DETERMINANT INEQUALITIES FOR POSITIVE DEFINITE MATRICES VIA HÖLDER, MINKOWSKI AND GRÜSS INTEGRAL INEQUALITIES

SILVESTRU SEVER DRAGOMIR[1,2]

Abstract. In this paper we prove among others that, if A, B, C are positive definite matrices, then

$$\left|\frac{\det(C)}{\det(C+A+B)}-\frac{\det(C)}{\det(C+A)}\frac{\det(C)}{\det(C+B)}\right| \leq \frac{1}{2}\left[\frac{\det(C)}{\det(C+2A)}-\frac{[\det(C)]^2}{[\det(C+A)]^2}\right]^{1/2}\leq\frac{1}{4}.$$

If $p,\ q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, then also

$$[\det(C+pA)]^{1/p}[\det(C+qB)]^{1/q}\geq[\det(C+A+B)].$$

1. Introduction

A real square matrix $A=(a_{ij})$, $i,\ j=1,...,n$ is *symmetric* provided $a_{ij}=a_{ji}$ for all $i,\ j=1,...,n$. A real symmetric matrix is said to be *positive definite* provided the quadratic form $Q(x)=\sum_{i,j=1}^{n}a_{ij}x_ix_j$ is positive for all $x=(x_1,...,x_n)\in\mathbb{R}^n\setminus\{0\}$. It is well known that a necessary and sufficient condition for the symmetric matrix A to be positive definite, and we write $A>0$, is that all determinants

$$\det(A_k)=\det(a_{ij}),\ \ i,\ \ j=1,...,k;\ \ k=1,...,n$$

are positive.

It is know that the following integral representation is valid, see [1, pp. 61-62] or [11, pp. 211-212]

$$J_n(A):=\int_{\mathbb{R}^n}\exp(-\langle Ax,x\rangle)dx:=\int_{-\infty}^{\infty}...\int_{-\infty}^{\infty}\exp(-\langle Ax,x\rangle)dx=\frac{\pi^{n/2}}{[\det(A)]^{1/2}}, \tag{1.1}$$

where A is a positive definite matrix of order n and $\langle\cdot,\cdot\rangle$ is the usual inner product on $\mathbb{R}^n$.

By utilizing the representation (1.1) and Hölder's integral inequality for multiple integrals one can prove the *logarithmic concavity* of the determinant that is due to Ky Fan ([1, p. 63] or [11, p. 212]), namely

$$\det((1-\lambda)A+\lambda B)\geq[\det(A)]^{1-\lambda}[\det(B)]^{\lambda} \tag{1.2}$$

for any positive definite matrices A, B and $\lambda\in[0,1]$.

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By mathematical induction we can get a generalization of (1.2) which was obtained by L. Mirsky in [10], see also [11, p. 212]

$$\det\left(\sum_{j=1}^{m}\lambda_j A_j\right)\geq\prod_{j=1}^{m}\left[\det\left(A_j\right)\right]^{\lambda_j},\ m\geq 2, \tag{1.3}$$

where $\lambda_j>0$, $j=1,...,m$ with $\sum_{j=1}^{m}\lambda_j=1$ and $A_j>0$, $j=1,...,m$.

If we write (1.3) for $A_j=B_j^{-1}$ we get

$$\det\left(\sum_{j=1}^{m}\lambda_j B_j^{-1}\right)\geq\prod_{j=1}^{m}\left[\det\left(B_j^{-1}\right)\right]^{\lambda_j}=\left(\prod_{j=1}^{m}\left[\det\left(B_j\right)\right]^{\lambda_j}\right)^{-1},$$

which also gives

$$\prod_{j=1}^{m}\left[\det\left(A_j\right)\right]^{\lambda_j}\geq\det\left[\left(\sum_{j=1}^{m}\lambda_j A_j^{-1}\right)^{-1}\right], \tag{1.4}$$

where $\lambda_j>0$, $j=1,...,m$ with $\sum_{j=1}^{m}\lambda_j=1$ and $A_j>0$, $j=1,...,m$.

