M\|x\|^2\right) \le \frac{b}{a} \le \exp\left(-m\|x\|^2\right)$$

If we apply the inequality (2.2) for $a=\exp\left(-\langle Ax,x\rangle\right)$, $b=\exp\left(-\langle Bx,x\rangle\right)$, $k=\exp\left(-M\|x\|^{2}\right)$ and $K=\exp\left(-m\|x\|^{2}\right)$, then we get

$$\begin{aligned}&\frac{1}{2}\left(\exp\left(-Mt\|x\|^{2}\right)+\exp\left(-M(1-t)\|x\|^{2}\right)\right)\exp\left(-\langle Ax,x\rangle\right)\\&\leq\frac{1}{2}\left(\exp\left(-\langle((1-t)A+tB)x,x\rangle\right)+\exp\left(-\langle(tA+(1-t)B)x,x\rangle\right)\right)\\&\leq\frac{1}{2}\left(\exp\left(-mt\|x\|^{2}\right)+\exp\left(-m(1-t)\|x\|^{2}\right)\right)\exp\left(-\langle Ax,x\rangle\right),\end{aligned}$$

namely

$$\begin{aligned}&\frac{1}{2}\left(\exp\left(-\langle(A+MtI_n)x,x\rangle\right)+\exp\left(-\langle(A+M(1-t)I_n)x,x\rangle\right)\right)\\&\leq\frac{1}{2}\left(\exp\left(-\langle((1-t)A+tB)x,x\rangle\right)+\exp\left(-\langle(tA+(1-t)B)x,x\rangle\right)\right)\\&\leq\frac{1}{2}\left(\exp\left(-\langle(A+mtI_n)x,x\rangle\right)+\exp\left(-\langle(A+m(1-t)I_n)x,x\rangle\right)\right),\end{aligned}\tag{2.6}$$

for $x\in\mathbb{R}^n$ and $t\in[0,1]$.

If we take the integral on $\mathbb{R}^n$, then we get

$$\begin{aligned}&\frac{1}{2}\left(\int_{\mathbb{R}^n}\exp\left(-\langle(A+MtI_n)x,x\rangle\right)dx+\int_{\mathbb{R}^n}\exp\left(-\langle(A+M(1-t)I_n)x,x\rangle\right)dx\right)\\&\leq\frac{1}{2}\left(\int_{\mathbb{R}^n}\exp\left(-\langle((1-t)A+tB)x,x\rangle\right)dx\right.\\&\left.+\int_{\mathbb{R}^n}\exp\left(-\langle(tA+(1-t)B)x,x\rangle\right)dx\right)\\&\leq\frac{1}{2}\left(\int_{\mathbb{R}^n}\exp\left(-\langle(A+mtI_n)x,x\rangle\right)dx+\int_{\mathbb{R}^n}\exp\left(-\langle(A+m(1-t)I_n)x,x\rangle\right)dx\right),\end{aligned}$$

and by the representation (1.5) we derive

$$\begin{aligned}&\frac{1}{2}\left(J_n(A+MtI_n)+J_n(A+M(1-t)I_n)\right)\\&\leq\frac{1}{2}\left(J_n((1-t)A+tB)+J_n(tA+(1-t)B)\right)\\&\leq\frac{1}{2}\left(J_n(A+mtI_n)+J_n(A+m(1-t)I_n)\right)\end{aligned}$$

for $t\in[0,1]$.

By using the second part of (1.5) we derive (2.4).

By taking the integral over t on $[0,1]$ in the inequality (2.4) and since

$$\int_0^1\det((1-t)A+tB)^{-1/2}dt=\int_0^1[\det(tA+(1-t)B)]^{-1/2}dt,$$

$$\int_0^1[\det(A+MtI_n)]^{-1/2}dt=\int_0^1[\det(A+M(1-t)I_n)]^{-1/2}dt,$$

and

$$\int_0^1[\det(A+mtI_n)]^{-1/2}dt=\int_0^1[\det(A+m(1-t)I_n)]^{-1/2}dt$$

then we obtain the desired result (2.5). □

Lemma 2. *Consider the function* $h_\nu : [0,\infty) \to \mathbb{R}$ *for* $t \in (0,1)$ *defined by*

$$h_t(u) = \frac{u+1}{2} - \frac{1}{2}\left(u^t + u^{1-t}\right) \geq 0. \tag{2.7}$$

Then h_t *is decreasing on* $[0,1)$ *and increasing on* $(1,\infty)$ *with* $u = 1$ *its global minimum. We have* $h_t(0) = \frac{1}{2}$, $\lim_{u\to\infty} h_t(u) = \infty$ *and* h_t *is convex on* $(0,\infty)$.

