SOME DETERMINANT INEQUALITIES FOR TWO POSITIVE DEFINITE MATRICES VIA A RESULT OF CARTWRIGHT AND FIELD

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Abstract. In this paper we prove among others that, if the positive definite matrices A, B satisfy the condition $A \le B$, then

$$\begin{aligned}(0 \le)\ &\frac{1}{12}\left[[\det(A)]^{-1} - 2[\det(B)]^{-1} + [\det(2B-A)]^{-1}\right] \\ &\le \frac{[\det(B)]^{-1} + [\det(A)]^{-1}}{2} - \int_0^1 [\det((1-t)B + tA)]^{-1} dt.\end{aligned}$$

If $A \le B < 2A$, then also

$$\begin{aligned}&\frac{[\det(B)]^{-1} + [\det(A)]^{-1}}{2} - \int_0^1 [\det((1-t)B + tA)]^{-1} dt \\ &\le \frac{1}{12}\left[[\det(2A-B)]^{-1} - 2[\det(A)]^{-1} + [\det(B)]^{-1}\right].\end{aligned}$$

1. Introduction

We have the following inequality that provides a refinement and a reverse for the celebrated Young's inequality

$$\frac{1}{2}\nu(1-\nu)\frac{(b-a)^2}{\max\{a,b\}} \le (1-\nu)a + \nu b - a^{1-\nu}b^{\nu} \le \frac{1}{2}\nu(1-\nu)\frac{(b-a)^2}{\min\{a,b\}} \tag{1.1}$$

for any $a, b > 0$ and $\nu \in [0,1]$.

This result was obtained in 1978 by Cartwright and Field [3] who established a more general result for n variables and gave an application for a probability measure supported on a finite interval.

A real square matrix $A = (a_{ij})$, $i, j = 1, ..., n$ is *symmetric* provided $a_{ij} = a_{ji}$ for all $i, j = 1, ..., n$. A real symmetric matrix is said to be *positive definite* provided the quadratic form $Q(x) = \sum_{i,j=1}^{n} a_{ij}x_i x_j$ is positive for all $x = (x_1, ..., x_n) \in \mathbb{R}^n \setminus \{0\}$. It is well known that a necessary and sufficient condition for the symmetric matrix A to be positive definite, and we write $A > 0$, is that all determinants

$$\det(A_k) = \det(a_{ij}),\ i,\ j = 1, ..., k;\ k = 1, ..., n$$

are positive.

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It is know that the following integral representation is valid, see [1, pp. 61-62] or [12, pp. 211-212]

$$J_n(A) := \int_{\mathbb{R}^n} \exp(-\langle Ax, x\rangle)dx := \int_{-\infty}^{\infty} ... \int_{-\infty}^{\infty} \exp(-\langle Ax, x\rangle)dx = \frac{\pi^{n/2}}{[\det(A)]^{1/2}}, \tag{1.2}$$

where A is a positive definite matrix of order n and $\langle\cdot,\cdot\rangle$ is the usual inner product on $\mathbb{R}^n$.

By utilizing the representation (1.2) and Hölder's integral inequality for multiple integrals one can prove the *logarithmic concavity* of the determinant that is due to Ky Fan ([1, p. 63] or [12, p. 212]), namely

$$\det((1-\lambda)A + \lambda B) \geq [\det(A)]^{1-\lambda}[\det(B)]^{\lambda} \tag{1.3}$$

for any positive definite matrices A, B and $\lambda \in [0,1]$.

By mathematical induction we can get a generalization of (1.3) which was obtained by L. Mirsky in [11], see also [12, p. 212]

$$\det\left(\sum_{j=1}^{m} \lambda_j A_j\right) \geq \prod_{j=1}^{m} [\det(A_j)]^{\lambda_j}, \ m \geq 2, \tag{1.4}$$

where $\lambda_j > 0$, $j = 1, ..., m$ with $\sum_{j=1}^{m} \lambda_j = 1$ and $A_j > 0$, $j = 1, ..., m$.

If we write (1.4) for $A_j = B_j^{-1}$ we get

$$\det\left(\sum_{j=1}^{m} \lambda_j B_j^{-1}\right) \geq \prod_{j=1}^{m} \left[\det\left(B_j^{-1}\right)\right]^{\lambda_j} = \left(\prod_{j=1}^{m} [\det(B_j)]^{\lambda_j}\right)^{-1},$$

which also gives

$$\prod_{j=1}^{m} [\det(A_j)]^{\lambda_j} \geq \det\left[\left(\sum_{j=1}^{m} \lambda_j A_j^{-1}\right)^{-1}\right], \tag{1.5}$$

where $\lambda_j > 0$, $j = 1, ..., m$ with $\sum_{j=1}^{m} \lambda_j = 1$ and $A_j > 0$, $j = 1, ..., m$.