Using the representation (1.1) one can also prove the result, see [11, p. 212],

$$\det\left(A\right)=\det\left(A_{1n}\right)\leq\det\left(A_{1k}\right)\det\left(A_{(k+1)n}\right),\ k=1,...,n; \tag{1.5}$$

where the determinant $\det\left(A_{rs}\right)$ is defined by

$$\det\left(A_{rs}\right)=\det\left(a_{ij}\right),i,\ j=r,...,s.$$

In particular,

$$\det\left(A\right)\leq a_{11}a_{22}...a_{nn}. \tag{1.6}$$

We recall also the Minkowski's type inequality,

$$\left[\det\left(A+B\right)\right]^{1/n}\geq\left[\det\left(A\right)\right]^{1/n}+\left[\det\left(B\right)\right]^{1/n} \tag{1.7}$$

for A, B positive definite matrices of order n. For other determinant inequalities see Chapter VIII of the classic book [11]. For some recent results see [5]-[9].

Motivated by the above results, in this paper we prove among others that, if A, B, C are positive definite matrices, then

$$\left|\frac{\det\left(C\right)}{\det\left(C+A+B\right)}-\frac{\det\left(C\right)}{\det\left(C+A\right)}\frac{\det\left(C\right)}{\det\left(C+B\right)}\right|$$
$$\leq\frac{1}{2}\left[\frac{\det\left(C\right)}{\det\left(C+2A\right)}-\frac{\left[\det\left(C\right)\right]^2}{\left[\det\left(C+A\right)\right]^2}\right]^{1/2}\leq\frac{1}{4}.$$

If p, $q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, then also

$$\left[\det\left(C+pA\right)\right]^{1/p}\left[\det\left(C+qB\right)\right]^{1/q}\geq\left[\det\left(C+A+B\right)\right].$$

2. Inequalities Via One Variable Representation

Let $(\Omega, \mathcal{A}, \mu)$ be a measurable space consisting of a set Ω, a σ-algebra $\mathcal{A}$ of parts of Ω and a countably additive and positive measure μ on $\mathcal{A}$ with values in $\mathbb{R}\cup\{\infty\}$. Consider the *Lebesgue space*

$$L(\Omega,\mu) := \{f : \Omega \to \mathbb{R},\ f \text{ is } \mu\text{-measurable and } \int_\Omega |f(t)|\, d\mu(t) < \infty\}.$$

For simplicity of notation we write everywhere in the sequel $\int_\Omega w d\mu$ instead of $\int_\Omega w(t)\, d\mu(t)$.

Theorem 1. *Assume that A, B are positive definite matrices, $C \geq 0$ and p, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, then*

$$[\det(C + pA)]^{1/p} [\det(C + qB)]^{1/q} \geq [\det(C + A + B)]. \tag{2.1}$$

In particular, for $p = q = 2$ we get

$$\det(C + 2A)\det(C + 2B) \geq [\det(C + A + B)]^2. \tag{2.2}$$

Proof. We use the following weighted Hölder's inequality for p, $q > 1$ with $\frac{1}{p}+\frac{1}{q} = 1$,

$$\left|\int_\Omega h(t) f(s) g(t)\, d\mu(t)\right| \leq \left(\int_\Omega h(t) |f(t)|^p d\mu(t)\right)^{1/p} \left(\int_\Omega h(t) |g(t)|^q d\mu(t)\right)^{1/q}$$

where h is measurable and nonnegative on Ω and $h|f|^p$, $h|g|^q \in L(\Omega,\mu)$.