Proof. We have

$$h_t'(u) = \frac{1}{2}\left(1 - \frac{t}{u^{1-t}} - \frac{1-t}{u^t}\right)$$

and

$$h_t''(u) = \frac{1}{2}t(1-t)\left(u^{t-2} + u^{-t-1}\right)$$

for any $u \in (0,\infty)$ and $t \in (0,1)$.

We observe that $h_t'(1) = 0$ and $h_t''(u) > 0$ for any $u \in (0,\infty)$ and $t \in (0,1)$. These imply that the equation $h_t'(u) = 0$ has only one solution on $(0,\infty)$, namely $u = 1$. Since $h_t'(u) < 0$ for $u \in (0,1)$ and $h_t'(u) > 0$ for $u \in (1,\infty)$, then we deduce the desired conclusion. □

We also have:

Lemma 3. *Assume that* $a, b > 0$ *and such that the condition (2.1) holds. Then we have*

$$0 \leq \delta_t(k,K)\, a \leq \frac{a+b}{2} - H_t(a,b) \leq \Delta_t(k,K)\, a, \tag{2.8}$$

where

$$\delta_t(k,K) := \begin{cases} h_t(K) & \text{if } K < 1, \\ 0 & \text{if } k \leq 1 \leq K, \\ h_t(k) & \text{if } 1 < k \end{cases} \tag{2.9}$$

and

$$\Delta_t(k,K) := \begin{cases} h_t(k) & \text{if } K < 1, \\ \max\{h_t(k), h_t(K)\} & \text{if } k \leq 1 \leq K, \\ h_t(K) & \text{if } 1 < k, \end{cases} \tag{2.10}$$

where h_t *is defined by (2.7).*

Proof. Using Lemma 2 we have

$$\begin{cases} h_t(K) \text{ if } K < 1, \\ 0 \text{ if } k \leq 1 \leq K, \\ h_t(k) \text{ if } 1 < k, \end{cases} \leq h_t(u) \leq \begin{cases} h_t(k) \text{ if } K < 1, \\ \max\{h_t(k), h_t(K)\} \text{ if } k \leq 1 \leq K, \\ h_t(K) \text{ if } 1 < k \end{cases}$$

for any $u \in [k,K]$ and $t \in (0,1)$.

This mean that for $u \in [k,K]$,

$$\delta_t(k,K) \leq \frac{u+1}{2} - \frac{1}{2}\left(u^t + u^{1-t}\right) \leq \Delta_t(k,K). \tag{2.11}$$

From (2.1) we have,

$$\delta_t(k,K) \le \frac{\frac{b}{a}+1}{2} - \frac{1}{2}\left(\left(\frac{b}{a}\right)^t + \left(\frac{b}{a}\right)^{1-t}\right) \le \Delta_t(k,K). \tag{2.12}$$

Finally, if we multiply both sides of (2.12) by a we get the desired result (2.8). □

Theorem 2. *Assume that the positive definite matrices A, B satisfy the condition (2.3), then*

$$\begin{aligned} 0 &\le \frac{1}{2}\left([\det(A+mI_n)]^{-1/2} + [\det(A)]^{-1/2}\right) \\ &\quad - \frac{1}{2}\left([\det(A+mtI_n)]^{-1/2} + [\det(A+m((1-t))I_n)]^{-1/2}\right) \\ &\le \frac{1}{2}\left[[\det(A)]^{-1/2} + [\det(B)]^{-1/2}\right] \\ &\quad - \frac{1}{2}\left[[\det((1-t)A+tB)]^{-1/2} + [\det(tA+(1-t)B)]^{-1/2}\right] \\ &\le \frac{1}{2}\left([\det(A+MI_n)]^{-1/2} + [\det(A)]^{-1/2}\right) \\ &\quad - \frac{1}{2}\left([\det(A+MtI_n)]^{-1/2} + [\det(A+M((1-t))I_n)]^{-1/2}\right), \end{aligned} \tag{2.13}$$

for all $t \in [0,1]$.