Using the representation (1.2) one can also prove the result, see [12, p. 212],

$$\det(A) = \det(A_{1n}) \leq \det(A_{1k}) \det\left(A_{(k+1)n}\right), \ k = 1, ..., n; \tag{1.6}$$

where the determinant $\det(A_{rs})$ is defined by

$$\det(A_{rs}) = \det(a_{ij}), i, \ j = r, ..., s.$$

In particular,

$$\det(A) \leq a_{11}a_{22}...a_{nn}. \tag{1.7}$$

We recall also the Minkowski's type inequality,

$$[\det(A+B)]^{1/n} \geq [\det(A)]^{1/n} + [\det(B)]^{1/n} \tag{1.8}$$

for A, B positive definite matrices of order n. For other determinant inequalities see Chapter VIII of the classic book [12]. For some recent results see [6]-[10].

Motivated by the above results, in this paper we prove among others that, if the positive definite matrices A, B satisfy the condition $A \leq B$, then

$$\begin{aligned}(0 \leq)\ &\frac{1}{12}\left[[\det(A)]^{-1} - 2[\det(B)]^{-1} + [\det(2B-A)]^{-1}\right] \\ &\leq \frac{[\det(B)]^{-1} + [\det(A)]^{-1}}{2} - \int_0^1 [\det((1-t)B + tA)]^{-1}\,dt.\end{aligned}$$

If $A \leq B < 2A$, then also

$$\begin{aligned}&\frac{[\det(B)]^{-1} + [\det(A)]^{-1}}{2} - \int_0^1 [\det((1-t)B + tA)]^{-1}\,dt \\ &\leq \frac{1}{12}\left[[\det(2A-B)]^{-1} - 2[\det(A)]^{-1} + [\det(B)]^{-1}\right].\end{aligned}$$

2. Main Results

Our firsr mai result is as follows:

Theorem 1. *Let A, B be positive definite matrices and $t \in [0,1]$. If $A \leq B$, then*

$$\begin{aligned}(0 \leq)\ &\frac{1}{2}t(1-t)\left[[\det(A)]^{-1/2} - 2[\det(B)]^{-1/2} + [\det(2B-A)]^{-1/2}\right] \\ &\leq (1-t)[\det(B)]^{-1/2} + t[\det(A)]^{-1/2} - [\det((1-t)B + tA)]^{-1/2},\end{aligned} \tag{2.1}$$

and

$$\begin{aligned}(0 \leq)\ &\frac{1}{2}t(1-t)\left[[\det(A)]^{-1/2} - 2[\det(B)]^{-1/2} + [\det(2B-A)]^{-1/2}\right] \\ &\leq \frac{[\det(B)]^{-1/2} + [\det(A)]^{-1/2}}{2} \\ &- \frac{[\det((1-t)B + tA)]^{-1/2} + [\det(tB + (1-t)A)]^{-1/2}}{2}.\end{aligned} \tag{2.2}$$

If $A \leq B < 2A$, then also

$$\begin{aligned}&(1-t)[\det(B)]^{-1/2} + t[\det(A)]^{-1/2} - [\det((1-t)B + tA)]^{-1/2} \\ &\leq \frac{1}{2}t(1-t)\left[[\det(2A-B)]^{-1/2} - 2[\det(A)]^{-1/2} + [\det(B)]^{-1/2}\right]\end{aligned} \tag{2.3}$$

and

$$\begin{aligned}&(1-t)[\det(B)]^{-1/2} + t[\det(A)]^{-1/2} - [\det((1-t)B + tA)]^{-1/2} \\ &\leq \frac{1}{2}t(1-t)\left[[\det(2A-B)]^{-1/2} - 2[\det(A)]^{-1/2} + [\det(B)]^{-1/2}\right].\end{aligned} \tag{2.4}$$

Proof. If $0 < a < b$, then by (1.1)

$$(0 \leq)\ \frac{1}{2}t(1-t)\frac{(b-a)^2}{b} \leq (1-t)a + tb - a^{1-t}b^t \leq \frac{1}{2}t(1-t)\frac{(b-a)^2}{a},$$

namely

$$\begin{aligned}(0 \leq)\ \frac{1}{2}t(1-t)\left(b - 2a + a^2b^{-1}\right) &\leq (1-t)a + tb - a^{1-t}b^t \\ &\leq \frac{1}{2}t(1-t)\left(b^2a^{-1} - 2b + a\right),\end{aligned} \tag{2.5}$$

for all $t \in [0,1]$.