If we take $\Omega = \mathbb{R}^n$, $h(x) = \exp(-\langle Cx, x\rangle)$, $f(x) = \exp(-\langle Ax, x\rangle)$ and $g(x) = \exp(-\langle Bx, x\rangle)$, then we get

$$\int_{\mathbb{R}^n} \exp(-\langle (C + A + B)x, x\rangle)\, dx \leq \left(\int_{\mathbb{R}^n} \exp(-\langle (C + pA)x, x\rangle)\, dx\right)^{1/p} \left(\int_{\mathbb{R}^n} \exp(-\langle (C + qB)x, x\rangle)\, dx\right)^{1/q}.$$

Using the representation (1.1) we get

$$J_n(C + A + B) \leq [J_n(C + pA)]^{1/p} [J_n(C + qB)]^{1/q},$$

namely, by the second equality in (1.1)

$$\frac{\pi^{n/2}}{[\det(C + A + B)]^{1/2}} \leq \left[\frac{\pi^{n/2}}{[\det(C + pA)]^{1/2}}\right]^{1/p} \left[\frac{\pi^{n/2}}{[\det(C + qB)]^{1/2}}\right]^{1/q},$$

which is equivalent to

$$[\det(C + pA)]^{1/(2p)} [\det(C + qB)]^{1/(2q)} \geq [\det(C + A + B)]^{1/2}$$

and the inequality (2.1) is proved. □

Remark 1. *If we take $C = 0$ in (2.1) then we get for p, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, that*

$$p^{n/p} q^{n/q} [\det(A)]^{1/p} [\det(B)]^{1/q} \geq \det(A + B). \tag{2.3}$$

In particular, for $p = q = 2$ we get

$$4^n \det(A)\det(B) \geq [\det(A + B)]^2. \tag{2.4}$$

Also, if we take $C = I_n$ in Theorem 1, then we get

$$[\det(I_n + pA)]^{1/p} [\det(I_n + qB)]^{1/q} \geq [\det(I_n + A + B)]. \tag{2.5}$$

for $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$.

In particular, for $p = q = 2$ we get

$$\det(I_n + 2A) \det(I_n + 2B) \geq [\det(I_n + A + B)]^2. \tag{2.6}$$

Remark 2. *Since, by Young's inequality*

$$\frac{1}{p}[\det(C + pA)] + \frac{1}{q}\det(C + qB) \geq [\det(C + pA)]^{1/p} [\det(C + qB)]^{1/q},$$

hence we get the additive inequality as well

$$\frac{1}{p}\det(C + pA) + \frac{1}{q}\det(C + qB) \geq [\det(C + A + B)] \tag{2.7}$$

for $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$.

In particular,

$$\frac{\det(C + 2A) + \det(C + 2B)}{2} \geq [\det(C + A + B)]. \tag{2.8}$$

We also have:

Theorem 2. *Assume that A, B are positive definite matrices, $C \geq 0$ and $p \geq 2$ a natural number, then*

$$\begin{aligned} &\left(\sum_{k=0}^{p} \binom{n}{k} [\det(C + (p-k)A + kB)]^{-1/2}\right)^{1/p} \\ &\leq (\det(C + pA))^{-1/(2p)} + (\det(C + pB))^{-1/(2p)}. \end{aligned} \tag{2.9}$$

In particular, for $p = 2$, we get

$$\begin{aligned} &\left([\det(C + 2A)]^{-1/2} + 2[\det(C + A + B)]^{-1/2} + [\det(C + 2B)]^{-1/2}\right)^{1/2} \\ &\leq (\det(C + 2A))^{-1/4} + (\det(C + 2B))^{-1/4}. \end{aligned} \tag{2.10}$$

Proof. We use weighted Minkowski's inequality for $p \geq 1$

$$\begin{aligned} &\left(\int_\Omega h(t) |f(t) + g(t)|^p d\mu(t)\right)^{1/p} \\ &\leq \left(\int_\Omega h(t) |f(t)|^p d\mu(t)\right)^{1/p} + \left(\int_\Omega h(t) |g(t)|^p d\mu(t)\right)^{1/p}. \end{aligned}$$