Also

$$\begin{aligned} 0 &\le \frac{1}{2}\left([\det(A+mI_n)]^{-1/2} + [\det(A)]^{-1/2}\right) \\ &\quad - \int_0^1 [\det(A+mtI_n)]^{-1/2}\,dt \\ &\le \frac{1}{2}\left[[\det(A)]^{-1/2} + [\det(B)]^{-1/2}\right] - \int_0^1 [\det((1-t)A+tB)]^{-1/2}\,dt \\ &\le \frac{1}{2}\left([\det(A+MI_n)]^{-1/2} + [\det(A)]^{-1/2}\right) \\ &\quad - \int_0^1 [\det(A+MtI_n)]^{-1/2}\,dt. \end{aligned} \tag{2.14}$$

Proof. If we apply the inequality (2.2) for $a = \exp(-\langle Ax, x\rangle)$, $b = \exp(-\langle Bx, x\rangle)$, $k = \exp\left(-M\|x\|^2\right)$ and $K = \exp\left(-m\|x\|^2\right) < 1$, then we get

$$\left[\frac{\exp\left(-m\|x\|^2\right)+1}{2} - \frac{1}{2}\left(\left(\exp\left(-m\|x\|^2\right)\right)^t + \left(\exp\left(-m\|x\|^2\right)\right)^{1-t}\right)\right]$$
$$\times \exp(-\langle Ax, x\rangle)$$
$$\leq \frac{1}{2}\left[\exp(-\langle Ax, x\rangle) + \exp(-\langle Bx, x\rangle)\right]$$
$$-\frac{1}{2}\left(\exp\left(-\langle((1-t)A + tB)x, x\rangle\right) + \exp\left(-\langle(tA + (1-t)B)x, x\rangle\right)\right)$$
$$\leq \left[\frac{\exp\left(-M\|x\|^2\right)+1}{2} - \frac{1}{2}\left(\left(\exp\left(-M\|x\|^2\right)\right)^t + \left(\exp\left(-M\|x\|^2\right)\right)^{1-t}\right)\right]$$
$$\times \exp(-\langle Ax, x\rangle),$$

namely

$$\begin{aligned}(2.15)\quad &\frac{1}{2}\left(\exp(-\langle A + mI_n x, x\rangle) + \exp(-\langle Ax, x\rangle)\right)\\ &-\frac{1}{2}\left(\exp(-\langle A + mtI_n x, x\rangle) + \left(\exp(-\langle A + m(1-t)I_n x, x\rangle)\right)\right)\\ &\leq \frac{1}{2}\left[\exp(-\langle Ax, x\rangle) + \exp(-\langle Bx, x\rangle)\right]\\ &-\frac{1}{2}\left(\exp\left(-\langle((1-t)A + tB)x, x\rangle\right) + \exp\left(-\langle(tA + (1-t)B)x, x\rangle\right)\right)\\ &\leq \frac{1}{2}\left(\exp(-\langle A + MI_n x, x\rangle) + \exp(-\langle Ax, x\rangle)\right)\\ &-\frac{1}{2}\left(\exp(-\langle A + MtI_n x, x\rangle) + \left(\exp(-\langle A + M(1-t)I_n x, x\rangle)\right)\right)\end{aligned}$$

for $x \in \mathbb{R}^n$ and $t \in [0, 1]$.

If we take the integral on $\mathbb{R}^n$ in the inequality (2.15), then we get

$$\frac{1}{2}\left(\int_{\mathbb{R}^n} \exp(-\langle(A + mI_n)x, x\rangle)\,dx + \int_{\mathbb{R}^n} \exp(-\langle Ax, x\rangle)\,dx\right)$$
$$-\frac{1}{2}\left(\int_{\mathbb{R}^n} \exp(-\langle(A + mtI_n)x, x\rangle)\,dx\right.$$
$$\left.+\int_{\mathbb{R}^n} \left(\exp(-\langle(A + m(1-t)I_n)x, x\rangle)\right)dx\right)$$