Since $0 < A \leq B$, hence $\exp(-\langle Bx, x\rangle) \leq \exp(-\langle Ax, x\rangle)$ for $x \in \mathbb{R}^n$. If we take in (2.5)

$$a = \exp(-\langle Bx, x\rangle) \text{ and } b = \exp(-\langle Ax, x\rangle),$$

then we get

$$\begin{aligned} (2.6)\qquad & (0 \leq) \frac{1}{2} t(1-t) \\ & \times (\exp(-\langle Ax, x\rangle) - 2\exp(-\langle Bx, x\rangle) + \exp(-\langle (2B-A)x, x\rangle)) \\ & \leq (1-t)\exp(-\langle Bx, x\rangle) + t\exp(-\langle Ax, x\rangle) \\ & - \exp(-\langle ((1-t)B + tA)x, x\rangle) \\ & \leq \frac{1}{2} t(1-t) \\ & \times (\exp(-\langle (2A-B)x, x\rangle) - 2\exp(-\langle Ax, x\rangle) + \exp(-\langle Bx, x\rangle)), \end{aligned}$$

for $x \in \mathbb{R}^n$ and $t \in [0,1]$.

Since $2B - A > 0$, hence we can take the integral on $\mathbb{R}^n$ in the first inequality in (2.6) to get

$$\begin{aligned} (2.7)\qquad & (0 \leq) \frac{1}{2} t(1-t) \left[\int_{\mathbb{R}^n} \exp(-\langle Ax, x\rangle) dx - 2\int_{\mathbb{R}^n} \exp(-\langle Bx, x\rangle) dx \right. \\ & \left. + \int_{\mathbb{R}^n} \exp(-\langle (2B-A)x, x\rangle) dx\right] \\ & \leq (1-t)\int_{\mathbb{R}^n} \exp(-\langle Bx, x\rangle) dx + t\int_{\mathbb{R}^n} \exp(-\langle Ax, x\rangle) dx \\ & - \int_{\mathbb{R}^n} \exp(-\langle ((1-t)B + tA)x, x\rangle) dx \end{aligned}$$

for $t \in [0,1]$.

Using representation (1.2) we get

$$\begin{aligned} (2.8)\qquad & (0 \leq) \frac{1}{2} t(1-t)[J_n(A) - 2J_n(B) + J_n(2B-A)] \\ & \leq (1-t)J_n(B) + tJ_n(A) - J_n((1-t)B + tA), \end{aligned}$$

which, by the second equality in (1.2), is equivalent to (2.1).

If we replace t by $1-t$ in (2.1), we get

$$\begin{aligned} (2.9)\qquad & (0 \leq) \frac{1}{2} t(1-t)\left[[\det(A)]^{-1/2} - 2[\det(B)]^{-1/2} + [\det(2B-A)]^{-1/2}\right] \\ & \leq t[\det(B)]^{-1/2} + (1-t)[\det(A)]^{-1/2} - [\det(tB + (1-t)A)]^{-1/2}. \end{aligned}$$

If we add (2.1) with (2.9) and divide by 2, then we get (2.2).

Now, if $B<2A$, then we can also take the integral in the second inequality in (2.6) to get

$$\begin{aligned}&(1-t)\int_{\mathbb{R}^n}\exp\left(-\left\langle Bx,x\right\rangle\right)dx+t\int_{\mathbb{R}^n}\exp\left(-\left\langle Ax,x\right\rangle\right)dx\\&-\int_{\mathbb{R}^n}\exp\left(-\left\langle\left(\left(1-t\right)B+tA\right)x,x\right\rangle\right)dx\\&\leq\frac{1}{2}t\left(1-t\right)\left(\int_{\mathbb{R}^n}\exp\left(-\left\langle\left(2A-B\right)x,x\right\rangle\right)dx\right.\\&\left.-2\int_{\mathbb{R}^n}\exp\left(-\left\langle Ax,x\right\rangle\right)dx+\int_{\mathbb{R}^n}\exp\left(-\left\langle Bx,x\right\rangle\right)dx\right),\end{aligned}$$

which gives (2.3). □

Corollary 1. *Let A, B be positive definite matrices. If $A\leq B$, then*

$$\begin{aligned}&(0\leq)\,\frac{1}{12}\left[\left[\det\left(A\right)\right]^{-1/2}-2\left[\det\left(B\right)\right]^{-1/2}+\left[\det\left(2B-A\right)\right]^{-1/2}\right]\\&\leq\frac{\left[\det\left(B\right)\right]^{-1/2}+\left[\det\left(A\right)\right]^{-1/2}}{2}-\int_0^1\left[\det\left(\left(1-t\right)B+tA\right)\right]^{-1/2}dt.\end{aligned}\tag{2.10}$$