If we take $\Omega = \mathbb{R}^n$, $h(x) = \exp(-\langle Cx, x\rangle)$, $f(x) = \exp(-\langle Ax, x\rangle)$ and $g(x) = \exp(-\langle Bx, x\rangle)$, then we get for natural number $p \geq 2$

$$\begin{aligned} &\left(\int_{\mathbb{R}^n} \exp(-\langle Cx, x\rangle)(\exp(-\langle Ax, x\rangle) + \exp(-\langle Bx, x\rangle))^p dx\right)^{1/p} \\ &\leq \left(\int_{\mathbb{R}^n} \exp(-\langle (C + pA)x, x\rangle) dx\right)^{1/p} \\ &+ \left(\int_{\mathbb{R}^n} \exp(-\langle (C + pB)x, x\rangle) dx\right)^{1/p}. \end{aligned} \tag{2.11}$$

Since

$$\begin{aligned}&\int_{\mathbb{R}^n} \exp\left(-\langle Cx, x\rangle\right)\left(\exp\left(-\langle Ax, x\rangle\right)+\exp\left(-\langle Bx, x\rangle\right)\right)^p dx\\&=\sum_{k=0}^{p}\binom{n}{k}\int_{\mathbb{R}^n}\exp\left(-\langle\left(C+(p-k)A+kB\right)x, x\rangle\right)dx,\end{aligned}$$

hence by (2.11) we get

$$\begin{aligned}&\left(\sum_{k=0}^{p}\binom{n}{k}\int_{\mathbb{R}^n}\exp\left(-\langle\left(C+(p-k)A+kB\right)x, x\rangle\right)dx\right)^{1/p}\\&\leq\left(\int_{\mathbb{R}^n}\exp\left(-\langle\left(C+pA\right)x, x\rangle\right)dx\right)^{1/p}+\left(\int_{\mathbb{R}^n}\exp\left(-\langle\left(C+pB\right)x, x\rangle\right)dx\right)^{1/p}.\end{aligned}$$

Using the representation (1.1) we get

$$\begin{aligned}&\left(\sum_{k=0}^{p}\binom{n}{k}J_n\left(C+(p-k)A+kB\right)\right)^{1/p}\\&\leq\left(J_n\left(C+pA\right)\right)^{1/p}+\left(J_n\left(C+pB\right)\right)^{1/p},\end{aligned}$$

which is equivalent to (2.9). □

Remark 3. *If we take the square in (2.10) and perform the required calculations, we also obtain (2.2). Also, if we take $C=0$, then we get*

$$\begin{aligned}&\left(\sum_{k=0}^{p}\binom{n}{k}\left[\det\left((p-k)A+kB\right)\right]^{-1/2}\right)^{1/p}\\&\leq p^{-n/(2p)}\left[\left(\det(A)\right)^{-1/(2p)}+\left(\det(B)\right)^{-1/(2p)}\right].\end{aligned}\tag{2.12}$$

Theorem 3. *Assume that A, B, C are positive definite matrices, then*

$$\begin{aligned}&\left|\frac{\left[\det(C)\right]^{1/2}}{\left[\det(C+A+B)\right]^{1/2}}-\frac{\left[\det(C)\right]^{1/2}}{\left[\det(C+A)\right]^{1/2}}\frac{\left[\det(C)\right]^{1/2}}{\left[\det(C+B)\right]^{1/2}}\right|\\&\leq\frac{1}{2}\left[\frac{\left[\det(C)\right]^{1/2}}{\left[\det(C+2A)\right]^{1/2}}-\frac{\left[\det(C)\right]}{\left[\det(C+A)\right]}\right]^{1/2}\leq\frac{1}{4}.\end{aligned}\tag{2.13}$$