$$\leq \frac{1}{2}\left[\int_{\mathbb{R}^n} \exp\left(-\langle Ax, x\rangle\right) dx + \int_{\mathbb{R}^n} \exp\left(-\langle Bx, x\rangle\right) dx\right]$$
$$-\frac{1}{2}\left(\int_{\mathbb{R}^n} \exp\left(-\langle\left(\left(1-t\right)A + tB\right)x, x\rangle\right) dx\right.$$
$$\left.+\int_{\mathbb{R}^n} \exp\left(-\langle\left(tA + \left(1-t\right)B\right)x, x\rangle\right) dx\right)$$
$$\leq \frac{1}{2}\left(\int_{\mathbb{R}^n} \exp\left(-\langle\left(A + MI_n\right)x, x\rangle\right) dx + \int_{\mathbb{R}^n} \exp\left(-\langle Ax, x\rangle\right) dx\right)$$
$$-\frac{1}{2}\left(\int_{\mathbb{R}^n} \exp\left(-\langle\left(A + MtI_n\right)x, x\rangle\right) dx\right.$$
$$\left.+\int_{\mathbb{R}^n} \left(\exp\left(-\langle\left(A + M\left(1-t\right)I_n\right)x, x\rangle\right)\right) dx\right),$$

which, by the representation (1.5) gives

$$0 \leq \frac{1}{2}\left(J_n\left(A + mI_n\right) + J_n\left(A\right)\right)$$
$$-\frac{1}{2}\left(J_n\left(A + mtI_n\right) + J_n\left(A + m\left(\left(1-t\right)\right)I_n\right)\right)$$
$$\leq \frac{1}{2}\left[J_n\left(A\right) + J_n\left(B\right)\right] - \frac{1}{2}\left[J_n\left(\left(1-t\right)A + tB\right) + J_n\left(tA + \left(1-t\right)B\right)\right]$$
$$\leq \frac{1}{2}\left(J_n\left(A + MI_n\right) + J_n\left(A\right)\right)$$
$$-\frac{1}{2}\left(J_n\left(A + MtI_n\right) + J_n\left(A + M\left(\left(1-t\right)\right)I_n\right)\right).$$

By making use of the second equality in (1.5) we derive (2.13).

The inequality (2.14) follows by integrating (2.13) over $t \in [0,1]$. □

We also have:

Lemma 4. *Consider the function* $g_t : [0,\infty) \to \mathbb{R}$ *for* $t \in (0,1)$ *defined by*

$$g_t\left(u\right) = \frac{1}{2}\left(u^t + u^{1-t}\right) - \sqrt{u} \geq 0. \tag{2.16}$$

Then $g_t\left(0\right) = g_t\left(1\right) = 0$, g_t *is increasing on* $\left(0, u_t\right)$ *with a local maximum in*

$$u_t := \left(\frac{t}{1-t}\right)^{\frac{2}{1-2t}} \in \left(0,1\right),\ t \neq 1/2 \tag{2.17}$$

is decreasing on $\left(u_t, 1\right)$ *with a local minimum in* $u = 1$ *and increasing on* $\left(1,\infty\right)$ *with* $\lim_{u\to\infty} g_t\left(u\right) = \infty$.

Proof. (i). If $t \in \left(0, \frac{1}{2}\right)$, then

$$g_t'\left(u\right) = \frac{1}{2}\left(\frac{t}{u^{1-t}} + \frac{1-t}{u^t} - \frac{1}{u^{1/2}}\right)$$
$$= \frac{1}{2}\frac{t + \left(1-t\right)u^{1-2t} - u^{\frac{1-2t}{2}}}{u^{1-t}}.$$

If we denote $w=u^{\frac{1-2t}{2}}$, then we have

$$\begin{aligned} t+(1-t)\,u^{1-2t}-u^{\frac{1-2t}{2}} &= (1-t)\,w^2-w+t. \\ &= (1-t)\left(w-\frac{t}{1-t}\right)(w-1) \\ &= (1-t)\left(u^{\frac{1-2t}{2}}-\frac{t}{1-t}\right)\left(u^{\frac{1-2t}{2}}-1\right). \end{aligned}$$

We observe that $g_t'(u)=0$ only for $u=1$ and $u_t=\left(\frac{t}{1-t}\right)^{\frac{2}{1-2t}}\in(0,1)$. Also $g_t'(u)>0$ for $u\in(0,u_t)\cup(1,\infty)$ and $g_t'(u)<0$ for $u\in(u_t,1)$. These imply the desired conclusion.