If $A\leq B<2A$, then also

$$\begin{aligned}&\frac{\left[\det\left(B\right)\right]^{-1/2}+\left[\det\left(A\right)\right]^{-1/2}}{2}-\int_0^1\left[\det\left(\left(1-t\right)B+tA\right)\right]^{-1/2}dt\\&\leq\frac{1}{12}\left[\left[\det\left(2A-B\right)\right]^{-1/2}-2\left[\det\left(A\right)\right]^{-1/2}+\left[\det\left(B\right)\right]^{-1/2}\right].\end{aligned}\tag{2.11}$$

The proof follows by Theorem 1 by taking the integral and observing that

$$\frac{1}{2}\int_0^1 t\left(1-t\right)dt=\frac{1}{12}.$$

If we take the square in the representation (1.2), then we get

$$\left(\int_{\mathbb{R}^n}\exp(-\left\langle x,Ax\right\rangle)dx\right)^2=\frac{\pi^n}{\det\left(A\right)}.$$

Since

$$\begin{aligned}\left(\int_{\mathbb{R}^n}\exp(-\left\langle Ax,x\right\rangle)dx\right)^2&=\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\exp(-\left\langle Ax,x\right\rangle)\exp(-\left\langle Ay,y\right\rangle)dxdy\\&=\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\exp(-\left\langle Ax,x\right\rangle-\left\langle Ay,y\right\rangle))dxdy,\end{aligned}$$

hence

$$K_n\left(A\right):=\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\exp(-\left\langle Ax,x\right\rangle-\left\langle Ay,y\right\rangle))dxdy=\frac{\pi^n}{\det\left(A\right)}\tag{2.12}$$

for A a positive definite matrix of order n and $\langle\cdot,\cdot\rangle$ is the usual inner product on $\mathbb{R}^n$.

Theorem 2. *Let A, B be positive definite matrices and $t\in[0,1]$. If $A\leq B$, then*

$$\begin{aligned}&(0\leq)\,\frac{1}{2}t\left(1-t\right)\left[\left[\det\left(A\right)\right]^{-1}-2\left[\det\left(B\right)\right]^{-1}+\left[\det\left(2B-A\right)\right]^{-1}\right]\\&\leq\left(1-t\right)\left[\det\left(B\right)\right]^{-1}+t\left[\det\left(A\right)\right]^{-1}-\left[\det\left(\left(1-t\right)B+tA\right)\right]^{-1},\end{aligned}\tag{2.13}$$

and

$$(0\leq)\,\frac{1}{2}t\,(1-t)\left[[\det(A)]^{-1}-2\,[\det(B)]^{-1}+[\det(2B-A)]^{-1}\right]\leq\frac{[\det(B)]^{-1}+[\det(A)]^{-1}}{2}-\frac{[\det((1-t)\,B+tA)]^{-1}+[\det(tB+(1-t)\,A)]^{-1}}{2}. \tag{2.14}$$

If $A\leq B<2A$, then also

$$(1-t)\,[\det(B)]^{-1}+t\,[\det(A)]^{-1}-[\det((1-t)\,B+tA)]^{-1}\leq\frac{1}{2}t\,(1-t)\left[[\det(2A-B)]^{-1}-2\,[\det(A)]^{-1}+[\det(B)]^{-1/2}\right] \tag{2.15}$$

and

$$(1-t)\,[\det(B)]^{-1}+t\,[\det(A)]^{-1}-[\det((1-t)\,B+tA)]^{-1}\leq\frac{1}{2}t\,(1-t)\left[[\det(2A-B)]^{-1}-2\,[\det(A)]^{-1}+[\det(B)]^{-1}\right]. \tag{2.16}$$