Proof. We use the following pre-Grüss inequality

$$\begin{aligned}&\left|\frac{\int_\Omega h(t)f(t)g(t)d\mu(t)}{\int_\Omega h(t)d\mu(t)}-\frac{\int_\Omega h(t)f(t)d\mu(t)}{\int_\Omega h(t)d\mu(t)}\frac{\int_\Omega h(t)g(t)d\mu(t)}{\int_\Omega h(t)d\mu(t)}\right|\\&\leq\frac{1}{2}(\Gamma-\gamma)\left[\frac{\int_\Omega h(t)f^2(t)d\mu(t)}{\int_\Omega h(t)d\mu(t)}-\left(\frac{\int_\Omega h(t)f(t)d\mu(t)}{\int_\Omega h(t)d\mu(t)}\right)^2\right]^{1/2}\\&\leq\frac{1}{4}(\Gamma-\gamma)(\Delta-\delta),\end{aligned}\tag{2.14}$$

provided that h is measurable, nonnegative, $\int_\Omega h(t)\,d\mu(t)>0$ and $\gamma\leq f\leq\Gamma$, $\delta\leq g\leq\Delta$ μ-a.e. on Ω for the constants γ, Γ, δ and Δ.

If we take $\Omega = \mathbb{R}^n$, $h(x) = \exp(-\langle Cx, x\rangle)$, $f(x) = \exp(-\langle Ax, x\rangle)$ and $g(x) = \exp(-\langle Bx, x\rangle)$, then $0 < f(x) \leq 1$, $0 < g(x) \leq 1$ for $x \in \mathbb{R}^n$, and by (2.14) we get

$$\begin{aligned}
&\left| \frac{\int_{\mathbb{R}^n} \exp(-\langle (C+A+B)x, x\rangle) dx}{\int_{\mathbb{R}^n} \exp(-\langle Cx, x\rangle) dx} \right. \\
&\left. - \frac{\int_{\mathbb{R}^n} \exp(-\langle (C+A)x, x\rangle) dx}{\int_{\mathbb{R}^n} \exp(-\langle Cx, x\rangle) dx} \frac{\int_{\mathbb{R}^n} \exp(-\langle (C+B)x, x\rangle) dx}{\int_{\mathbb{R}^n} \exp(-\langle Cx, x\rangle) dx} \right| \\
&\leq \frac{1}{2}\left[\frac{\int_{\mathbb{R}^n} \exp(-\langle (C+2A)x, x\rangle) dx}{\int_{\mathbb{R}^n} \exp(-\langle Cx, x\rangle) dx} - \left(\frac{\int_{\mathbb{R}^n} \exp(-\langle (C+A)x, x\rangle) dx}{\int_{\mathbb{R}^n} \exp(-\langle Cx, x\rangle) dx} \right)^2 \right]^{1/2} \\
&\leq \frac{1}{4},
\end{aligned}$$

which, by the representation (1.1), we derive

$$\begin{aligned}
&\left| \frac{J_n(C+A+B)}{J_n(C)} - \frac{J_n(C+A)}{J_n(C)} \frac{J_n(C+B)}{J_n(C)} \right| \\
&\leq \frac{1}{2}\left[\frac{J_n(C+2A)}{J_n(C)} - \left(\frac{J_n(C+A)}{J_n(C)} \right)^2 \right]^{1/2} \leq \frac{1}{4}.
\end{aligned} \tag{2.15}$$

Using the second identity in (1.1), we have

$$\begin{aligned}
&\left| \frac{[\det(C+A+B)]^{-1/2}}{[\det(C)]^{-1/2}} - \frac{[\det(C+A)]^{-1/2}}{[\det(C)]^{-1/2}} \frac{[\det(C+B)]^{-1/2}}{[\det(C)]^{-1/2}} \right| \\
&\leq \frac{1}{2}\left[\frac{[\det(C+2A)]^{-1/2}}{[\det(C)]^{-1/2}} - \left(\frac{[\det(C+A)]^{-1/2}}{[\det(C)]^{-1/2}} \right)^2 \right]^{1/2} \leq \frac{1}{4},
\end{aligned}$$

which is equivalent to (2.13). □

Remark 4. *If we take $C = I_n$ in (2.13), then we obtain*

$$\begin{aligned}
&\left| \frac{1}{[\det(I_n+A+B)]^{1/2}} - \frac{1}{[\det(I_n+A)]^{1/2}} \frac{1}{[\det(I_n+B)]^{1/2}} \right| \\
&\leq \frac{1}{2}\left[\frac{1}{[\det(I_n+2A)]^{1/2}} - \frac{1}{[\det(I_n+A)]} \right]^{1/2} \leq \frac{1}{4}.
\end{aligned} \tag{2.16}$$