(ii) If $t\in\left(\frac{1}{2},1\right)$, then

$$g_t'(u)=\frac{1}{2}\frac{1-t+tu^{2t-1}-u^{\frac{2t-1}{2}}}{u^t}.$$

If we denote $z=u^{\frac{2t-1}{2}}$, then we have

$$\begin{aligned} 1-t+tu^{2t-1}-u^{\frac{2t-1}{2}} &= tz^2-z+1-t \\ &= t\left(z-\frac{1-t}{t}\right)(z-1) \\ &= t\left(u^{\frac{2t-1}{2}}-\frac{1-t}{t}\right)\left(u^{\frac{2t-1}{2}}-1\right). \end{aligned}$$

We observe that $g_t'(u)=0$ only for $u=1$ and $u_t=\left(\frac{1-t}{t}\right)^{\frac{2}{2t-1}}=\left(\frac{t}{1-t}\right)^{\frac{2}{1-2t}}\in(0,1)$. Also $g_t'(u)>0$ for $u\in(0,u_t)\cup(1,\infty)$ and $g_t'(u)<0$ for $u\in(u_t,1)$. These imply the desired conclusion. □

Lemma 5. *Assume that $a,b>0$ with $0<\frac{b}{a}\leq 1$, then for $t\in(0,1)\setminus\{1/2\}$*

$$0\leq H_t(a,b)-\sqrt{ab}\leq D(t)\,a, \tag{2.18}$$

where

$$D(t):=\frac{1}{2}\left(\left(\frac{t}{1-t}\right)^{\frac{2t}{1-2t}}+\left(\frac{t}{1-t}\right)^{\frac{2(1-t)}{1-2t}}\right)-\left(\frac{t}{1-t}\right)^{\frac{1}{1-2t}}. \tag{2.19}$$

Proof. From Lemma 4 we have for $u\in(0,1)$ and $t\in(0,1)\setminus\{1/2\}$ that

$$\begin{aligned} 0 &\leq \frac{1}{2}\left(u^t+u^{1-t}\right)-\sqrt{u} \\ &\leq \frac{1}{2}\left(\left(\frac{t}{1-t}\right)^{\frac{2t}{1-2t}}+\left(\frac{t}{1-t}\right)^{\frac{2(1-t)}{1-2t}}\right)-\left(\frac{t}{1-t}\right)^{\frac{1}{1-2t}} \\ &= D(t). \end{aligned}$$

By taking $u=\frac{b}{a}$ we get the desired result (2.18). □

Theorem 3. *Assume that A, B are positive definite matrices with $A \le B$, then*

$$0 \le \frac{1}{2}\left[\left[\det\left(\left(1-t\right)A+tB\right)\right]^{-1/2}+\left[\det\left(tA+\left(1-t\right)B\right)\right]^{-1/2}\right] - \left[\det\left(\frac{A+B}{2}\right)\right]^{-1/2} \le D\left(t\right)\left[\det\left(A\right)\right]^{-1/2} \tag{2.20}$$

for all $t \in (0,1) \setminus \{1/2\}$.

Also,

$$0 \le \int_0^1 \left[\det\left(\left(1-t\right)A+tB\right)\right]^{-1/2} dt - \left[\det\left(\frac{A+B}{2}\right)\right]^{-1/2} \le \left[\det\left(A\right)\right]^{-1/2}\int_0^1 D\left(t\right)dt. \tag{2.21}$$

Proof. For $a = \exp\left(-\left\langle Ax,x\right\rangle\right)$, $b = \exp\left(-\left\langle Bx,x\right\rangle\right)$ we have for $x \in \mathbb{R}^n$ that

$$\frac{b}{a} = \frac{\exp\left(-\left\langle Bx,x\right\rangle\right)}{\exp\left(-\left\langle Ax,x\right\rangle\right)} = \exp\left(-\left\langle\left(B-A\right)x,x\right\rangle\right) \le 1.$$

From (2.18) we get

$$\begin{aligned} 0 &\le \frac{1}{2}\left(\int_{\mathbb{R}^n}\exp\left(-\left\langle\left(\left(1-t\right)A+tB\right)x,x\right\rangle\right)dx \right.\\ &\left.+\int_{\mathbb{R}^n}\exp\left(-\left\langle\left(tA+\left(1-t\right)B\right)x,x\right\rangle\right)dx\right) \\ &-\int_{\mathbb{R}^n}\exp\left(-\left\langle\left(\frac{A+B}{2}\right)x,x\right\rangle\right)dx \\ &\le D\left(t\right)\int_{\mathbb{R}^n}\exp\left(-\left\langle Ax,x\right\rangle\right)dx, \end{aligned}$$

and by a similar argument as above, we get (2.20). □

3. Related Results

If we take the square in the representation (1.5), then we get

$$\left(\int_{\mathbb{R}^n}\exp(-\left\langle x,Ax\right\rangle)dx\right)^2 = \frac{\pi^n}{\det\left(A\right)}.$$