Proof. Since $0<A\leq B$, hence $\exp(-\langle Bx,x\rangle-\langle By,y\rangle)\leq\exp(-\langle Ax,x\rangle-\langle Ay,y\rangle)$ for $x,\,y\in\mathbb{R}^n$. If we take in (2.5)

$$a=\exp(-\langle Bx,x\rangle-\langle By,y\rangle)\ \text{ and }\ b=\exp(-\langle Ax,x\rangle-\langle Ay,y\rangle),$$

then we get

$$\begin{aligned}(0\leq)\,&\frac{1}{2}t\,(1-t)\,(\exp(-\langle Ax,x\rangle-\langle Ay,y\rangle)\\&-2\exp(-\langle Bx,x\rangle-\langle By,y\rangle)+\exp(-\langle(2B-A)\,x,x\rangle-\langle(2B-A)\,y,y\rangle))\\&\leq(1-t)\exp(-\langle Bx,x\rangle-\langle By,y\rangle)+t\exp(-\langle Ax,x\rangle-\langle Ay,y\rangle)\\&-\exp(-\langle((1-t)\,B+tA)\,x,x\rangle-\langle((1-t)\,B+tA)\,y,y\rangle)\\&\leq\frac{1}{2}t\,(1-t)\,(\exp(-\langle(2A-B)\,x,x\rangle-\langle(2A-B)\,y,y\rangle)\\&-2\exp(-\langle Ax,x\rangle-\langle Ay,y\rangle)+\exp(-\langle Bx,x\rangle-\langle By,y\rangle)),\end{aligned} \tag{2.17}$$

for $x,\,y\in\mathbb{R}^n$ and $t\in[0,1]$.

Since $2B-A>0$, hence we can take the double integral on $\mathbb{R}^n\times\mathbb{R}^n$ in the first inequality in (2.6) to get

$$\begin{aligned}(0\leq)\,&\frac{1}{2}t\,(1-t)\left(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\exp(-\langle Ax,x\rangle-\langle Ay,y\rangle)\,dxdy\right.\\&-2\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\exp(-\langle Bx,x\rangle-\langle By,y\rangle)\,dxdy\\&+\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\exp(-\langle(2B-A)\,x,x\rangle-\langle(2B-A)\,y,y\rangle)\,dxdy\\&\leq(1-t)\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\exp(-\langle Bx,x\rangle-\langle By,y\rangle)\,dxdy\\&+t\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\exp(-\langle Ax,x\rangle-\langle Ay,y\rangle)\,dxdy\\&-\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\exp(-\langle((1-t)\,B+tA)\,x,x\rangle-\langle((1-t)\,B+tA)\,y,y\rangle)\,dxdy,\end{aligned}$$

for $t \in [0,1]$.

By utilising the representation (2.12) we get

$$(0 \leq) \frac{1}{2} t(1-t)[K_n(A) - 2K_n(B) + K_n(2B-A)] \leq (1-t) K_n(B) + tK_n(A) - K_n((1-t)B + tA),$$

which is equivalent to (2.13). □

Corollary 2. *Let A, B be positive definite matrices. If $A \leq B$, then*

$$(0 \leq) \frac{1}{12}\left[[\det(A)]^{-1} - 2[\det(B)]^{-1} + [\det(2B-A)]^{-1}\right] \leq \frac{[\det(B)]^{-1} + [\det(A)]^{-1}}{2} - \int_0^1 [\det((1-t)B + tA)]^{-1} dt. \tag{2.18}$$

If $A \leq B < 2A$, then also

$$\frac{[\det(B)]^{-1} + [\det(A)]^{-1}}{2} - \int_0^1 [\det((1-t)B + tA)]^{-1} dt \leq \frac{1}{12}\left[[\det(2A-B)]^{-1} - 2[\det(A)]^{-1} + [\det(B)]^{-1}\right]. \tag{2.19}$$

3. The Case of Hermitian Matrices

A complex square matrix $H = (h_{ij})$, $i,j = 1,...,n$ is said to be Hermitian provided $h_{ij} = \overline{h_{ji}}$ for all i, $j = 1,...,n$. A Hermitian matrix is said to be positive definite if the Hermitian form $P(z) = \sum_{i,j=1}^{n} a_{ij} z_i \overline{z_j}$ is positive for all $z = (z_1,...,z_n) \in \mathbb{C}^n \setminus \{0\}$.

It is known that, see for instance [12, p. 215], for a positive definite Hermitian matrix H, we have

$$K_n(H) := \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \exp(-\langle \overline{z}, Hz \rangle) \, dxdy = \frac{\pi^n}{\det(H)}, \tag{3.1}$$

where $z = x + iy$ and dx and dy denote integration over real n-dimensional space $\mathbb{R}^n$. Here the inner product $\langle x, y \rangle$ is understood in the real sense, i.e. $\langle x, y \rangle = \sum_{k=1}^{n} x_k y_k$.

On making use of a similar argument to the one in Theorem 2 for the representation $K_n(\cdot)$ we can state the same inequalities for positive definite Hermitian matrices H and K.

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