3. Inequalities Via Two Variables Representation

If we take the square in the representation (1.1), then we get

$$\left(\int_{\mathbb{R}^n} \exp(-\langle x, Ax\rangle) dx \right)^2 = \frac{\pi^n}{\det(A)}.$$

Since

$$\begin{aligned}
\left(\int_{\mathbb{R}^n} \exp(-\langle Ax, x\rangle) dx \right)^2 &= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \exp(-\langle Ax, x\rangle) \exp(-\langle Ay, y\rangle) dx dy \\
&= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \exp(-\langle Ax, x\rangle - \langle Ay, y\rangle)) dx dy,
\end{aligned}$$

hence

$$K_n(A) := \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \exp(-\langle Ax, x\rangle - \langle Ay, y\rangle))dxdy = \frac{\pi^n}{\det(A)} \tag{3.1}$$

for A a positive definite matrix of order n and $\langle\cdot,\cdot\rangle$ is the usual inner product on $\mathbb{R}^n$.

Theorem 4. *Assume that* A, B, C *are positive definite matrices, then*

$$\left|\frac{\det(C)}{\det(C+A+B)} - \frac{\det(C)}{\det(C+A)}\frac{\det(C)}{\det(C+B)}\right| \le \frac{1}{2}\left[\frac{\det(C)}{\det(C+2A)} - \frac{[\det(C)]^2}{[\det(C+A)]^2}\right]^{1/2} \le \frac{1}{4}. \tag{3.2}$$

In particular, for $C = I_n$*, we get*

$$\left|\frac{1}{\det(I_n+A+B)} - \frac{1}{\det(I_n+A)}\frac{1}{\det(I_n+B)}\right| \le \frac{1}{2}\left[\frac{1}{\det(I_n+2A)} - \frac{1}{[\det(I_n+A)]^2}\right]^{1/2} \le \frac{1}{4}. \tag{3.3}$$

Proof. If we take $\Omega = \mathbb{R}^n \times \mathbb{R}^n$, $h(x,y) = \exp(-\langle Cx, x\rangle - \langle Cy, y\rangle)$, $f(x,y) = \exp(-\langle Ax, x\rangle - \langle Ay, y\rangle)$ and $g(x,y) = \exp(-\langle Bx, x\rangle - \langle By, y\rangle)$, then $0 < f(x,y) \le 1$, $0 < g(x,y) \le 1$ for $(x,y) \in \mathbb{R}^n \times \mathbb{R}^n$, and by (2.14) we get

$$\begin{aligned}
&\left|\frac{\int_{\mathbb{R}^n}\exp(-\langle (C+A+B)x, x\rangle - \langle (C+A+B)y, y\rangle)dxdy}{\int_{\mathbb{R}^n}\exp(-\langle Cx, x\rangle - \langle Cy, y\rangle)dxdy}\right.\\
&- \frac{\int_{\mathbb{R}^n}\exp(-\langle (C+A)x, x\rangle - \langle (C+A)y, y\rangle)dxdy}{\int_{\mathbb{R}^n}\exp(-\langle Cx, x\rangle - \langle Cy, y\rangle)dxdy}\\
&\left.\times\frac{\int_{\mathbb{R}^n}\exp(-\langle (C+B)x, x\rangle - \langle (C+B)y, y\rangle)dxdy}{\int_{\mathbb{R}^n}\exp(-\langle Cx, x\rangle - \langle Cy, y\rangle)dxdy}\right|\\
&\le \frac{1}{2}\left[\frac{\int_{\mathbb{R}^n}\exp(-\langle (C+2A)x, x\rangle - \langle (C+2A)y, y\rangle)dxdy}{\int_{\mathbb{R}^n}\exp(-\langle Cx, x\rangle - \langle Cy, y\rangle)dxdy}\right.\\
&\left.-\left(\frac{\int_{\mathbb{R}^n}\exp(-\langle (C+A)x, x\rangle - \langle (C+A)y, y\rangle)dxdy}{\int_{\mathbb{R}^n}\exp(-\langle Cx, x\rangle - \langle Cy, y\rangle)dxdy}\right)^2\right]^{1/2}\\
&\le \frac{1}{4},
\end{aligned}$$