Since

$$\begin{aligned} \left(\int_{\mathbb{R}^n}\exp(-\left\langle Ax,x\right\rangle)dx\right)^2 &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\exp(-\left\langle Ax,x\right\rangle)\exp(-\left\langle Ay,y\right\rangle)dxdy \\ &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\exp(-\left\langle Ax,x\right\rangle-\left\langle Ay,y\right\rangle))dxdy, \end{aligned}$$

hence

$$K_n\left(A\right) := \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\exp(-\left\langle Ax,x\right\rangle-\left\langle Ay,y\right\rangle))dxdy = \frac{\pi^n}{\det\left(A\right)} \tag{3.1}$$

for A a positive definite matrix of order n and $\langle\cdot,\cdot\rangle$ is the usual inner product on $\mathbb{R}^n$.

We have:

Theorem 4. *Assume that the positive definite matrices A, B satisfy the condition (2.3), then*

$$\begin{aligned} (3.2)\qquad 0 &\leq \frac{1}{2}\left([\det(A+mI_n)]^{-1}+[\det(A)]^{-1}\right) \\ &-\frac{1}{2}\left([\det(A+mtI_n)]^{-1}+[\det(A+m((1-t))I_n)]^{-1}\right) \\ &\leq \frac{1}{2}\left[[\det(A)]^{-1}+[\det(B)]^{-1}\right] \\ &-\frac{1}{2}\left[[\det((1-t)A+tB)]^{-1}+[\det(tA+(1-t)B)]^{-1}\right] \\ &\leq \frac{1}{2}\left([\det(A+MI_n)]^{-1}+[\det(A)]^{-1}\right) \\ &-\frac{1}{2}\left([\det(A+MtI_n)]^{-1}+[\det(A+M((1-t))I_n)]^{-1}\right), \end{aligned}$$

for all $t\in[0,1]$.

Also

$$\begin{aligned} (3.3)\qquad 0 &\leq \frac{1}{2}\left([\det(A+mI_n)]^{-1}+[\det(A)]^{-1}\right) \\ &-\int_0^1[\det(A+mtI_n)]^{-1}dt \\ &\leq \frac{1}{2}\left[[\det(A)]^{-1}+[\det(B)]^{-1}\right]-\int_0^1[\det((1-t)A+tB)]^{-1}dt \\ &\leq \frac{1}{2}\left([\det(A+MI_n)]^{-1}+[\det(A)]^{-1}\right) \\ &-\int_0^1[\det(A+MtI_n)]^{-1}dt. \end{aligned}$$

Proof. If we apply the inequality (2.2) for $a=\exp(-\langle Ax,x\rangle-\langle Ay,y\rangle)$, $b=\exp(-\langle Bx,x\rangle-\langle By,y\rangle)$,

$$k=\exp\left(-M\left(\|x\|^2+\|y\|^2\right)\right)$$

and

$$K=\exp\left(-m\left(\|x\|^2+\|y\|^2\right)\right)<1,$$

then we get

$$\begin{aligned} &\left[\frac{\exp\left(-m\left(\|x\|^2+\|y\|^2\right)\right)+1}{2}\right. \\ &\left.-\frac{1}{2}\left(\left(\exp\left(-m\left(\|x\|^2+\|y\|^2\right)\right)\right)^t+\left(\exp\left(-m\left(\|x\|^2+\|y\|^2\right)\right)\right)^{1-t}\right)\right] \\ &\times\exp(-\langle Ax,x\rangle-\langle Ay,y\rangle) \end{aligned}$$

$$\begin{aligned}
&\leq \frac{1}{2}\left[\exp\left(-\langle Ax,x\rangle-\langle Ay,y\rangle\right)+\exp\left(-\langle Bx,x\rangle-\langle By,y\rangle\right)\right]\\
&-\frac{1}{2}\left[\exp\left(-\left\langle\left(\left(1-t\right)A+tB\right)x,x\right\rangle-\left\langle\left(\left(1-t\right)A+tB\right)y,y\right\rangle\right)\right.\\
&\left.+\exp\left(-\left\langle\left(tA+\left(1-t\right)B\right)x,x\right\rangle-\left\langle\left(tA+\left(1-t\right)B\right)y,y\right\rangle\right)\right]\\
&\leq\left[\frac{\exp\left(-M\left(\|x\|^2+\|y\|^2\right)\right)+1}{2}\right.\\
&\left.-\frac{1}{2}\left(\left(\exp\left(-M\left(\|x\|^2+\|y\|^2\right)\right)\right)^t+\left(\exp\left(-M\left(\|x\|^2+\|y\|^2\right)\right)\right)^{1-t}\right)\right]\\
&\times\exp\left(-\langle Ax,x\rangle-\langle Ay,y\rangle\right),
\end{aligned}$$

for $x,y\in\mathbb{R}^n$ and $t\in[0,1]$.

If we take the double integral on $\mathbb{R}^n\times\mathbb{R}^n$ then we get by (3.1)

$$\begin{aligned}
0&\leq\frac{1}{2}\left(K_n\left(A+mI_n\right)+K_n\left(A\right)\right)\\
&-\frac{1}{2}\left(K_n\left(A+mtI_n\right)+K_n\left(A+m\left(\left(1-t\right)\right)I_n\right)\right)\\
&\leq\frac{1}{2}\left[K_n\left(A\right)+K_n\left(B\right)\right]-\frac{1}{2}\left[K_n\left(\left(1-t\right)A+tB\right)+K_n\left(tA+\left(1-t\right)B\right)\right]\\
&\leq\frac{1}{2}\left(K_n\left(A+MI_n\right)+K_n\left(A\right)\right)\\
&-\frac{1}{2}\left(K_n\left(A+MtI_n\right)+K_n\left(A+M\left(\left(1-t\right)\right)I_n\right)\right).
\end{aligned}$$

By making use of the second equality in (3.1) we derive (3.2).

The inequality (3.3) follows by integrating (3.2) over $t\in[0,1]$. □

In a similar manner we can prove the following result as well:

Theorem 5. *Assume that A, B are positive definite matrices with $A\leq B$, then*

$$\begin{aligned}
0&\leq\frac{1}{2}\left[\left[\det\left(\left(1-t\right)A+tB\right)\right]^{-1}+\left[\det\left(tA+\left(1-t\right)B\right)\right]^{-1}\right]\\
&-\left[\det\left(\frac{A+B}{2}\right)\right]^{-1}\\
&\leq D\left(t\right)\left[\det\left(A\right)\right]^{-1}
\end{aligned}\tag{3.4}$$

for all $t\in(0,1)\setminus\{1/2\}$.

Also,

$$\begin{aligned}
0&\leq\int_0^1\left[\det\left(\left(1-t\right)A+tB\right)\right]^{-1}dt-\left[\det\left(\frac{A+B}{2}\right)\right]^{-1}\\
&\leq\left[\det\left(A\right)\right]^{-1}\int_0^1 D\left(t\right)dt.
\end{aligned}\tag{3.5}$$

A complex square matrix $H=(h_{ij})$, $i,j=1,...,n$ is said to be Hermitian provided $h_{ij}=\overline{h_{ji}}$ for all i, $j=1,...,n$. A Hermitian matrix is said to be positive definite if the Hermitian form $P(z)=\sum_{i,j=1}^n a_{ij}z_i\overline{z_j}$ is positive for all $z=(z_1,...,z_n)\in\mathbb{C}^n\setminus\{0\}$.

It is known that, see for instance [11, p. 215], for a positive definite Hermitian matrix H, we have

$$K_n(H) := \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \exp\left(-\langle \overline{z}, Hz\rangle\right) dxdy = \frac{\pi^n}{\det(H)}, \tag{3.6}$$

where $z = x + iy$ and dx and dy denote integration over real n-dimensional space $\mathbb{R}^n$. Here the inner product $\langle x, y\rangle$ is understood in the real sense, i.e. $\langle x, y\rangle = \sum_{k=1}^{n} x_k y_k$.

On making use of a similar argument to the one in Theorem 4 and Theorem 5 for the representation $K_n(\cdot)$ we can state the same inequalities for positive definite Hermitian matrices H and K.

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