which, by the representation (1.1), we derive

$$\left|\frac{K_n(C+A+B)}{K_n(C)} - \frac{K_n(C+A)}{K_n(C)}\frac{K_n(C+B)}{K_n(C)}\right| \le \frac{1}{2}\left[\frac{K_n(C+2A)}{K_n(C)} - \left(\frac{K_n(C+A)}{K_n(C)}\right)^2\right]^{1/2} \le \frac{1}{4}.$$

Using the second identity in (3.1), we have

$$\left|\frac{[\det(C+A+B)]^{-1}}{[\det(C)]^{-1}}-\frac{[\det(C+A)]^{-1}}{[\det(C)]^{-1}}\frac{[\det(C+B)]^{-1}}{[\det(C)]^{-1}}\right|$$
$$\leq\frac{1}{2}\left[\frac{[\det(C+2A)]^{-1}}{[\det(C)]^{-1}}-\left(\frac{[\det(C+A)]^{-1}}{[\det(C)]^{-1}}\right)^{2}\right]^{1/2}\leq\frac{1}{4},$$

which is equivalent to (2.13). □

We also have:

Theorem 5. *Assume that* A, B *are positive definite matrices,* $C\geq 0$ *and* $p\geq 2$ *a natural number, then*

$$\left(\sum_{k=0}^{p}\binom{n}{k}[\det(C+(p-k)A+kB)]^{-1}\right)^{1/p}\leq(\det(C+pA))^{-1/p}+(\det(C+pB))^{-1/p}. \tag{3.4}$$

In particular, we also have

$$\left(\sum_{k=0}^{p}\binom{n}{k}[\det((p-k)A+kB)]^{-1}\right)^{1/p}\leq p^{-n/p}\left[(\det(A))^{-1/p}+(\det(B))^{-1/p}\right]. \tag{3.5}$$

A complex square matrix $H=(h_{ij})$, $i,j=1,...,n$ is said to be Hermitian provided $h_{ij}=\overline{h_{ji}}$ for all i, $j=1,...,n$. A Hermitian matrix is said to be positive definite if the Hermitian form $P(z)=\sum_{i,j=1}^{n}a_{ij}z_i\overline{z_j}$ is positive for all $z=(z_1,...,z_n)\in\mathbb{C}^n\setminus\{0\}$.

It is known that, see for instance [11, p. 215], for a positive definite Hermitian matrix H, we have

$$K_n(H):=\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\exp(-\langle\overline{z},Hz\rangle)\,dxdy=\frac{\pi^n}{\det(H)}, \tag{3.6}$$

where $z=x+iy$ and dx and dy denote integration over real n-dimensional space $\mathbb{R}^n$. Here the inner product $\langle x,y\rangle$ is understood in the real sense, i.e. $\langle x,y\rangle=\sum_{k=1}^{n}x_ky_k$.

On making use of a similar argument to the one in Theorem 4 for the representation $K_n(\cdot)$ we can state the same inequalities for positive definite Hermitian matrices H and K.

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[1]Mathematics, College of Engineering & Science, Victoria University, PO Box 14428, Melbourne City, MC 8001, Australia.

E-mail address: sever.dragomir@vu.edu.au

URL: http://rgmia.org/dragomir

[2]DST-NRF Centre of Excellence in the Mathematical, and Statistical Sciences, School of Computer Science, & Applied Mathematics, University of the Witwatersrand,, Private Bag 3, Johannesburg 2050, South Africa